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Complete moment convergence of weighted sums for arrays of negatively dependent random variables and its applications

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ABSTRACT

The authors study the complete moment convergence of weighted sums for arrays of rowwise negatively dependent random variables. The obtained results improve the corresponding results of Baek and Park (2010). Convergence of weighted sums for arrays of negatively dependent random variables and its applications. As an application, the authors obtain the complete moment convergence of linear processes based on pairwise negatively dependent random variables. In addition, the authors point out a gap of the proof in Baek and Park (2010) and raise an open problem.

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1. Introduction

Lehmann (1966) introduced the following concept of pairwise negatively dependent (ND).

Definition 1.1. Two random variables X and Y are said to be negatively dependent if

$$P(X \le x, Y \le y) \le P(X \le x)P(Y \le y) \quad \text{for all } x \text{ and } y.$$

A sequence of random variables $\{X_n, n \ge 1\}$ is said to be pairwise negatively dependent if every pair of random variables in the sequence are negatively dependent.

Definition 1.2. The random variables $X_1, \ldots, X_k$ are said to be upper negatively dependent(UND) if for all real $x_1, \ldots, x_k$,

$$P(X_i > x_i, i = 1, 2, \ldots, k) \le \prod_{i=1}^{k} P(X_i > x_i),$$

and lower negatively dependent (LND) if

$$P(X_i \le x_i, i = 1, 2, \ldots, k) \le \prod_{i=1}^{k} P(X_i \le x_i).$$

Random variables $X_1, \ldots, X_k$ are said to be negatively dependent (ND) if they are both UND and LND. This concept was introduced by Ebrahimi and Ghosh (1981).

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Recently, many authors studied various properties for sequences of pairwise ND random variables. We refer the reader to Matula (1992), Wu (2002), Liang et al. (2002), Cabrera and Volodin (2005), Li and Yang (2008), Gan and Chen (2008), Baek et al. (2008), Sung et al. (2008), Meng and Lin (2009), Baek and Park (2010), Wu and Guan (2011), Sung (2012), and Wu and Wang (2012).

Meanwhile, sequences of ND random variables also have been an attractive research topic in the recent literature; see Taylor et al. (2001, 2002), Volodin (2002), Volodin and Cabrera (2006), Ko et al. (2006), Gan and Chen (2008), Wu and Zhu (2010), Qiu et al. (2011), and Wu et al. (2012).

A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant a if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty.$$

This notion was given firstly by Hsu and Robbins (1947).

Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \quad \text{for some or all } \varepsilon > 0,$$

then the above result was called the complete moment convergence by Chow (1988).

It is worthy to point out that the complete moment convergence is the more general version of the complete convergence, which will been shown in Sec. 3.

An array of random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is said to be stochastically dominated by a random variable X (write $\{X_{ni}\} \prec X$) if there exists a constant $C > 0$ such that

$$\sup_{n,i} P(|X_{ni}| > x) \leq CP(|X| > x), \quad \forall x > 0.$$

Stochastic dominance of $\{X_{ni}, i \geq 1, n \geq 1\}$ by the random variable X implies $E|X_{ni}|^p \leq CE|X|^p$ if the p-moment of $|X|$ exists, i.e., if $E|X|^p < \infty$.

Baek and Park (2010) studied the complete convergence of weighted sums for arrays of rowwise negatively dependent random variables. They obtained the following results.

Theorem 1.1. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise pairwise ND random variables with $EX_{ni} = 0$ and $\{X_{ni}\} \prec X$. Suppose that $\beta \geq -1$ and that $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of constants such that*

$$\sup_{i \geq 1} |a_{ni}| = O(n^{-\gamma}) \quad \textit{for some} \quad \gamma > 0 \tag{1.1}$$

and

$$\sum_{i=1}^{\infty} |a_{ni}| = O(n^{\alpha}) \quad \textit{for some} \quad \alpha \in [0, \gamma). \tag{1.2}$$

(a) *If $1 + \alpha + \beta > 0$ and there exists some $\delta > 0$ such that $\alpha/\gamma + 1 < \delta \leq 2$, $s = \max\{1 + (1 + \alpha + \beta)/\gamma,\ \delta\}$, and $E|X|^s < \infty$, then we have*

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right| > \varepsilon\right) < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{1.3}$$

(b) *If $1 + \alpha + \beta = 0$ and $E(|X| \log(1 + |X|)) < \infty$, then (1.3) remains true.*

Remark 1.1. We find that there exists a gap in Baek and Park (2010). When they proved $I_2^* < \infty$ in their paper, they presented $I_2^* \leq C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E|\varepsilon'_{ni}|^2$ and $\sum_{i=1}^{\infty} E|\varepsilon'_{ni}|^2 \leq Cn^{-\gamma[\delta-(1+\alpha/\gamma)]}$. We think that they can only get $I_2^* \leq C\sum_{n=1}^{\infty} n^{\beta-\gamma[\delta-(1+\alpha/\gamma)]}$ instead of $I_2^* \leq C\sum_{n=1}^{\infty} n^{\beta-\gamma[\delta-(1+\alpha/\gamma)]M}$. In this case, to obtain $I_2^* < \infty$, it should be ensured that $\beta - \gamma[\delta - (1+\alpha/\gamma)] < -1$ holds, namely, $1 + (1+\alpha+\beta)/\gamma < \delta$. Then $1+\alpha+\beta < \gamma$ follows by $1 + (1+\alpha+\beta)/\gamma < \delta$ and $\delta \leq 2$.

Then we clarify why the proof of Beak and Park (2010) only holds true when $1+\alpha+\beta < \gamma$. If we assume that $1+\alpha+\beta \geq \gamma$, then $\beta \geq \gamma - 1 - \alpha$. Hence, we obtain

$$\beta - \gamma[\delta - (1+\alpha/\gamma)] = \beta - \gamma\delta + \gamma + \alpha \geq (2-\delta)\gamma - 1.$$

Since Beak and Park (2010) assume $\delta \leq 2$, we can only derive that

$$\beta - \gamma[\delta - (1+\alpha/\gamma)] \geq -1.$$

Hence, we derive that

$$I_2^* \leq C\sum_{n=1}^{\infty} n^{\beta-\gamma[\delta-(1+\alpha/\gamma)]} = \infty,$$

which does not imply that $I_2^* < \infty$. Therefore, by means of the method of Baek and Park (2010), we can not make sure whether (1.3) remains true for the case $1+\alpha+\beta \geq \gamma$.

Theorem 1.2. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise ND random variables with $EX_{ni} = 0$ and $\{X_{ni}\} \prec X$, where $\{k_n, n \geq 1\}$ is a sequence of positive integers. Assume that $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ is an array of real numbers satisfying*

$$\max_{1\leq i\leq k_n} |a_{ni}| = O(\log^{-1} k_n) \tag{1.4}$$

and

$$\sum_{i=1}^{k_n} a_{ni}^2 = o(\log^{-1} k_n). \tag{1.5}$$

If $Ee^{\eta|X|} < \infty$ for all $\eta > 0$, then

$$\sum_{n=1}^{\infty} k_n^{\beta} P\left(\left|\sum_{i=1}^{k_n} a_{ni}X_{ni}\right| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0 \text{ and } \beta \geq 0. \tag{1.6}$$

The main goal of this article is to obtain some results on the complete moment convergence for arrays of rowwise ND random variables. These results improve Theorems 1.1 and 1.2 (see Remarks 3.1 and 3.2). As an application, we establish the complete moment convergence of linear processes based on pairwise negatively dependent random variables, which improves Theorem 4.1 of Baek and Park (2010).

Throughout this article, the symbol C represents positive constants whose values may change from one place to another.

2. Preliminaries

In this section, we state some lemmas which will be used in the proof of our main result.

Lemma 2.1. (Ebrahimi and Ghosh, 1981) *If $\{X_n, n \geq 1\}$ is a sequence of ND random variables and $\{f_n, n \geq 1\}$ is a sequence of monotone increasing, Borel functions, then $\{f_n(X_n), n \geq 1\}$ is a sequence of ND random variables.*

Lemma 2.2. (Bozorgnia et al., 1996) *If $X_1, \ldots, X_n$ be a finite sequence of ND random variables, and $t_1, \ldots, t_n$ be all nonnegative (nonpositive), then*

$$Ee^{\sum_{i=1}^{n} t_i X_i} \leq \prod_{i=1}^{n} Ee^{t_i X_i}.$$

Lemma 2.3. (Wu, 2002) *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise ND random variables with mean zero and $EX_n^2 < \infty$, and $T_j(k) = \sum_{i=j+1}^{j+k} X_i$, $j \geq 0$. Then,*

$$E(T_j(k))^2 \leq C \sum_{i=j+1}^{j+k} EX_i^2, \quad E \max_{1\leq k\leq n} (T_j(k))^2 \leq C \log^2 n \sum_{i=j+1}^{j+n} EX_i^2.$$

Lemma 2.4. (Burton and Dehling, 1990) *Let $\sum_{i=-\infty}^{\infty} a_i$ be an absolutely convergent series of real numbers with $a = \sum_{i=-\infty}^{\infty} a_i$ and $b = \sum_{i=-\infty}^{\infty} |a_i|$. Suppose $\Phi : [-b, b] \to R$ is a function satisfying the following conditions.*

(i) Φ is bounded and continuous at a.

(ii) There exist $\delta > 0$ and $C > 0$ such that for all $|x| \leq \delta$, $|\Phi(x)| \leq C|x|$. Then,

$$\lim_{n\to\infty} n^{-1} \sum_{i=-\infty}^{\infty} \Phi\left(\sum_{j=i+1}^{i+n} a_j \right) = \Phi(a).$$

Lemma 2.5. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables with $\{X_{ni}\} \prec X$. Then there exists a constant C such that, for all $q > 0$ and $x > 0$:*

(i) $E|X_{ni}|^q I(|X_{ni}| \leq x) \leq C\{E|X|^q I(|X| \leq x) + x^q P(|X| > x)\}$ and

(ii) $E|X_{ni}|^q I(|X_{ni}| > x) \leq CE|X|^q I(|X| > x)$.

This lemma can be easily proved by using integration by parts. We omit the details.

3. Complete moment convergence of the weighted sums

Theorem 3.1. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise pairwise ND random variables with $EX_{ni} = 0$ and $\{X_{ni}\} \prec X$. Suppose that $\beta \geq -1$ and that $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of constants satisfying (1.1) and (1.2):*

(a) If $0 < 1 + \alpha + \beta < \gamma$ and $E|X|^{1+(1+\alpha+\beta)/\gamma} < \infty$, then

$$\sum_{n=1}^{\infty} n^{\beta} E\left\{ \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| - \varepsilon \right\}_{+} < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{3.1}$$

(b) If $1 + \alpha + \beta = 0$ and $E(|X| \log(1 + |X|)) < \infty$, then (3.1) remains true.

Remark 3.1. As stated in Remark 1.1, since we can not make sure whether (1.3) holds for $1 + \alpha + \beta \geq \gamma$, we can only consider the cases $0 < 1 + \alpha + \beta < \gamma$ and $1 + \alpha + \beta = 0$.

As we point out in Remark 1.1, Theorem 1.1(a) remains true for $0 < 1 + \alpha + \beta < \gamma$ and $1 + (1 + \alpha + \beta)/\gamma < \delta$. Thus $s = \max\{1 + (1 + \alpha + \beta)/\gamma, \ \delta\} = \delta$. Noting that $E|X|^s < \infty$ implies $E|X|^{1+(1+\alpha+\beta)/\gamma} < \infty$ and

$$\sum_{n=1}^{\infty} n^{\beta} E\left\{ \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| - \varepsilon \right\}_{+} = \sum_{n=1}^{\infty} n^{\beta} \int_0^{\infty} P\left(\left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon + t \right) dt$$

$$\geq \sum_{n=1}^{\infty} n^{\beta} \int_0^{\varepsilon} P\left(\left|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right| > \varepsilon + t\right)\mathrm{d}t \geq \varepsilon \sum_{n=1}^{\infty} n^{\beta} P\left(\left|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right| > 2\varepsilon\right),$$

we know that Theorem 3.1 improves Theorem 1.1.

Theorem 3.2. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise ND random variables with $EX_{ni} = 0$ and $\{X_{ni}\} \prec X$, where $\{k_n, n \geq 1\}$ is a sequence of positive integers such that $k_n \uparrow \infty$ strictly. Assume that $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ is an array of real numbers satisfying (1.4) and (1.5). If $Ee^{\eta|X|} < \infty$ for all $\eta > 0$, then*

$$\sum_{n=1}^{\infty} k_n^{\beta} E\left\{\left|\sum_{i=1}^{k_n} a_{ni}X_{ni}\right| - \varepsilon\right\}_+ < \infty \quad \text{for all} \quad \varepsilon > 0 \quad \text{and} \quad \beta \geq 0. \tag{3.2}$$

Remark 3.2. Since the conditions of Theorems 3.2 and 1.2 are same and

$$\sum_{n=1}^{\infty} k_n^{\beta} E\left\{\left|\sum_{i=1}^{k_n} a_{ni}X_{ni}\right| - \varepsilon\right\}_+ \geq \varepsilon \sum_{n=1}^{\infty} k_n^{\beta} P\left(\left|\sum_{i=1}^{k_n} a_{ni}X_{ni}\right| > 2\varepsilon\right),$$

Theorem 3.2 improves Theorem 1.2.

Proof of Theorem 3.1. Let $S_n = \sum_{i=1}^{\infty} a_{ni}X_{ni}$. For any given $\varepsilon > 0$,

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\beta} E\{|S_n| - \varepsilon\}_+ &= \sum_{n=1}^{\infty} n^{\beta} \int_0^{\infty} P(|S_n| > \varepsilon + t)\mathrm{d}t \\
&= \sum_{n=1}^{\infty} n^{\beta} \int_0^{1} P(|S_n| > \varepsilon + t)\mathrm{d}t + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P(|S_n| > \varepsilon + t)\mathrm{d}t \\
&\leq \sum_{n=1}^{\infty} n^{\beta} P(|S_n| > \varepsilon) + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P(|S_n| > t)\mathrm{d}t \\
&=: I_1 + I_2.
\end{aligned}$$

To prove (3.1), it need only to show that $I_1 < \infty$ and $I_2 < \infty$. By Theorem 1.1, we have $I_1 < \infty$. Next, we prove $I_2 < \infty$. For all $t \geq 1$, let $Y_{ni} = -tI(a_{ni}X_{ni} < -t) + a_{ni}X_{ni}I(|a_{nk}X_{ni}| \leq t) + tI(a_{ni}X_{ni} > t)$, $Z_{ni} = a_{ni}X_{ni} - Y_{ni}$, $S'_n = \sum_{i=1}^{\infty} Y_{ni}$. Then we have

$$\begin{aligned}
I_2 &\leq \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} \int_1^{\infty} P(|a_{ni}X_{ni}| > t)\mathrm{d}t + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P(|S'_n| > t)\mathrm{d}t \\
&=: I_3 + I_4.
\end{aligned}$$

First, we prove $I_3 < \infty$. From (1.1) and (1.2), without loss of generality, we assume

$$\sup_{i \geq 1} |a_{ni}| = n^{-\gamma} \tag{3.3}$$

and

$$\sum_{i=1}^{\infty} |a_{ni}| = n^{\alpha}. \tag{3.4}$$

Noting that $\int_1^{\infty} P(|a_{ni}X_{ni}| > t)\mathrm{d}t \leq E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}| > 1)$. By Lemma 2.5, (3.3), and (3.4), we have

$$I_3 \leq \sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}| > 1)$$

$$\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{\infty} E|a_{ni}X|I(|X|>|a_{ni}|^{-1})\\
&\le C\sum_{n=1}^{\infty} n^{\alpha+\beta}E|X|I(|X|>n^{\gamma})\\
&= C\sum_{n=1}^{\infty} n^{\alpha+\beta}\sum_{m=n}^{\infty} E|X|I(m^{\gamma}<|X|\le(m+1)^{\gamma})\\
&= C\sum_{m=1}^{\infty} E|X|I(m^{\gamma}<|X|\le(m+1)^{\gamma})\sum_{n=1}^{m} n^{\alpha+\beta}.
\end{aligned}$$

If $1+\alpha+\beta>0$, by $E|X|^{1+(1+\alpha+\beta)/\gamma}<\infty$, we get

$$\begin{aligned}
I_3 &\le C\sum_{m=1}^{\infty} m^{1+\alpha+\beta}\ E|X|I(m^{\gamma}<|X|\le(m+1)^{\gamma})\\
&\le CE|X|^{1+(1+\alpha+\beta)/\gamma}<\infty.
\end{aligned}$$

If $1+\alpha+\beta=0$, by $E(|X|\log(1+|X|))<\infty$, we also get

$$\begin{aligned}
I_3 &\le C\sum_{m=1}^{\infty} \log m\ E|X|I(m^{\gamma}<|X|\le(m+1)^{\gamma})\\
&\le CE(|X|\log(1+|X|))<\infty.
\end{aligned}$$

Then we consider I_4. Noting that $|Z_{ni}|\le|a_{ni}X_{ni}|I(|a_{ni}X_{ni}|>t)$. By $EX_{ni}=0$, Lemma 2.5, (3.3), and (3.4),

$$\begin{aligned}
\sup_{t\ge1} t^{-1}\left|ES'_n\right| = \sup_{t\ge1} t^{-1}\left|\sum_{i=1}^{\infty} EZ_{ni}\right| &\le \sup_{t\ge1} t^{-1}\sum_{i=1}^{\infty} E|Z_{ni}|\\
&\le \sup_{t\ge1} t^{-1}\sum_{i=1}^{\infty} E|a_{ni}X_{ni}|I(|a_{ni}X_{ni}|>t)\\
&\le C\sup_{t\ge1} t^{-1}\sum_{i=1}^{\infty} E|a_{ni}X|I(|a_{ni}X|>t)\\
&\le C\sum_{i=1}^{\infty} E|a_{ni}X|I(|a_{ni}X|>1)\\
&\le C\sum_{i=1}^{\infty} E|a_{ni}X|^{1+(1+\alpha+\beta)/\gamma}I(|a_{ni}X|>1)\\
&\le Cn^{-(\beta+1)}E|X|^{1+(1+\alpha+\beta)/\gamma}I(|X|>n^{\gamma}).
\end{aligned}$$

If $1+\alpha+\beta>0$, by $\beta\ge-1$ and $E|X|^{1+(1+\alpha+\beta)/\gamma}<\infty$, we get

$$\sup_{t\ge1} t^{-1}\left|ES'_n\right|\to 0 \quad \text{as} \quad n\to\infty \tag{3.5}$$

If $1+\alpha+\beta=0$, by $\beta\ge-1$ and $E|X|<\infty$, we also get (3.5). Therefore, while n is sufficiently large, $|ES'_n|\le t/2$ holds uniformly for $t\ge1$. Then by the Markov inequality, Lemma

2.3, and Lemma 2.5, we have

$$
\begin{aligned}
I_4 &\leq \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P(|S'_n - ES'_n| > t/2)\mathrm{d}t \\
&\leq C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} \int_1^{\infty} t^{-2} EY_{ni}^2 \mathrm{d}t \\
&= C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} \int_1^{\infty} t^{-2}\left\{E(a_{ni}X_{ni})^2 I(|a_{ni}X_{ni}| \leq t) + t^2 P(|a_{ni}X_{ni}| > t)\right\}\mathrm{d}t \\
&\leq C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} \int_1^{\infty} t^{-2}\left\{E(a_{ni}X)^2 I(|a_{ni}X| \leq t) + 2t^2 P(|a_{ni}X| > t)\right\}\mathrm{d}t \\
&= C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} \int_1^{\infty} t^{-2} E(a_{ni}X)^2 I(|a_{ni}X| \leq 1)\mathrm{d}t \\
&\quad + C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} \int_1^{\infty} t^{-2} E(a_{ni}X)^2 I(1 < |a_{ni}X| \leq t)\mathrm{d}t \\
&\quad + C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} \int_1^{\infty} P(|a_{ni}X| > t)\mathrm{d}t \\
&=: I_5 + I_6 + I_7.
\end{aligned}
$$

By a similar argument as in the proof of $I_3 < \infty$, we can prove $I_7 < \infty$. For I_5, we have

$$
\begin{aligned}
I_5 &\leq C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E(a_{ni}X)^2 I(|X| \leq |a_{ni}|^{-1}) \\
&= C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E(a_{ni}X)^2 I(|X| \leq n^{\gamma}) + C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E(a_{ni}X)^2 I(n^{\gamma} < |X| \leq |a_{ni}|^{-1}) \\
&=: I_8 + I_9.
\end{aligned}
$$

By (3.3) and (3.4), we have

$$
\begin{aligned}
I_8 &\leq C\sum_{n=1}^{\infty} n^{\alpha+\beta-\gamma} EX^2 I(|X| \leq n^{\gamma}) \\
&= C\sum_{n=1}^{\infty} n^{\alpha+\beta-\gamma} \sum_{m=1}^{n} EX^2 I((m-1)^{\gamma} < |X| \leq m^{\gamma}) \\
&= C\sum_{m=1}^{\infty} EX^2 I((m-1)^{\gamma} < |X| \leq m^{\gamma}) \sum_{n=m}^{\infty} n^{\alpha+\beta-\gamma} \quad (\text{since } \alpha+\beta-\gamma < -1) \\
&\leq C\sum_{m=1}^{\infty} (m-1)^{1+\alpha+\beta-\gamma} EX^2 I((m-1)^{\gamma} < |X| \leq m^{\gamma}) \\
&\leq CE|X|^{1+(1+\alpha+\beta)/\gamma} < \infty.
\end{aligned}
$$

By a similar argument as in the proof of $I_3 < \infty$, we have

$$
I_9 \leq C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E|a_{ni}X| I(n^{\gamma} < |X| \leq |a_{ni}|^{-1})
$$

$$\leq C\sum_{n=1}^{\infty} n^{\alpha+\beta} E|X|I(|X| > n^{\gamma}) < \infty.$$

Finally, we prove $I_6 < \infty$. Noting that

$$\begin{aligned}
&\int_1^{\infty} t^{-2}E(a_{ni}X)^2 I(1 < |a_{ni}X| \leq t)\mathrm{d}t \\
&= \sum_{m=1}^{\infty}\int_m^{m+1} t^{-2}E(a_{ni}X)^2 I(1 < |a_{ni}X| \leq t)\mathrm{d}t \\
&\leq \sum_{m=1}^{\infty} m^{-2}E(a_{ni}X)^2 I(1 < |a_{ni}X| \leq m+1) \\
&= \sum_{m=1}^{\infty} m^{-2}\sum_{s=1}^{m} E(a_{ni}X)^2 I(s < |a_{ni}X| \leq s+1) \\
&= \sum_{s=1}^{\infty} E(a_{ni}X)^2 I(s < |a_{ni}X| \leq s+1)\sum_{m=s}^{\infty} m^{-2} \\
&\leq C\sum_{s=1}^{\infty}(s-1)^{-1}E(a_{ni}X)^2 I(s < |a_{ni}X| \leq s+1) \\
&\leq CE|a_{ni}X|I(|a_{ni}X| > 1).
\end{aligned}$$

Hence, by a similar argument as in the proof of $I_3 < \infty$, we have

$$I_6 \leq C\sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{\infty} E|a_{ni}X|I(|a_{ni}X| > 1) < \infty.$$

The proof is completed. □

Proof of Theorem 3.2. Since $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$, without loss of generality, we assume $a_{ni} \geq 0$. Noting that

$$\begin{aligned}
&\sum_{n=1}^{\infty} k_n^{\beta} E\left\{\left|\sum_{i=1}^{k_n} a_{ni}X_{ni}\right| - \varepsilon\right\}_{+} \\
&= \sum_{n=1}^{\infty} k_n^{\beta}\int_0^{\infty} P\left(\left|\sum_{i=1}^{k_n} a_{ni}X_{ni}\right| > \varepsilon + t\right)\mathrm{d}t \\
&\leq \sum_{n=1}^{\infty} k_n^{\beta} P\left(\left|\sum_{i=1}^{k_n} a_{ni}X_{ni}\right| > \varepsilon\right) + \sum_{n=1}^{\infty} k_n^{\beta}\int_1^{\infty} P\left(\left|\sum_{i=1}^{k_n} a_{ni}X_{ni}\right| > t\right)\mathrm{d}t \\
&=: I_{10} + I_{11}.
\end{aligned}$$

By Theorem 1.2, we get $I_{10} < \infty$. To prove (3.2), it suffices to show that $I_{11} < \infty$. From Definition 1.2, we know that $\{a_{ni}X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ is still an array of rowwise ND random variables. Noting that the inequality $e^x \leq 1 + x + \frac{1}{2}x^2 e^{|x|}$ for all $x \in R$. Taking $\lambda = q\log k_n$, where q is a large constant and its value will be specified below.

By Lemma 2.2, (1.4), and (1.5), we have

$$\sum_{n=1}^{\infty} k_n^{\beta}\int_1^{\infty} P\left(\sum_{i=1}^{k_n} a_{ni}X_{ni} > t\right)\mathrm{d}t$$

$$
\begin{aligned}
&\le \sum_{n=1}^{\infty} k_n^{\beta} \int_1^{\infty} e^{-\lambda t} E e^{\lambda \sum_{i=1}^{k_n} a_{ni} X_{ni}} \mathrm{d}t \\
&\le C \sum_{n=1}^{\infty} k_n^{\beta}\ \lambda^{-1} e^{-\lambda} E e^{\lambda \sum_{i=1}^{k_n} a_{ni} X_{ni}} \\
&\le C \sum_{n=1}^{\infty} k_n^{\beta-q}\ (\log k_n)^{-1} \prod_{i=1}^{k_n} E e^{\lambda a_{ni} X_{ni}} \\
&\le C \sum_{n=1}^{\infty} k_n^{\beta-q}\ (\log k_n)^{-1} \prod_{i=1}^{k_n} E\Big(1 + \lambda a_{ni} X_{ni} + \frac{1}{2}\lambda^2 a_{ni}^2 X_{ni}^2 e^{\lambda a_{ni}|X_{ni}|}\Big) \\
&= C \sum_{n=1}^{\infty} k_n^{\beta-q}\ (\log k_n)^{-1} \prod_{i=1}^{k_n} \Big(1 + \frac{1}{2}\lambda^2 a_{ni}^2 E X_{ni}^2 e^{\lambda a_{ni}|X_{ni}|}\Big) \\
&\le C \sum_{n=1}^{\infty} k_n^{\beta-q}\ (\log k_n)^{-1} \prod_{i=1}^{k_n} \Big(1 + C(\log k_n)^2 a_{ni}^2 E e^{(1+C)|X|}\Big) \\
&\le C \sum_{n=1}^{\infty} k_n^{\beta-q}\ (\log k_n)^{-1} \exp\Big\{C(\log k_n)^2 \sum_{i=1}^{k_n} a_{ni}^2\Big\} \qquad (\text{ taking } \ q > \beta + \varepsilon + 1 \ ) \\
&\le \ C \sum_{n=1}^{\infty} k_n^{\beta-q+\varepsilon}\ (\log k_n)^{-1} < \infty.
\end{aligned} \tag{3.6}
$$

By replacing X_{ni} into $-X_{ni}$ in the above statement, we can also get

$$
\sum_{n=1}^{\infty} k_n^{\beta} \int_1^{\infty} P\Big(-\sum_{i=1}^{k_n} a_{ni} X_{ni} > t\Big) \mathrm{d}t \ < \ \infty. \tag{3.7}
$$

From (3.6) and (3.7), we have $I_{11} < \infty$. The proof is completed. □

4. Complete moment convergence of linear processes

In this section, we state one result about the complete moment convergence of linear processes which follows from Theorem 3.1. The result improves Theorem 4.1 of Baek and Park (2010).

Theorem 4.1. *Assume that $\{Y_i, -\infty < i < \infty\}$ is a sequence of rowwise pairwise ND random variables with $EY_i = 0$ and $\{Y_i\} \prec Y$ for some random variable Y. Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and $X_k = \sum_{i=-\infty}^{\infty} a_{i+k} Y_i$, $k \ge 1$.*

(1) Let $\beta > -1$, $1 \le p < 2$ and $(\beta + 2)p < 2$. If $E|Y|^{(\beta+2)p} < \infty$, then

$$
\sum_{n=1}^{\infty} n^{\beta - 1/p} E\left\{\left|\sum_{k=1}^{n} X_k\right| - \varepsilon n^{1/p}\right\}_+ < \infty \quad \textit{for all} \ \ \varepsilon > 0.
$$

(2) Let $1 < p < 2$. If $E|Y|^p < \infty$, then

$$
\sum_{n=1}^{\infty} n^{-1 - 1/p} E\left\{\left|\sum_{k=1}^{n} X_k\right| - \varepsilon n^{1/p}\right\}_+ < \infty \quad \textit{for all} \ \ \varepsilon > 0.
$$

(3) If $E(|Y|\log(1+|Y|)) < \infty$, then

$$\sum_{n=1}^{\infty} n^{-2} E\left\{\left|\sum_{k=1}^{n} X_k\right| - \varepsilon n\right\}_{+} < \infty \quad for \ \ all \ \ \varepsilon > 0.$$

Proof. Let $X_{ni} = Y_i$ and $a_{ni} = n^{-1/p}\sum_{k=1}^{n} a_{i+k}$ for $i \geq 1$ and $n \geq 1$. The result follows by Theorem 3.1 with $\alpha = 1 - 1/p$, $\gamma = 1/p$ and $1 \leq p < 2$ (see the proof of Theorem 4.1 in Baek and Park (2010)). □

Remark 4.1. Similarly to the statement of Remark 3.1, we know that this result improves Theorem 4.1 in Baek and Park (2010).

Finally, we present a difficult but very interesting problem as follows.

Open problem. As stated in Remarks 1.1 and 3.1, it is still unknown whether Theorem 1.1 or Theorem 3.1 remains true for the case $1 + \alpha + \beta \geq \gamma$. Despite our efforts to solve this problem, it is still an open problem.

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