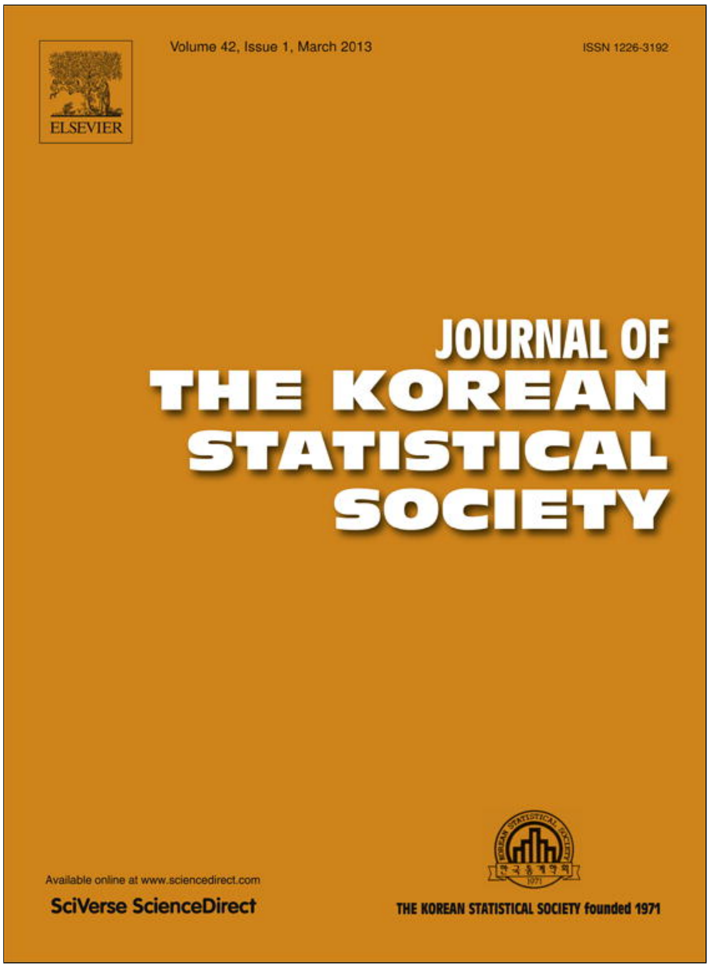
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On limiting behavior for arrays of rowwise negatively orthant dependent random variables

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ABSTRACT

In this paper, the authors study limiting behavior for arrays of rowwise negatively orthant dependent random variables and obtain some new results which extend and improve the corresponding theorems by Hu, Móricz, and Taylor (1989), Taylor, Patterson, and Bozorgnia (2002) and Wu and Zhu (2010).

1. Introduction

The concept of negatively orthant dependent (NOD) random variables was introduced by Ebrahimi and Ghosh (1981).

Definition 1.1. The random variables $X_1, \ldots, X_k$ are said to be negatively upper orthant dependent (NUOD) if for all real $x_1, \ldots, x_k$,

$$P(X_i > x_i,\ i = 1, 2, \ldots, k) \le \prod_{i=1}^{k} P(X_i > x_i), \tag{1.1}$$

and negatively lower orthant dependent (NLOD) if

$$P(X_i \le x_i,\ i = 1, 2, \ldots, k) \le \prod_{i=1}^{k} P(X_i \le x_i). \tag{1.2}$$

Random variables $X_1, \ldots, X_k$ are said to be negatively orthant dependent (NOD) if they are both NUOD and NLOD.

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It is easily seen that independent random variables and negatively associated (NA, in short, cf. Joag-Dev & Proschan, 1983) random variables are NOD. Since the paper of Ebrahimi and Ghosh (1981) appeared, the convergence properties of NOD random sequences have been studied in some papers. We refer the reader to Bozorgnia, Patterson, and Taylor (1996), Gan and Chen (2008), Ko, Han, and Kim (2006), Taylor et al. (2002), Volodin, Ordóñez Cabrera, and Hu (2006) and Wu and Zhu (2010).

A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant a if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty.$$

In this case we write $U_n \to a$ completely. This notion was given first by Hsu and Robbins (1947).

Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \quad \text{for some or all } \varepsilon > 0,$$

then the above result was called the complete moment convergence by Chow (1988).

An array of rowwise random variables $\{X_{nk},\ 1 \leq k \leq n, n \geq 1\}$ is said to be uniformly bounded by a random variable X (write $\{X_{nk}\} \prec X$) if there exists a constant $C > 0$ such that

$$\sup_{n,k} P(|X_{nk}| > x) \leq CP(|X| > x), \quad \forall x > 0.$$

Clearly if $\{X_{nk}\} \prec X$, for $0 < p < \infty$ and any $1 \leq k \leq n, n \geq 1$, $E|X_{nk}|^p \leq CE|X|^p$.

Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise NOD random variables and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Also, let $\{\Psi_n(t), n \geq 1\}$ be a sequence of nonnegative even functions satisfying

$$\frac{\Psi_n(|t|)}{|t|} \uparrow \quad \text{and} \quad \frac{\Psi_n(|t|)}{|t|^p} \downarrow \quad \text{as } |t| \uparrow. \tag{1.3}$$

Introduce the conditions

$$EX_{nk} = 0, \quad 1 \leq k \leq n,\ n \geq 1, \tag{1.4}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)} < \infty, \tag{1.5}$$

$$\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n} \frac{E|X_{nk}|^r}{a_n^r}\right]^s < \infty, \tag{1.6}$$

where $0 < r \leq 2$, $s > 0$.

Wu and Zhu (2010) obtained the following theorems.

Theorem A. *Let $\{X_{nk},\ 1 \leq k \leq n,\ n \geq 1\}$ be an array of rowwise NOD random variables and $\{a_n,\ n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Also, let $\{\Psi_n(t), n \geq 1\}$ be a sequence of nonnegative even functions satisfying (1.3) for some real number $1 < p \leq 2$. Then conditions (1.4) and (1.5) imply*

$$\frac{1}{a_n}\sum_{k=1}^{n} X_{nk} \to 0 \quad \textit{completely}. \tag{1.7}$$

Theorem B. *Let $\{X_{nk},\ 1 \leq k \leq n,\ n \geq 1\}$ be an array of rowwise NOD random variables and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Also, let $\{\Psi_n(t),\ n \geq 1\}$ be a sequence of nonnegative even functions satisfying (1.3) for $p > 2$. Then conditions (1.4)–(1.6) imply (1.7).*

Theorem C. *Let $\{X_{nk},\ 1 \leq k \leq n,\ n \geq 1\}$ be an array of rowwise NOD random variables and $\{a_n,\ n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Also, let $\{\Psi_n(t), n \geq 1\}$ be a sequence of nonnegative even functions satisfying (1.3) for some real number $1 < p \leq 2$. Then conditions (1.4) and (1.5) imply*

$$\sum_{n=1}^{\infty} a_n^{-1} E\left\{\left|\sum_{k=1}^{n} X_{nk}\right| - \varepsilon a_n\right\}_+ < \infty, \quad \forall \varepsilon > 0. \tag{1.8}$$

Theorem D. *Let $\{X_{nk},\ 1 \leq k \leq n,\ n \geq 1\}$ be an array of rowwise NOD random variables and $\{a_n, n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. Also, let $\{\Psi_n(t), n \geq 1\}$ be a sequence of nonnegative even functions satisfying (1.3) for $p > 2$. Then conditions (1.4)–(1.6) imply (1.8).*

In addition, Hu et al. (1989) obtained the following result in complete convergence.

Theorem E. *Let* $\{X_{nk},\ 1 \le k \le n,\ n \ge 1\}$ *be an array of rowwise independent random variables with* $EX_{nk} = 0$ *and* $\{X_{nk}\} \prec X$. *If* $E|X|^{2p} < \infty$ *for some* $1 \le p < 2$, *then*

$$n^{-1/p} \sum_{k=1}^{n} X_{nk} \to 0 \quad \textit{completely}. \tag{1.9}$$

In this work, we shall improve Theorems A–E under some weaker conditions. In addition, we study L^1 convergence and convergence in probability for the arrays of NOD random variables under some appropriate conditions.

Below, C will denote generic positive constants, whose value may vary from one application to another, $I(A)$ will indicate the indicator function of A.

2. Main results

Now we will present the main results of the paper. The proofs will be detailed in the next section.

Theorem 2.1. *Let* $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ *be an array of rowwise NOD random variables with* (1.4). *Suppose the following conditions hold:*

(i) *for every* $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} P(|X_{nk}| > \varepsilon) < \infty, \tag{2.1}$$

(ii) *for some* $\delta > 0$ *and* $\eta > 1$

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \le \delta) \right)^{\eta} < \infty. \tag{2.2}$$

(iii)

$$\sum_{k=1}^{n} E|X_{nk}| I(|X_{nk}| > \delta) \to 0 \quad \textit{as } n \to \infty. \tag{2.3}$$

Then

$$\sum_{k=1}^{n} X_{nk} \to 0 \quad \textit{completely}. \tag{2.4}$$

By a similar argument as the proof of Theorem 1 in Qiu, Chang, Antonini, and Volodin (2011), we can prove Theorem 2.1. Therefore, we will omit the details of the proof.

Let $\{a_n, n \ge 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$, and take X_{nk}/a_n instead of X_{nk} in Theorem 2.1, we can get the following corollary.

Corollary 2.1. *Let* $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ *be an array of rowwise NOD random variables with* (1.4), *and* $\{a_n, n \ge 1\}$ *be a sequence of positive real numbers with* $a_n \uparrow \infty$. *Suppose the following conditions hold:*

(i) *for every* $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} P(|X_{nk}| > a_n \varepsilon) < \infty, \tag{2.5}$$

(ii) *for some* $\delta > 0$ *and* $\eta > 1$

$$\sum_{n=1}^{\infty} \left(a_n^{-2} \sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \le a_n \delta) \right)^{\eta} < \infty, \tag{2.6}$$

(iii)

$$a_n^{-1} \sum_{k=1}^{n} E|X_{nk}| I(|X_{nk}| > a_n \delta) \to 0 \quad \textit{as } n \to \infty. \tag{2.7}$$

Then (1.7) *holds.*

Remark 2.1. The following statements show that the conditions of Corollary 2.1 are weaker than those of Theorems A and B.

First, without loss of generality we may assume $0 < \varepsilon < 1$. By the Markov inequality, (1.3) and (1.5), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=1}^{n}P(|X_{nk}|>a_n\varepsilon) &\le \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E|X_{nk}|}{a_n\varepsilon}I(|X_{nk}|>a_n\varepsilon)\\ &\le \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E|X_{nk}|^p}{(a_n\varepsilon)^p}I(a_n\varepsilon<|X_{nk}|\le a_n)+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E|X_{nk}|}{a_n\varepsilon}I(|X_{nk}|>a_n)\\ &\le (\varepsilon^{-p}+\varepsilon^{-1})\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)}<\infty.\end{aligned}$$

Second, we state that (1.3), (1.5) and (1.6) implies (2.6). For $\delta < 1$ and $1 < p \le 2$, by (1.3) and (1.5), we can get easily

$$\sum_{n=1}^{\infty}\left(a_n^{-2}\sum_{k=1}^{n}EX_{nk}^2I(|X_{nk}|\le a_n\delta)\right)^{\eta}\le\sum_{n=1}^{\infty}\left(a_n^{-2}\sum_{k=1}^{n}EX_{nk}^2I(|X_{nk}|\le a_n)\right)^{\eta}\le\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)}\right)^{\eta}<\infty.$$

For $\delta < 1$ and $p > 2$, take $0 < r \le 2$, $s > 0$ and $\eta > \max\{1, s\}$. By (1.3) and (1.6), we can get

$$\sum_{n=1}^{\infty}\left(a_n^{-2}\sum_{k=1}^{n}EX_{nk}^2I(|X_{nk}|\le a_n\delta)\right)^{\eta}\le\sum_{n=1}^{\infty}\left(a_n^{-2}\sum_{k=1}^{n}EX_{nk}^2I(|X_{nk}|\le a_n)\right)^{\eta}\le\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{E|X_{nk}|^r}{a_n^r}\right)^{s}\right)^{\eta/s}<\infty.$$

For $\delta \ge 1$ and $1 < p \le 2$, by (1.3) and (1.5), we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(a_n^{-2}\sum_{k=1}^{n}EX_{nk}^2I(|X_{nk}|\le a_n\delta)\right)^{\eta}\\ &=\sum_{n=1}^{\infty}\left(a_n^{-2}\sum_{k=1}^{n}EX_{nk}^2I(|X_{nk}|\le a_n)\right)^{\eta}+\sum_{n=1}^{\infty}\left(a_n^{-2}\sum_{k=1}^{n}EX_{nk}^2I(a_n<|X_{nk}|\le a_n\delta)\right)^{\eta}\\ &\le\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)}\right)^{\eta}+\delta^{\eta}\sum_{n=1}^{\infty}\left(a_n^{-1}\sum_{k=1}^{n}E|X_{nk}|I(a_n<|X_{nk}|\le a_n\delta)\right)^{\eta}\\ &\le(1+\delta^{\eta})\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)}\right)^{\eta}<\infty.\end{aligned}$$

For $\delta \ge 1$ and $p > 2$, by (1.3), (1.5) and (1.6), we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(a_n^{-2}\sum_{k=1}^{n}EX_{nk}^2I(|X_{nk}|\le a_n\delta)\right)^{\eta}\\ &=\sum_{n=1}^{\infty}\left(a_n^{-2}\sum_{k=1}^{n}EX_{nk}^2I(|X_{nk}|\le a_n)\right)^{\eta}+\delta^{\eta}\sum_{n=1}^{\infty}\left(a_n^{-1}\sum_{k=1}^{n}E|X_{nk}|I(a_n<|X_{nk}|\le a_n\delta)\right)^{\eta}\\ &\le\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{E|X_{nk}|^r}{a_n^r}\right)^{s}\right)^{\eta/s}+\delta^{\eta}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)}\right)^{\eta}<\infty.\end{aligned}$$

Finally, we state that (1.3), (1.5) and (1.6) implies (2.7). For $\delta \ge 1$, by (1.3) and (1.5), we have

$$\sum_{k=1}^{n}\frac{E|X_{nk}|}{a_n}I(|X_{nk}|>a_n\delta)\le\sum_{k=1}^{n}\frac{E|X_{nk}|}{a_n}I(|X_{nk}|>a_n)\le\sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)}\to 0\quad\text{as } n\to\infty.$$

For $\delta < 1$, by (1.3) and (1.5), we also have

$$\begin{aligned}\sum_{k=1}^{n}\frac{E|X_{nk}|}{a_n}I(|X_{nk}|>a_n\delta)&\le\sum_{k=1}^{n}\frac{E|X_{nk}|}{a_n}I(|X_{nk}|>a_n)+\delta^{1-p}\sum_{k=1}^{n}\frac{E|X_{nk}|^p}{a_n^p}I(a_n\delta<|X_{nk}|\le a_n)\\ &\le(1+\delta^{1-p})\sum_{k=1}^{n}\frac{E\Psi_k(X_{nk})}{\Psi_k(a_n)}\to 0\quad\text{as } n\to\infty.\end{aligned}$$

To sum up, we know that Corollary 2.1 improve Theorems A and B.

Take $X_{nk}/n^{1/p}$ $(1 \le p < 2)$ instead of X_{nk} in Theorem 2.1, we can get the following corollary.

Corollary 2.2. *Let $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of rowwise NOD random variables with (1.4). Suppose the following conditions hold:*

(i) *for every $\varepsilon > 0$*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} P(|X_{nk}| > n^{1/p}\varepsilon) < \infty, \tag{2.8}$$

(ii) *for some $\delta > 0$ and $\eta > p/(2-p)$*

$$\sum_{n=1}^{\infty}\left(n^{-2/p}\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \le n^{1/p}\delta)\right)^{\eta} < \infty, \tag{2.9}$$

(iii)

$$n^{-1/p}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > n^{1/p}\delta) \to 0 \quad \text{as } n \to \infty, \tag{2.10}$$

where $1 \le p < 2$. Then (1.9) holds.

Remark 2.2. The following statements show that the conditions of Corollary 2.2 are weaker than those of Theorem E.

First, by $\{X_{nk}\} \prec X$ and $E|X|^{2p} < \infty$, we have

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} P(|X_{nk}| > n^{1/p}\varepsilon) \le \sum_{n=1}^{\infty} nP(|X| > n^{1/p}\varepsilon) \le CE|X|^{2p} < \infty.$$

Second, since $E|X|^{2p} < \infty$ for $1 \le p < 2$, we know $E|X|^2 < \infty$. Hence, by $\eta > p/(2-p)$ and $\{X_{nk}\} \prec X$, we have

$$\sum_{n=1}^{\infty}\left(n^{-2/p}\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \le n^{1/p}\delta)\right)^{\eta} \le C\sum_{n=1}^{\infty} n^{(1-2/p)\eta}(E|X|^2)^{\eta} < \infty.$$

Finally, by $\{X_{nk}\} \prec X$ and $E|X|^{2p} < \infty$, we have

$$n^{-1/p}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > n^{1/p}\delta) \le \delta^{1-2p}\sum_{k=1}^{n}\frac{E|X_{nk}|^{2p}}{n^2}I(|X_{nk}| > n^{1/p}\delta) \le \delta^{1-2p}n^{-1}E|X|^{2p} \to 0 \quad \text{as } n \to \infty.$$

To sum up, we know that Corollary 2.2 extends and improves Theorem E. In addition, Corollary 2.2 also improves partially Theorem 3.1 by Taylor et al. (2002).

The following theorem shows that, under some appropriate conditions, we can obtain complete moment convergence for the array of rowwise NOD random variables.

Theorem 2.2. *Let $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of rowwise NOD random variables with (1.4). Suppose that for some $\delta > 0$ and $\eta > 1$*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > \delta/4\eta) < \infty. \tag{2.11}$$

Then (2.1), (2.2) and (2.11) imply

$$\sum_{n=1}^{\infty} E\left\{\left|\sum_{k=1}^{n} X_{nk}\right| - \varepsilon\right\}_{+} < \infty, \quad \text{for all } \varepsilon > 0. \tag{2.12}$$

Let $\{a_n, n \ge 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$, and take X_{nk}/a_n instead of X_{nk} in Theorem 2.2, we can get the following corollary.

Corollary 2.3. *Let $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of rowwise NOD random variables with (1.4). Suppose that for some $\delta > 0$ and $\eta > 1$*

$$\sum_{n=1}^{\infty} a_n^{-1}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > a_n\delta/4\eta) < \infty. \tag{2.13}$$

Then (2.5), (2.6) and (2.13) imply (1.8).

Remark 2.3. By a similar argument as in Remark 2.1, we can show that the conditions of Corollary 2.3 are weaker than those of Theorems C and D. Here we omit the details.

Theorem 2.3. *Let* $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ *be an array of rowwise NOD random variables with* (1.4)*. Suppose that for some* $\delta > 0$

$$\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > \delta) \to 0 \quad \text{as } n \to \infty, \tag{2.14}$$

$$\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \le \delta) \to 0 \quad \text{as } n \to \infty. \tag{2.15}$$

Then

$$\sum_{k=1}^{n} X_{nk} \xrightarrow{L^1} 0. \tag{2.16}$$

Let $\{a_n, n \ge 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$, and take X_{nk}/a_n instead of X_{nk} in Theorem 2.3, we can get the following corollary.

Corollary 2.4. *Let* $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ *be an array of rowwise NOD random variables with* (1.4)*. Suppose that for some* $\delta > 0$

$$a_n^{-1} \sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > a_n\delta) \to 0 \quad \text{as } n \to \infty, \tag{2.17}$$

$$a_n^{-2} \sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \le a_n\delta) \to 0 \quad \text{as } n \to \infty. \tag{2.18}$$

Then

$$a_n^{-1} \sum_{k=1}^{n} X_{nk} \xrightarrow{L^1} 0.$$

Remark 2.4. By a similar argument as in Remark 2.1, we can show that the conditions of Corollary 2.4 are weaker than those of Theorem 1.5 by Wu and Zhu (2010). Here we omit the details.

The following theorem shows that, under some weaker conditions, we can obtain convergence in probability for the array of rowwise NOD random variables.

Theorem 2.4. *Let* $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ *be an array of rowwise NOD random variables with* (1.4)*. Suppose that for some* $\delta > 0$

$$\sum_{k=1}^{n} P(|X_{nk}| > \delta) \to 0 \quad \text{as } n \to \infty. \tag{2.19}$$

Then (2.15) *and* (2.19) *imply*

$$\sum_{k=1}^{n} X_{nk} \xrightarrow{P} 0. \tag{2.20}$$

3. Proofs

To prove main results in this paper, we need the following lemmas.

Lemma 3.1 (*Cf. Bozorgnia et al., 1996*)**.** *Let random variables* $X_1, X_2, \ldots, X_n$ *be NOD and* $f_1, f_2, \ldots, f_n$ *be all nondecreasing (or nonincreasing) functions, then random variables* $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ *are NOD.*

Lemma 3.2 (*Cf. Wu & Zhu, 2010*)**.** *Let* $\{X_n, n \ge 1\}$ *be a sequence of NOD random variable with mean zero and* $0 < B_n = \sum_{k=1}^{n} EX_k^2 < \infty$*. Let* $S_n = \sum_{k=1}^{n} X_k$*, then*

$$P(|S_n| \geq x) \leq \sum_{k=1}^{n} P(|X_k| \geq y) + 2\exp\left(\frac{x}{y} - \frac{x}{y}\log\left(1 + \frac{xy}{B_n}\right)\right)$$

for all $x > 0, y > 0$.

Lemma 3.3 (*Cf. Gan, Chen, & Qiu, 2011*). *Let* $\{X_n, n \geq 1\}$ *be a sequence of NOD mean zero random variables.* $S_n = \sum_{k=1}^{n} X_k,\ n \geq 1,\ p \geq 2$. *Then for any* $n \geq 1$,

$$E|S_n|^p \leq C\left\{\sum_{k=1}^{n} E|X_k|^p + \left(\sum_{k=1}^{n} EX_k^2\right)^{p/2}\right\},$$

where C is a positive constant depending only on p. Especially we have

$$E|S_n|^2 \leq C\sum_{k=1}^{n} EX_k^2.$$

Remark before the proof of Theorem 2.2. A reviewer of the paper suggested an interesting idea to provide a proof of Theorem 2.2 based on Theorem 2.1 from the paper (Gan et al., 2011). But in this case we must add some stronger conditions than we have in Theorem 2.2. For example, $\sum_{n=1}^{\infty}\sum_{k=1}^{n} E|X_{nk}|^p I(|X_{nk}| \leq \delta) < \infty$ should be required. □

Proof of Theorem 2.2. Since

$$\begin{aligned}\sum_{n=1}^{\infty} E\left\{\left|\sum_{k=1}^{n} X_{nk}\right| - \varepsilon\right\}_+ &= \sum_{n=1}^{\infty}\int_0^{\infty} P\left(\left|\sum_{k=1}^{n} X_{nk}\right| - \varepsilon > t\right)\mathrm{d}t \\ &= \sum_{n=1}^{\infty}\left\{\int_0^{\delta} P\left(\left|\sum_{k=1}^{n} X_{nk}\right| > \varepsilon + t\right)\mathrm{d}t + \int_{\delta}^{\infty} P\left(\left|\sum_{k=1}^{n} X_{nk}\right| > \varepsilon + t\right)\mathrm{d}t\right\} \\ &\leq \delta\sum_{n=1}^{\infty} P\left(\left|\sum_{k=1}^{n} X_{nk}\right| > \varepsilon\right) + \sum_{n=1}^{\infty}\int_{\delta}^{\infty} P\left(\left|\sum_{k=1}^{n} X_{nk}\right| > t\right)\mathrm{d}t \\ &=: I_1 + I_2,\end{aligned}$$

to prove (2.12), it is enough to prove that $I_1 < \infty$ and $I_2 < \infty$. Noting that (2.11) implies (2.3), by Theorem 2.1 in this paper, we have $I_1 < \infty$. To prove (2.12), it suffices to prove $I_2 < \infty$. Let

$$\begin{aligned} Y_{nk} &= -tI(X_{nk} < -t) + X_{nk}I(|X_{nk}| \leq t) + tI(X_{nk} > t), \\ Z_{nk} &= X_{nk} - Y_{nk} = (X_{nk} + t)I(X_{nk} < -t) + (X_{nk} - t)I(X_{nk} > t).\end{aligned}$$

Obviously

$$P\left(\left|\sum_{k=1}^{n} X_{nk}\right| > t\right) \leq \sum_{k=1}^{n} P(|X_{nk}| > t) + P\left(\left|\sum_{k=1}^{n} Y_{nk}\right| > t\right).$$

Hence

$$\begin{aligned} I_2 &\leq \sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_{\delta}^{\infty} P(|X_{nk}| > t)\mathrm{d}t + \sum_{n=1}^{\infty}\int_{\delta}^{\infty} P\left(\left|\sum_{k=1}^{n} Y_{nk}\right| > t\right)\mathrm{d}t \\ &=: I_3 + I_4.\end{aligned}$$

For I_3, by (2.11), we have

$$I_3 \leq \sum_{n=1}^{\infty}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > \delta) < \infty.$$

By (1.4) and (2.11), we have

$$\begin{aligned}\max_{t\geq\delta} t^{-1}\left|\sum_{k=1}^{n} EY_{nk}\right| &= \max_{t\geq\delta} t^{-1}\left|\sum_{k=1}^{n} EZ_{nk}\right| \leq \max_{t\geq\delta} t^{-1}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > t) \\ &\leq \delta^{-1}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > \delta) \to 0 \quad \text{as } n \to \infty. \qquad (3.1)\end{aligned}$$

Therefore, while n is sufficiently large, $|\sum_{k=1}^{n} EY_{nk}| \le t/2$ holds uniformly for $t \ge \delta$. Then

$$P\left(\left|\sum_{k=1}^{n} Y_{nk}\right| > t\right) \le P\left(\left|\sum_{k=1}^{n} (Y_{nk} - EY_{nk})\right| > t/2\right). \tag{3.2}$$

Let $B_n'' = \sum_{k=1}^{n} E(Y_{nk} - EY_{nk})^2$. Take $x = t/2$, $y = t/2\eta$. By Lemma 3.2 and (3.2), we have

$$\begin{aligned}
I_4 &\le \sum_{n=1}^{\infty}\int_{\delta}^{\infty} P\left(\left|\sum_{k=1}^{n}\left(Y_{nk} - EY_{nk}\right)\right| > t/2\right)\mathrm{d}t \\
&\le \sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_{\delta}^{\infty} P\left(|Y_{nk} - EY_{nk}| > t/2\eta\right)\mathrm{d}t + C\sum_{n=1}^{\infty}\int_{\delta}^{\infty}\left(\frac{B_n''}{B_n'' + t^2/4\eta}\right)^{\eta}\mathrm{d}t \\
&=: I_5 + I_6.
\end{aligned}$$

By a similar argument as in the proof of (3.1), we can prove $\max_{t\ge\delta} t^{-1}|EY_{nk}| \to 0$ as $n \to \infty$. Therefore, while n is sufficiently large, by $|Y_{nk}| \le |X_{nk}|$ and (2.11), we have

$$\begin{aligned}
I_5 &\le \sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_{\delta}^{\infty} P\big(|Y_{nk}| > t/4\eta\big)\mathrm{d}t \le \sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_{\delta}^{\infty} P\big(|X_{nk}| > t/4\eta\big)\mathrm{d}t \\
&\le C\sum_{n=1}^{\infty}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > \delta/4\eta) < \infty.
\end{aligned}$$

Then we prove $I_6 < \infty$. By C_r-inequality, we have

$$\begin{aligned}
I_6 &\le C\sum_{n=1}^{\infty}\int_{\delta}^{\infty}\left(t^{-2}B_n''\right)^{\eta}\mathrm{d}t \le C\sum_{n=1}^{\infty}\int_{\delta}^{\infty}\left(t^{-2}\sum_{k=1}^{n} EY_{nk}^2\right)^{\eta}\mathrm{d}t \\
&= C\sum_{n=1}^{\infty}\int_{\delta}^{\infty}\left(t^{-2}\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \le t) + \sum_{k=1}^{n} P(|X_{nk}| > t)\right)^{\eta}\mathrm{d}t \\
&\le C\sum_{n=1}^{\infty}\int_{\delta}^{\infty}\left(t^{-2}\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \le \delta)\right)^{\eta}\mathrm{d}t \\
&\quad + C\sum_{n=1}^{\infty}\int_{\delta}^{\infty}\left(t^{-1}\sum_{k=1}^{n} E|X_{nk}|I(\delta < |X_{nk}| \le t)\right)^{\eta}\mathrm{d}t + C\sum_{n=1}^{\infty}\int_{\delta}^{\infty}\left(\sum_{k=1}^{n} P(|X_{nk}| > t)\right)^{\eta}\mathrm{d}t \\
&=: I_{61} + I_{62} + I_{63}.
\end{aligned}$$

By $\eta > 1$ and (2.2), we have

$$\begin{aligned}
I_{61} &\le C\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \le \delta)\right)^{\eta}\int_{\delta}^{\infty} t^{-2\eta}\mathrm{d}t \\
&\le C\frac{1}{2\eta - 1}\delta^{1-2\eta}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} EX_{nk}^2 I(|X_{nk}| \le \delta)\right)^{\eta} < \infty.
\end{aligned}$$

By $\eta > 1$ and (2.11), we have

$$\begin{aligned}
I_{62} &\le C\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > \delta)\right)^{\eta}\int_{\delta}^{\infty} t^{-\eta}\mathrm{d}t \\
&\le C\frac{1}{\eta - 1}\delta^{1-\eta}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n} E|X_{nk}|I(|X_{nk}| > \delta)\right)^{\eta} < \infty.
\end{aligned}$$

For $t \ge \delta$, by (2.11), we know

$$\max_{t\ge\delta}\sum_{k=1}^{n}P(|X_{nk}|>t)\le\sum_{k=1}^{n}P(|X_{nk}|>\delta)$$
$$\le\delta^{-1}\sum_{k=1}^{n}E|X_{nk}|I(|X_{nk}|>\delta)\to 0\quad\text{as } n\to\infty.$$

Therefore, while n is sufficiently large, we know that $\sum_{k=1}^{n}P(|X_{nk}|>t)<1$ holds uniformly for $t\ge\delta$. Hence, by a similar argument as in the proof of $I_3<\infty$, we have

$$I_{63}\le C\sum_{n=1}^{\infty}\int_{\delta}^{\infty}\sum_{k=1}^{n}P(|X_{nk}|>t)\mathrm{d}t\le C\sum_{n=1}^{\infty}\sum_{k=1}^{n}E|X_{nk}|I(|X_{nk}|>\delta)<\infty.$$

The proof is complete. □

Proof of Theorem 2.3. Let

$$Y_{nk}=-\delta I(X_{nk}<-\delta)+X_{nk}I(|X_{nk}|\le\delta)+\delta I(X_{nk}>\delta),$$
$$Z_{nk}=X_{nk}-Y_{nk}=(X_{nk}+\delta)I(X_{nk}<-\delta)+(X_{nk}-\delta)I(X_{nk}>\delta).$$

Then

$$E\left|\sum_{k=1}^{n}X_{nk}\right|\le E\left|\sum_{k=1}^{n}(Z_{nk}-EZ_{nk})\right|+E\left|\sum_{k=1}^{n}(Y_{nk}-EY_{nk})\right|$$
$$\le E\left|\sum_{k=1}^{n}(Z_{nk}-EZ_{nk})\right|+\left\{E\left(\sum_{k=1}^{n}(Y_{nk}-EY_{nk})\right)^2\right\}^{1/2}$$
$$=: I_7+I_8.$$

Noting that $|Z_{nk}|\le|X_{nk}|I(|X_{nk}|>\delta)$. By (2.14), we have

$$I_7\le 2\sum_{k=1}^{n}E|X_{nk}|I(|X_{nk}|>\delta)\to 0\quad\text{as } n\to\infty.$$

Then we prove $I_8\to 0$ as $n\to\infty$. By Lemma 3.3 and C_r-inequality, we have

$$I_8^2\le C\sum_{k=1}^{n}E(Y_{nk}-EY_{nk})^2\le C\sum_{k=1}^{n}EY_{nk}^2$$
$$=C\sum_{k=1}^{n}EX_{nk}^2I(|X_{nk}|\le\delta)+C\sum_{k=1}^{n}P(|X_{nk}|>\delta)$$
$$\le C\sum_{k=1}^{n}EX_{nk}^2I(|X_{nk}|\le\delta)+C\sum_{k=1}^{n}E|X_{nk}|I(|X_{nk}|>\delta)$$
$$=: I_{81}+I_{82}.$$

By (2.14) and (2.15), we have $I_{81}\to 0$ and $I_{82}\to 0$ as $n\to\infty$. Hence we get $I_8\to 0$ as $n\to\infty$. The proof is complete. □

Proof of Theorem 2.4. Following the notations of the proof in Theorem 2.3. For all $\varepsilon>0$, we have

$$P\left(\left|\sum_{k=1}^{n}X_{nk}\right|>2\varepsilon\right)\le P\left(\left|\sum_{k=1}^{n}(Y_{nk}-EY_{nk})\right|>\varepsilon\right)+P\left(\left|\sum_{k=1}^{n}(Z_{nk}-EZ_{nk})\right|>\varepsilon\right)$$
$$=: I_9+I_{10}.$$

By the Markov inequality, Lemma 3.3 and C_r-inequality, we have

$$I_9\le C\sum_{k=1}^{n}E(Y_{nk}-EY_{nk})^2\le C\sum_{k=1}^{n}EY_{nk}^2$$
$$=C\sum_{k=1}^{n}EX_{nk}^2I(|X_{nk}|\le\delta)+C\sum_{k=1}^{n}P(|X_{nk}|>\delta).$$

By (2.15) and (2.19), we have $I_9\to 0$ as $n\to\infty$.

Taking into account the definition of Z_{nk} and (2.19), we have

$$\begin{aligned} I_{10} &\le P\Big(\exists k;\ 1 \le k \le n,\ \text{such that } |X_{nk}| > \delta\Big) \\ &\le \sum_{k=1}^{n} P(|X_{nk}| > \delta) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

The proof is complete. □

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