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Convergence of series of dependent φ-subgaussian random variables

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Abstract

The almost sure convergence of weighted sums of φ-subgaussian m-acceptable random variables is investigated. As corollaries, the main results are applied to the case of negatively dependent and m-dependent subgaussian random variables. Finally, an application to random Fourier series is presented.

Keywords: m-Acceptability; m-Dependence; Negative dependence; φ-Subgaussian random variables; Almost sure convergence; Weighted sums; Strong law of large numbers

1. Introduction

The main focus of the present investigation is to obtain convergence theorems for weighted sums of dependent φ-subgaussian random variables. The roots of the subject are in classical probability theory, and certainly can be traced back to Kolmogorov's theory of summation of independent random variables. We refer to the book of Kwapien and Woyczynski [13] that contains classical results as well as more recent results on sums of independent random vectors.

Subgaussianity (or, more generally, φ-subgaussianity) properties of random variables and random processes (see Buldygin and Kozachenko [3]) are important features, since they allow us to derive results concerning, for instance, large deviations inequalities, asymptotic behaviour of particular processes or the behaviour of their supremum (see Buldygin and Kozachenko [3], Castellucci and Giuliano Antonini [4], or Giuliano Antonini and Kozachenko [7]).

This paper was inspired by a celebrated paper by Chow [5] which deals with almost sure convergence of series of independent classical subgaussian random variables. Chow [5] provides interesting applications of his results to the strong law of large numbers and also to Fourier analysis.

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There are three more papers that are closely related to our investigation: Ouy [17] and Amini et al. [1,2]. First of all, we point out that our results are more general than all of the above quoted papers since we consider the wider class of φ-subgaussian random variables. Moreover, sequences of m-acceptable random variables are considered, which is a more general case than that of independent random variables considered in Chow [5] and the m-dependent case considered in Ouy [17]. In this paper more accurate estimation of subgaussian standard is obtained than in the paper Ouy [17]. In the papers [1] and [2] the case of negatively dependent subgaussian random variables is considered. Our results are more general and different from the results of Amini et al. [1] since we do not need the assumption that a particular conditional expectation of the considered random variables is equal to zero, as it is required in Amini et al. [1]. Our results are different from the results of Amini et al. [2] since we consider the wider class of φ-subgaussian random variables.

One of the most interesting applications of series of subgaussian random variables can be obtained in Fourier analysis. The study of random Fourier series has been exploited by Kahane in his celebrated book Kahane [11], and has remarkable applications for instance in harmonic analysis (see Marcus and Pisier [15]). In Kahane [11] some important examples of random trigonometric series are given: Rademacher series, Steinhaus series, gaussian series. Here we consider the case of φ-subgaussian random series; note that all the above examples are particular cases of our general situation. In this setting we prove an extension of the classical Salem and Zygmund Theorem, cf. [19].

The plan of the paper is as follows. In Section 2 we present all important definitions and provide six lemmas to be used in the proofs of the main results. The main results are two theorems presented in Section 3. In Section 4 we give applications to the case of negatively dependent random variables and show how our statements are related to those in Amini et al. [1,2] in the special case of classical subgaussian random variables; this type of variables is also discussed in Section 5, where we mainly consider the case of m-dependent random variables and show that our results are stronger than those in Ouy [17]. In Section 6 we present other corollaries. In Section 7 we give an interesting application to Fourier analysis.

2. Definitions and technical lemmas

In this section we present definitions and a few technical results to be used in the proofs of our main results.

A continuous even convex function $\varphi(x)$, $x \in \mathbf{R}$, is called an *N-function*, if

(a) $\varphi(0) = 0$ and $\varphi(x)$ monotone increasing for $x > 0$;
(b) $\lim_{x\to 0} \frac{\varphi(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$.

The following condition is important to ensure that the class of φ-subgaussian random variables (cf. definition below) is nonempty. An N-function $\varphi(x)$ satisfies *condition Q* if $\lim_{x\to 0} \frac{\varphi(x)}{x^2} = C > 0$, where C is not necessarily finite.

In the following the notation $\varphi(x)$ always stands for an N-function with condition Q.

The function $\varphi^*(x)$, $x \in \mathbf{R}$, defined by $\varphi^*(x) = \sup_{y\in\mathbf{R}}(xy - \varphi(y))$ is called the *Young–Fenchel transform* of $\varphi(x)$. It is well known that $\varphi^*(x)$ is an N-function, too, and if $\varphi(x) = |x|^p/p$, $p > 1$ for sufficiently large x, then $\varphi^*(x) = |x|^q/q$ for sufficiently large x, where $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 1. *Let $\varphi(x)$ be an N-function such that $\varphi(x^{1/r})$, $x \geqslant 0$, is convex. Let $0 < p < r$, then the function $\varphi(x^{1/p})$, $x \geqslant 0$, is convex, too.*

Proof. Since the function $x^{r/p}$ is convex, then for any $0 \leqslant \alpha \leqslant 1$,

$$\big(\alpha x_1 + (1-\alpha)x_2\big)^{r/p} \leqslant \alpha x_1^{r/p} + (1-\alpha)x_2^{r/p}.$$

Therefore

$$\begin{aligned}\varphi\big(\big(\alpha x_1 + (1-\alpha)x_2\big)^{1/p}\big) &= \varphi\big(\big(\big(\alpha x_1 + (1-\alpha)x_2\big)^{r/p}\big)^{1/r}\big)\\ &\leqslant \varphi\big(\big(\alpha x_1^{r/p} + (1-\alpha)x_2^{r/p}\big)^{1/r}\big) \leqslant \alpha\varphi\big(\big(x_1^{r/p}\big)^{1/r}\big) + (1-\alpha)\varphi\big(\big(x_2^{r/p}\big)^{1/r}\big)\\ &= \alpha\varphi\big(x_1^{1/p}\big) + (1-\alpha)\varphi\big(x_2^{1/p}\big). \qquad \square\end{aligned}$$

Now we introduce a class of N-function that plays an important role in the main results of this paper. Let p and q be positive constants. We say that an N-function $\varphi(x)$ belongs to the class $N(p,q)$ ($\varphi \in N(p,q)$ in short), if the functions $\varphi(|x|^{1/p})$ and $\varphi^*(|x|^{1/q})$ are both convex.

Before we formulate a proposition that gives a sufficient condition for an N-function to belong to the class $N(p,q)$, we introduce some notations.

By the well-known criterion of convexity (cf. Krasnosel'skij and Rutitskij [12, Theorem 1.1]) every continuous convex function φ satisfying $\varphi(0)=0$ can be represented as follows

$$\varphi(x) = \int_0^{|x|} f_\varphi(t)\,dt,$$

where $f_\varphi(t)$, $t \geqslant 0$, is a nondecreasing right continuous function. Consider the generalized inverse function $g(t)$, $t \geqslant 0$, of $f_\varphi(t)$, defined by the formula

$$g(t) = \sup\{u \geqslant 0:\ f_\varphi(u) \leqslant t\}.$$

Then the Young–Fenchel transform of $\varphi(x)$ can be represented as follows

$$\varphi^*(x) = \int_0^{|x|} g(t)\,dt.$$

Proposition 1. *Let $\varphi(x)$ be an N-function such that $f_\varphi(t)t^{-r}$ is nonincreasing for some $r>0$ and $\varphi(|x|^{1/p})$ is convex for some $p>1$; then $\varphi \in N(p,q)$ where $q = p/r$.*

Proof. Note that

$$\varphi\big(|x|^{1/p}\big) = \int_0^{|x|^{1/p}} f_\varphi(t)\,dt = \frac{1}{p}\int_0^{|x|} f_\varphi\big(u^{1/p}\big)u^{(1/p)-1}\,du.$$

Since $\varphi(|x|^{1/p})$ is convex, then $f_\varphi(u^{1/p})u^{(1/p)-1}$ is nondecreasing. Consider the change of variables

$$v^{1/q} = f_\varphi\big(u^{1/p}\big), \quad \text{that is,} \quad u = g^p\big(v^{1/q}\big).$$

Then

$$f_\varphi\big(u^{1/p}\big)u^{(1/p)-1} = v^{1/q}g^{1-p}\big(v^{1/q}\big) \quad \text{is nondecreasing.}$$

We have

$$\varphi^*\big(|x|^{1/q}\big) = \int_0^{|x|^{1/q}} g(t)\,dt = \frac{1}{q}\int_0^{|x|} g\big(v^{1/q}\big)v^{(1/q)-1}\,dv,$$

hence in order to show that $\varphi^*(|x|^{1/q})$ is convex we prove that

$$g\big(v^{1/q}\big)v^{(1/q)-1} \quad \text{is nondecreasing.}$$

We can write

$$g\big(v^{1/q}\big)v^{(1/q)-1} = \big[v^{1/q}g^{1-p}\big(v^{1/q}\big)\big]\big[g^p\big(v^{1/q}\big)v^{-1}\big].$$

The function in the first square brackets is nondecreasing by the above arguments. Hence it is sufficient to consider the function in the second square brackets. Letting $g^p(v^{1/q}) = t$, hence $v = f_\varphi^q(t^{1/p})$, we have

$$g^p\big(v^{1/q}\big)v^{-1} = \left(\frac{t^{r/p}}{f_\varphi(t^{1/p})}\right)^{p/r},$$

which is nondecreasing by the assumption of the proposition. □

A random variable X is said to be *φ-subgaussian* if there exists a constant $a > 0$ such that, for every $\lambda \in \mathbf{R}$, we have $E\exp\{\lambda X\} \leqslant \exp\{\varphi(a\lambda)\}$. The *$\varphi$-subgaussian standard* $\tau_\varphi(X)$ is defined as

$$\tau_\varphi(X) = \inf\bigl\{a > 0\colon\ E\exp\{\lambda X\} \leqslant \exp\bigl\{\varphi(a\lambda)\bigr\},\ \lambda \in \mathbf{R}\bigr\}.$$

We refer to the monograph Buldygin and Kozachenko [3] and the paper Giuliano Antonini et al. [8] where this notion is discussed in detail and important examples are provided. In the case $\varphi(t) = t^2/2$, φ-subgaussianity is simply the subgaussianity in the *classical* sense, cf. for example Hoffmann-Jørgensen [9].

The following lemma gives us an important estimation for the tail probabilities of a φ-subgaussian random variable (cf. Buldygin and Kozachenko [3, Chapter 2, Lemma 4.3]).

Lemma 2. *If a random variable X is φ-subgaussian, then for every $\epsilon > 0$ we have*

$$P\bigl\{|X| > \epsilon\bigr\} \leqslant 2\exp\left\{-\varphi^*\left(\frac{\epsilon}{\tau_\varphi(X)}\right)\right\}.$$

We say that a finite family of random variables $X_1, X_2, \ldots, X_n$ is *acceptable* if for any real λ,

$$E\exp\left\{\lambda\sum_{k=1}^n X_k\right\} \leqslant \prod_{k=1}^n E\exp\{\lambda X_k\}.$$

A sequence of random variables $\{X_k,\ k \geqslant 1\}$ is *acceptable* if every finite subfamily is acceptable.

The notion of m-dependent random variables is well known (cf. Section 5). The notion of m-acceptable associated random variables seems to be new.

Let $m > 1$ be a fixed integer. A sequence of random variables $\{X_n,\ n \geqslant 1\}$ is said to be *m-acceptable* if for any $n \geqslant 2$ and any $i_1, \ldots, i_n$ such that $|i_k - i_j| \geqslant m$ for all $1 \leqslant k \neq j \leqslant n$, we have that $X_{i_1}, \ldots, X_{i_n}$ are acceptable.

An array of random variables $\{X_{nk},\ k \geqslant 1,\ n \geqslant 1\}$ is *rowwise m-acceptable* if for any fixed $n \geqslant 1$ the row $\{X_{nk},\ k \geqslant 1\}$ is a sequence of m-acceptable random variables.

A sequence of negatively dependent random variables (cf. Section 4) as well as a sequence of m-dependent random variables (cf. Section 5) provide us two examples of sequences of m-acceptable random variables.

Another interesting example of a sequence $\{Z_n,\ n \geqslant 1\}$ of acceptable random variables can be constructed in the following way. Feller [6, Problem III.1] (cf. also Romano and Siegel [18, Section 4.30]) provides an example of two random variables X and Y such that the density of their sum is the convolution of their densities, yet they are not independent. It is simple to see that X and Y are not negatively dependent either. Since they are bounded, their Laplace transforms $E\exp\{\lambda X\}$ and $E\exp\{\lambda Y\}$ are finite for any λ. Next, since the density of their sum is the convolution of their densities, we have

$$E\exp\bigl\{\lambda(X+Y)\bigr\} = E\exp\{\lambda X\}E\exp\{\lambda Y\}.$$

The announced sequence of acceptable random variables $\{Z_n,\ n \geqslant 1\}$ can be now constructed in the following way. Let (X_k, Y_k) be independent copies of the random vector (X, Y), $k \geqslant 1$. For any $n \geqslant 1$ set $Z_n = X_k$ if $n = 2k+1$ and $Z_n = Y_k$ if $n = 2k$.

The next lemma allows us to estimate the φ-subgaussian standard of sums of acceptable random variables.

Lemma 3. *Let $X_1, \ldots, X_n$ be φ-subgaussian random variables. Then*

(a) *the variable $\sum_{k=1}^n X_k$ is φ-subgaussian and $\tau_\varphi(\sum_{k=1}^n X_k) \leqslant \sum_{k=1}^n \tau_\varphi(X_k)$;*
(b) *if in addition $X_1, \ldots, X_n$ are acceptable and the function $\varphi(|x|^{1/p})$ is convex for some $p \in [1,2]$, then $\tau_\varphi^p(\sum_{k=1}^n X_k) \leqslant \sum_{k=1}^n \tau_\varphi^p(X_k)$.*

Proof. Part (a) can be found in Buldygin and Kozachenko [3, p. 69]. Part (b) can be proved in the same way as Buldygin and Kozachenko [3, Theorem 5.2]. □

The next lemma is an important technical tool in the proof of our main result. Before we formulate and prove it, we introduce the following notations. Let $n \geqslant 0$ and $l \geqslant 1$ be integers, and $\{a_k,\ k \geqslant 1\}$, $\{\alpha_k,\ k \geqslant 1\}$ be sequences of real numbers such that $\alpha_k \geqslant 0$ for all $k \geqslant 1$. For integer $j \geqslant 1$ denote

$$\delta_{nl}(j)=\begin{cases}1 & \text{if } n+1\leqslant j\leqslant n+l,\\ 0 & \text{elsewhere.}\end{cases}$$

Let $m\geqslant 1$ be a fixed integer and $L=\lfloor\frac{l}{2m}\rfloor+1$, where $\lfloor\cdot\rfloor$ denotes the integer part function, that is the largest integer that is less or equal to the given value. For any $k\geqslant 1$ denote the sets of integers:

$$I_k=\{j\colon\ n+1+2(k-1)m\leqslant j\leqslant n+(2k-1)m\},$$
$$J_k=\{j\colon\ n+1+(2k-1)m\leqslant j\leqslant n+2km\},$$

and

$$N_k=I_k\cup J_k=\{n+1+2(k-1)m,\dots,n+2km\}.$$

If $j\in I_k$ for some k, then let $s=j-(n+2(k-1)m)$ and hence $1\leqslant s\leqslant m$. Denote

$$a^I_{sk}=a_j,\qquad \alpha^I_{sk}=\alpha_j\quad\text{and}\quad \delta_{nl}(s,k)=\delta_{nl}(j).$$

If $j\in J_k$ for some k, then let $s=j-(n+(2k-1)m)$ and hence $1\leqslant s\leqslant m$. Denote

$$a^J_{sk}=a_j,\qquad \alpha^J_{sk}=\alpha_j\quad\text{and}\quad \delta_{nl}(s,k)=\delta_{nl}(j).$$

Next, let $p\geqslant 1$ be a positive constant. Denote

$$B^{(p)}_{nl}=\sum_{s=1}^{m}\left(\sum_{k=1}^{L}\big(\alpha^I_{sk}\big|a^I_{sk}\big|\delta_{nl}(s,k)\big)^p\right)^{1/p}+\left(\sum_{k=1}^{L}\big(\alpha^J_{sk}\big|a^J_{sk}\big|\delta_{nl}(s,k)\big)^p\right)^{1/p}$$

and

$$A^{(p)}_{nl}=\sum_{j=n+1}^{n+l}\big(\alpha_j|a_j|\big)^p.$$

We can write the following

Estimate. $(A^{(p)}_{nl})^{1/p}\leqslant B^{(p)}_{nl}\leqslant(2m)^{1-1/p}(A^{(p)}_{nl})^{1/p}$.

Proof of the estimate. Recall that if $z_i\geqslant 0$, $i=1,\dots,m$, and $p\geqslant 1$, then $(\sum_{i=1}^m z_i)^p\leqslant m^{p-1}\sum_{i=1}^m z_i^p$.

This inequality can be proved using classical multivariate calculus procedure: Search for the maximum of the function $f(z_1,\dots,z_m)=(\sum_{i=1}^m z_i)^p$ on the constraint $\sum_{i=1}^m z_i^p=1$ using the method of Lagrange multipliers. We leave the details to the reader.

By the above inequality we obtain that

$$\big(B^{(p)}_{nl}\big)^p\leqslant 2^{p-1}\left(\left(\sum_{s=1}^{m}\left(\sum_{k=1}^{L}\big|a^I_{sk}\alpha^I_{sk}\delta_{nl}(s,k)\big|^p\right)^{1/p}\right)^p+\left(\sum_{s=1}^{m}\left(\sum_{k=1}^{L}\big(a^J_{sk}\alpha^J_{sk}\delta_{nl}(s,k)\big)^p\right)^{1/p}\right)^p\right).$$

We estimate each term of this expression separately; for the first summand we get

$$\left(\sum_{s=1}^{m}\left(\sum_{k=1}^{L}\big|a^I_{sk}\alpha^I_{sk}\delta_{nl}(s,k)\big|^p\right)^{1/p}\right)^p\leqslant m^{p-1}\left(\sum_{s=1}^{m}\sum_{k=1}^{L}\big|a^I_{sk}\alpha^I_{sk}\delta_{nl}(s,k)\big|^p\right).$$

By the same arguments we may estimate the second term. We conclude that

$$\big(B^{(p)}_{nl}\big)^p\leqslant(2m)^{p-1}\sum_{s=1}^{m}\sum_{k=1}^{L}\big(\big|a^I_{sk}\alpha^I_{sk}\delta_{nl}(s,k)\big|^p+\big|a^J_{sk}\alpha^J_{sk}\delta_{nl}(s,k)\big|^p\big)=A^{(p)}_{nl}.$$

Hence $B^{(p)}_{nl}\leqslant(2m)^{1-\frac{1}{p}}(A^{(p)}_{nl})^{1/p}$. The inequality $B^{(p)}_{nl}\geqslant(A^{(p)}_{nl})^{1/p}$ is obvious since

$$\left(\sum_{i=1}^{N} z_i\right)^{1/p} \leqslant \sum_{i=1}^{N} z_i^{1/p}$$

for $p > 1$, any positive integer N and $z_i > 0$. □

Lemma 4. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of m-acceptable φ-subgaussian random variables and $\alpha_n = \tau_\varphi(X_n)$. Assume that the function $\varphi(|x|^{1/p})$ is convex for some $p \in [1,2]$ and let $\{a_n,\ n \geqslant 1\}$ be a sequence of constants. For $n \geqslant 0, l \geqslant 1$ denote*

$$T_{nl} = \sum_{j=n+1}^{n+l} a_j X_j.$$

Then T_{nl} is φ-subgaussian and $\tau_\varphi(T_{nl}) \leqslant B_{nl}^{(p)}$.

Proof. In order to estimate the sum $T_{nl} = \sum_{j=n+1}^{n+l} a_j X_j$ consider the largest union of the sets N_k which is a subset of the segment $[n+1, \ldots, n+l]$,

$$\bigcup_{k=1}^{L-1} N_k \subset [n+1, \ldots, n+l] \subset \bigcup_{k=1}^{L} N_k.$$

Put

$$U_k = \sum_{j \in I_k} a_j X_j \delta_{nl}(j), \quad V_k = \sum_{j \in J_k} a_j X_j \delta_{nl}(j).$$

In addition to our previous notations consider the following.

If $j \in I_k$ for some k, then let $s = j - (n + 2(k-1)m)$ and hence $1 \leqslant s \leqslant m$. Denote $X_{sk} = X_j$.
If $j \in J_k$ for some k, then let $s = j - (n + (2k-1)m)$ and hence $1 \leqslant s \leqslant m$. Denote $X_{sk} = X_j$.

Fix s with $1 \leqslant s \leqslant m$, then the sequence $\{X_{sk},\ 1 \leqslant k \leqslant L\}$ is acceptable, and by Lemma 3 we have

$$\tau_\phi\left(\sum_{k=1}^{L} U_k\right) \leqslant \sum_{s=1}^{m} \tau_\phi\left(\sum_{k=1}^{L} a_{sk} X_{sk} \delta_{nl}(s,k)\right) \leqslant \sum_{s=1}^{m}\left(\sum_{k=1}^{L}\left(\left|a_{sk}^I\right| \alpha_{sk}^I \delta_{nl}(s,k)\right)^p\right)^{1/p}.$$

The estimation of $\tau_\phi(\sum_{k=1}^{L} V_k)$ can be established in the same way. Hence

$$\begin{aligned}\tau_\phi(T_{nl}) &\leqslant \tau_\phi\left(\sum_{k=1}^{L} U_k\right) + \tau_\phi\left(\sum_{k=1}^{L} V_k\right)\\ &\leqslant \sum_{s=1}^{m}\left(\sum_{k=1}^{L}\left(\left|a_{sk}^I\right| \alpha_{sk}^I \delta_{nl}(s,k)\right)^p\right)^{1/p} + \sum_{s=1}^{m}\left(\sum_{k=1}^{L}\left(\left|a_{sk}^J\right| \alpha_{sk}^J \delta_{nl}(s,k)\right)^p\right)^{1/p} = B_{nl}^{(p)}. \qquad \square\end{aligned}$$

Lemma 5, stated below (cf. Móricz [16, Theorem 2]), will be used to establish the almost sure convergence for our main result. We need the following notations. Let $\psi(\cdot)$ be a positive strictly increasing continuous function, $\psi(0) = 0$ and $\{Y_k,\ k \geqslant 1\}$ be a sequence of random variables (it is not assumed that the random variables are independent or that they are identically distributed). For any $n \geqslant 0$ and $l \geqslant 1$ set

$$T_{nl} = \sum_{j=n+1}^{n+l} Y_j \quad \text{and} \quad M_{nl} = \max_{1 \leqslant k \leqslant l} |T_{nk}|.$$

Let moreover $g(n,l)$ be a nonnegative function depending on the joint distribution of $Y_{n+1}, \ldots, Y_{n+l}$, and having the property

$$g(n,h) + g(n+h, l-h) \leqslant g(n,l)$$

for all $n \geqslant 0$, $l \geqslant 1$, and $1 \leqslant h < l$.

Lemma 5. *If there exists a constant $C > 0$ such that for all $\epsilon > 0$, $n \geqslant 0$, and $l \geqslant 1$,*

$$P\{|T_{nl}| > \epsilon\} \leqslant C \exp\left\{-\frac{\psi(\epsilon)}{g(n,l)}\right\},$$

then there exist positive constants C_1 and C_2 such that

$$P\{|M_{nl}| > \epsilon\} \leqslant C_1 \exp\left\{-\frac{C_2\psi(\epsilon)}{g(n,l)}\right\}.$$

Remark 1. Lemma 5 is presented as Theorem 2 in Móricz [16] without a proof. We are not sure that this result is true with the only restrictions on $\psi(\cdot)$ that it is a positive strictly increasing continuous function, $\psi(0) = 0$. But we were able to prove this result under the additional assumption $\psi(tx) \leqslant t\psi(x)$ for $0 < t < 1$. Lemma 5 is used in Theorem 1 for $\psi(x) = x^p$, $p \geqslant 1$ and hence this additional assumption is satisfied.

The last lemma in this section is a simple generalization of Chow [5, Lemma 4]. The only difference is that instead of Chow's [5] Lemma 3 (estimation of the derivative of a trigonometric polynomial) we use Bernstein's theorem (cf., for example, Zygmund [22, p. 11]). Chow [5] says that the Bernstein's theorem can be used, but for his result the simpler estimation was sufficient.

Lemma 6. *Let $\{X_n,\ n \geqslant 1\}$ and $\{D_n,\ n \geqslant 1\}$ be two sequences of random variables. For a sequence of constants $\{a_n,\ n \geqslant 1\}$, a positive integer N, and any $t \in [0, 2\pi]$ define $Q(t) = \sum_{n=1}^{N} a_n X_n \cos(nt + D_n)$ and $M = \|Q\|_\infty = \max_{0 \leqslant t \leqslant 2\pi} |Q(t)|$. Then for any $K \geqslant 0$,*

$$\{M \geqslant K\} \subset \bigcup_{l=0}^{\lfloor 2\pi N \rfloor} \{|Q(l/N)| \geqslant K/2\}.$$

Proof. Let $M(\omega) = |Q(t(\omega), \omega)|$, then for any $t \in [0, 2\pi]$,

$$\left|Q(t) - Q\big(t(\omega), \omega\big)\right| \leqslant \left|t - t(\omega)\right| \|Q'\|_\infty \leqslant \left|t - t(\omega)\right| NM$$

since $\|Q'\|_\infty \leqslant N\|Q\|_\infty$ by Bernstein's theorem (Zygmund [22, p. 11]).

Hence, for $|t - t(\omega)| \leqslant \frac{1}{2N}$ we have that $|Q(t)| \geqslant M/2 \geqslant K/2$ if $M \geqslant K$. The length of the interval $|t - t(\omega)| \leqslant \frac{1}{2N}$ is $1/N$, and we have $\lfloor 2\pi N \rfloor + 1$ of these intervals in the segment $[0, 2\pi]$, namely $t \in [\frac{l}{N}, \frac{l+1}{N}]$, $l = 0, \ldots, \lfloor 2\pi N \rfloor$. □

3. Main results

We need some more notations. Let $\{a_k,\ k \geqslant 1\}$, $\{\alpha_k,\ k \geqslant 1\}$ be sequences of real numbers such that $\alpha_k \geqslant 0$ for all $k \geqslant 1$. For $p > 0$ denote

$$A^{(p)} = \sum_{j=1}^{\infty} \big(\alpha_j |a_j|\big)^p,$$

$$B^{(p)} = \sum_{k=1}^{\infty} \left(\sum_{j=(k-1)m+1}^{km} \big(\alpha_j |a_j|\big)^p \right)^{1/p}.$$

Note that the series $A^{(p)}$ and $B^{(p)}$ converge simultaneously, and by the estimate (p. 1192)

$$\big(A^{(p)}\big)^{1/p} \leqslant B^{(p)} \leqslant (2m)^{1-1/p} \big(A^{(p)}\big)^{1/p}.$$

The following proposition gives us a result concerning the convergence in probability.

Proposition 2. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of m-acceptable φ-subgaussian random variables and $\alpha_n = \tau_\varphi(X_n)$. Assume that the function φ belong to $N(p,q)$ for some $p \in [1,2]$ and $q > 1$. Let $\{a_n,\ n \geqslant 1\}$ be a sequence of constants such that $A^{(p)} < \infty$. Then the series $T = \sum_{k=1}^{\infty} a_k X_k$ converges in probability.*

Proof. Let $A_{nl}^{(p)}$ and $B_{nl}^{(p)}$ be as in Lemma 4. According to the same lemma, the φ-subgaussian standard of the random variable $T_{nl} = \sum_{j=n+1}^{n+l} a_j X_j$ is bounded by $\tau_\varphi(T_{nl}) \leqslant B_{nl}^{(p)}$.

By Lemmas 2 and 4 and by the estimate (p. 1192) of $B_{nl}^{(p)}$, we can state that for any $\epsilon > 0$,

$$P\{|T_{nl}| > \epsilon\} \leqslant 2\exp\left\{-\varphi^*\left(\frac{\epsilon}{\tau_\varphi(T_{nl})}\right)\right\} \leqslant 2\exp\left\{-\varphi^*\left(\frac{\epsilon}{B_{nl}^{(p)}}\right)\right\} \leqslant 2\exp\left\{-\varphi^*\left(\frac{\epsilon}{(2m)^{1-1/p}(A_{nl}^{(p)})^{1/p}}\right)\right\}.$$

Since the function φ belongs to $N(p,q)$, there exists a positive constant C such that $\varphi^*(x) \geqslant Cx^q - 1$ (even the better estimate $\varphi^*(x) \geqslant Cx^q$ holds for $|x| > 1$).

Then for sufficiently small $A_{nl}^{(p)}$,

$$P\{|T_{nl}| > \epsilon\} \leqslant 2\exp\left\{-\frac{C\epsilon^q}{(2m)^{q(1-1/p)}(A_{nl}^{(p)})^{q/p}}\right\}.$$

According to the assumptions of the theorem, $A_{n,l}^{(p)} \to 0$ as $n \to \infty$. Hence $P\{|\sum_{j=n+1}^{n+l} a_j X_j| > \epsilon\} \to 0$ for all $\epsilon > 0$ as $n \to \infty$, that is, $S_n = \sum_{k=1}^{n} a_k X_k$ converges in probability to a random variable $T = \sum_{k=1}^{\infty} a_k X_k$. □

With these preliminaries accounted for, we can now state and prove the main results of the paper.

Theorem 1. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of m-acceptable φ-subgaussian random variables and $\alpha_n = \tau_\varphi(X_n)$. Assume that the function φ belongs to $N(r,q)$ for some $r, q \in [1,2]$. Let $p \leqslant \min(r,q)$ and let $\{a_n,\ n \geqslant 1\}$ be a sequence of constants such that $A^{(p)} < \infty$. Then*

(i) *the series $T = \sum_{k=1}^{\infty} a_k X_k$ converges a.s.*;
(ii) $\tau_\varphi(T) \leqslant B^{(r)} \leqslant (2m)^{1-1/r}(A^{(p)})^{1/p}$.

Proof. According to Proposition 2, $S_n = \sum_{k=1}^{n} a_k X_k$ converges in probability to a random variable $T = \sum_{k=1}^{\infty} a_k X_k$. Then there exists a subsequence $\{n_k,\ k \geqslant 1\}$ such that S_{n_k} converges to T almost surely. According to the assumptions of the theorem $A_{nl}^{(p)} \to 0$ as $n \to \infty$ uniformly in l. Choose n so big that $A_{nl}^{(p)} \leqslant 1$ for all $l \geqslant 1$; then

$$\left(A_{nl}^{(r)}\right)^{q/r} \leqslant \left(A_{nl}^{(p)}\right)^{q/p} \leqslant A_{nl}^{(p)}$$

since $q/p \geqslant 1$. According to Lemma 1 we have that φ belongs to $N(p,q)$, too.

Put $M_k = \max_{n_k \leqslant l < n_{k+1}} |T_{nl}|$ and denote $g(n,l) = \frac{(2m)^{q(1-1/p)}}{C} A_{nl}^{(p)}$, then $g(n,h) + g(n+h, l-h) \leqslant g(n,l)$ for all $n \geqslant 0, l \geqslant 1$, and $1 \leqslant h < l$. The following inequality was established in the proof of Proposition 2

$$P\{|T_{nl}| > \epsilon\} \leqslant 2\exp\left\{-\frac{C\epsilon^q}{(2m)^{q(1-1/p)}(A_{nl}^{(p)})^{q/p}}\right\}.$$

According to the remarks and notations above we can rewrite it as

$$P\{|T_{nl}| > \epsilon\} \leqslant 2\exp\left\{-\left(\frac{\epsilon^q}{g(n,l)}\right)\right\}.$$

By Lemma 5 with $\psi(t) = t^q$ we have that

$$P\{M_k > \epsilon\} \leqslant C_1 \exp\left\{-\frac{C_2\epsilon^q}{g(n_k, n_{k+1} - n_k)}\right\} \leqslant \frac{C_1}{C_2\epsilon^q} g(n_k, n_{k+1} - n_k),$$

where we have used the relation $e^{-x} \leqslant 1/x$ which holds for all $x > 0$. Hence

$$\sum_{k=1}^{\infty} P\{|M_k| > \epsilon\} \leqslant \frac{C_1}{C_2\epsilon^q} \sum_{k=1}^{\infty} g(n_k, n_{k+1} - n_k) = CA^{(p)} < \infty.$$

By Borel–Cantelli lemma $M_k \to 0$ as $k \to \infty$ almost surely. Hence, for every n, with $n_k \leqslant n < n_{k+1}$, we have

$$|S_n - T| \leqslant |S_n - S_{n_{k+1}}| + |S_{n_{k+1}} - T| \leqslant M_k + |S_{n_{k+1}} - T|,$$

so that, almost surely

$$\lim_{n\to\infty} |S_n - T| = 0.$$

This shows that $S_n = \sum_{k=1}^{n} a_k X_k$ converges almost surely to a random variable $T = \sum_{k=1}^{\infty} a_k X_k$.

In order to prove that T is a φ-subgaussian and estimate its φ-subgaussian standard, we apply Lemma 4 with $n = 0$ and $l \to \infty$. We obtain that $\tau_\varphi(T) \leqslant B^{(q)}$. □

The next theorem generalizes Theorem 1 for the case of arrays. Let $p > 0$ and let $\{a_{nk},\ k \geqslant 1,\ n \geqslant 1\}$, $\{\alpha_{nk},\ k \geqslant 1,\ n \geqslant 1\}$ be arrays of real numbers such that $\alpha_{nk} \geqslant 0$ for all $k \geqslant 1$ and $n \geqslant 1$. Denote

$$A_n^{(p)} = \sum_{j=1}^{\infty} \bigl(\alpha_{nj}|a_{nj}|\bigr)^p,$$

$$B_n^{(p)} = \sum_{k=1}^{\infty} \left(\sum_{j=(k-1)m+1}^{km} \bigl(\alpha_{nj}|a_{nj}|\bigr)^p \right)^{1/p}.$$

Theorem 2. *Let $\{X_{nk},\ k \geqslant 1,\ n \geqslant 1\}$ be an array of rowwise m-acceptable φ-subgaussian random variables with $\alpha_{nk} = \tau_\varphi(X_{nk})$. Assume that the function φ belongs to $N(r,q)$ for some $r, q \in [1,2]$. Let $p \leqslant \min(r,q)$ and let $\{a_{nk},\ k \geqslant 1,\ n \geqslant 1\}$ be an array of constants such that $A_n^{(p)} < \infty$ for all $n \geqslant 1$.*

If for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \exp\left\{ -\varphi^*\left(\frac{\epsilon}{B_n^{(r)}} \right) \right\} < \infty,$$

then the sequence of random variables $T_n = \sum_{k=1}^{\infty} a_{nk} X_{nk}$ (which is well defined by Theorem 1) converges to zero a.s.

Proof. By Theorem 1 we have that $\tau_\varphi(T_n) \leqslant B_n^{(r)}$ and by Lemma 2 for any $\epsilon > 0$,

$$P\bigl\{|T_n| > \epsilon\bigr\} \leqslant 2\exp\left\{ -\varphi^*\left(\frac{\epsilon}{B_n^{(r)}} \right) \right\}.$$

By the hypothesis $\sum_{n=1}^{\infty} P\{|T_n| > \epsilon\} < \infty$, hence in view of Borel–Cantelli lemma $T_n \to 0$ a.s. as $n \to \infty$. □

4. Case of negatively dependent random variables

Recall that random variables $X_1, \ldots, X_n$ are said to be *negatively dependent* if

$$P\left\{ \bigcap_{j=1}^{n} [X_j \leqslant x_j] \right\} \leqslant \prod_{j=1}^{n} P\{X_j \leqslant x_j\}$$

and

$$P\left\{ \bigcap_{j=1}^{n} [X_j > x_j] \right\} \leqslant \prod_{j=1}^{n} P\{X_j > x_j\}$$

for any $x_1, \ldots, x_n \in \mathbf{R}$.

A sequence of random variables $\{X_n,\ n \geqslant 1\}$ is said to be negatively dependent, if any finite subsequence of the sequence forms negatively dependent random variables. The notion of negatively dependent random variables was introduced by Lehmann [14] and developed in Joag-Dev and Proschan [10].

The notion of m-dependent random variables is well known (cf. Section 5). The notion of m-negatively dependent random variables seems to be new.

Let $m > 1$ be a fixed integer. A sequence of random variables $\{X_n,\ n \geqslant 1\}$ is said to be *m-negatively dependent* if for any $n \geqslant 2$ and any $i_1, \ldots, i_n$ such that $|i_k - i_j| \geqslant m$ for all $1 \leqslant k \neq j \leqslant n$, we have that $X_{i_1}, \ldots, X_{i_n}$ are negatively dependent.

An array of random variables $\{X_{nk},\ k \geqslant 1,\ n \geqslant 1\}$ is *rowwise m-negatively dependent* if for any fixed $n \geqslant 1$ the row $\{X_{nk},\ k \geqslant 1\}$ is a sequence of m-negatively dependent random variables.

Note that the m-negatively dependence assumption is stronger than the assumption of m-acceptability. This means that Theorems 1 and 2 are true for a sequence of m-negatively dependent random variables, too. Now we consider the case $m = 1$.

The following lemma can be found in Joag-Dev and Proschan [10]. The proof is obvious.

Lemma 7. *Let $X_1, \ldots, X_n$ be negatively dependent random variables.*

(a) *If $f_1, \ldots, f_n$ is a sequence of measurable functions which are all monotone increasing (or all are monotone decreasing), then $f_1(X_1), \ldots, f_n(X_n)$ are negatively dependent random variables, too.*
(b) *$E(X_1 \cdots X_n) \leqslant E(X_1) \cdots E(X_n)$, provided the expectations exist.*

The next lemma allows us to estimate the φ-subgaussian standard of sums of random variables.

Lemma 8. *Let $X_1, \ldots, X_n$ be negatively dependent φ-subgaussian random variables. If in addition the function $\varphi(|x|^{1/p})$ is convex for some $p \in [1,2]$, then $\tau_\varphi^p(\sum_{k=1}^n X_k) \leqslant \sum_{k=1}^n \tau_\varphi^p(X_k)$.*

Proof. The lemma can be proved in the same way as Buldygin and Kozachenko [3, Theorem 5.2], with the only difference that the validity of the estimation

$$E\exp\left\{\lambda \sum_{k=1}^n X_k\right\} \leqslant \prod_{k=1}^n E\exp\{\lambda X_k\}$$

(that is, negative dependent random variables are acceptable) can be justified by Lemma 7. First of all by Lemma 7(a) with $f_k(t) = \exp\{\lambda t\}$ for all $1 \leqslant k \leqslant n$ we can state that $\exp\{\lambda X_1\}, \ldots, \exp\{\lambda X_n\}$ are negatively dependent, and after apply Lemma 7(b). □

The next lemma is an analog of Lemma 4 for the case of negatively dependent random variables.

Lemma 9. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of negatively dependent φ-subgaussian random variables and $\alpha_n = \tau_\varphi(X_n)$. Assume that the function $\varphi(|x|^{1/p})$ is convex for some $p \in [1,2]$ and let $\{a_n,\ n \geqslant 1\}$ be a sequence of constants. For $n \geqslant 0$, $l \geqslant 1$ denote*

$$T_{nl} = \sum_{j=n+1}^{n+l} a_j X_j.$$

Then T_{nl} is φ-subgaussian and $\tau_\varphi(T_{n,l}) \leqslant \widehat{B}_{nl}^{(p)}$, where

$$\widehat{B}_{nl}^{(p)} = \left(\sum_{j=n+1}^{n+l} \left(\alpha_j |a_j|^p\right)\right)^{1/p}.$$

The proof of Lemma 9 repeats the proof of Lemma 4 with $m = 1$ and hence is omitted.

The following two corollaries are more general than results of Amini et al. [2] since a wider class of φ-subgaussian random variables is considered.

Corollary 1. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of negatively dependent φ-subgaussian random variables and $\alpha_n = \tau_\varphi(X_n)$. Assume that the function φ belongs to $N(r,q)$ for some $r,q \in [1,2]$. Let $p \leqslant \min(r,q)$ and let $\{a_n,\ n \geqslant 1\}$ be a sequence of constants such that*

$$A^{(p)} = \sum_{j=1}^{\infty} \alpha_j^p |a_j|^p < \infty.$$

Then

(i) *the series $T = \sum_{k=1}^{\infty} a_k X_k$ converges a.s.*;
(ii) $\tau_\varphi(T) \leqslant (A^{(p)})^{1/p}$.

Corollary 2. *Let $\{X_{nk},\ k \geqslant 1,\ n \geqslant 1\}$ be an array of rowwise negatively dependent φ-subgaussian random variables with $\alpha_{nk} = \tau_\varphi(X_{nk})$. Assume that the function φ belongs to $N(r,q)$ for some $r,q \in [1,2]$. Let $p \leqslant \min(r,q)$ and let $\{a_{nk},\ k \geqslant 1,\ n \geqslant 1\}$ be an array of constants such that*

$$A_n^{(p)} = \sum_{j=1}^{\infty} \alpha_{nj}^p |a_{nj}|^p < \infty.$$

If for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \exp\left\{-\varphi^*\left(\frac{\epsilon}{(A_n^{(p)})^{1/p}}\right)\right\} < \infty,$$

then the sequence of random variables $T_n = \sum_{k=1}^{\infty} a_{nk} X_{nk}$ converges to zero a.s.

The proofs of these corollaries repeat those of Theorems 1 and 2 with obvious changes and hence are omitted.

In order to compare Corollary 1 with the result of Amini et al. [1,2] we introduce the following notation. Classical subgaussian random variables are the special case of φ-subgaussian random variables with $\varphi(t) = t^2/2$. For a classical subgaussian random variable X, we denote the subgaussian standard as $\tau(X) = \tau_{t^2/2}(X)$. Note that $\varphi(t) = t^2/2$ belongs to $N(2,2)$. The following corollary is a reformulation of Corollary 1 for the classical subgaussian case.

Corollary 3. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of negatively dependent classical subgaussian random variables and $\alpha_n = \tau(X_n)$. Let $\{a_n,\ n \geqslant 1\}$ be a sequence of constants such that*

$$A^{(2)} = \sum_{j=1}^{\infty} \alpha_j^2 |a_j|^2 < \infty.$$

Then

(i) *the series $T = \sum_{k=1}^{\infty} a_k X_k$ converges a.s.*;
(ii) $\tau(T) \leqslant \sqrt{A^{(2)}}$.

In order to compare Corollary 3 with existing results, we first remark that Theorem 1 of Amini et al. [2] is more general than Theorem 5 of Amini et al. [1] (a special conditional expectation is not required to be zero). Theorem 1 from Amini et al. [2] can be slightly reformulated in the following way.

Proposition 3. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of negatively dependent classical subgaussian random variables such that $\tau(X_n) \leqslant \alpha$ for some $\alpha > 0$ and all $n \geqslant 1$. If $\sum_{j=1}^{\infty} a_j^2 < \infty$, then the series $T = \sum_{k=1}^{\infty} a_k X_k$ converges a.s.*

Corollary 3 is more general than Proposition 3 since the sequence of gaussian standards is not necessarily uniformly bounded. Moreover, our result provides an estimation of the subgaussian standard.

5. The case of m-dependent random variables and m-acceptable random variables

The first corollary in this section reformulates Theorem 1 for the case of classical subgaussian random variables.

Corollary 4. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of m-acceptable subgaussian random variables and $\alpha_n = \tau(X_n)$. Let $\{a_n,\ n \geqslant 1\}$ be a sequence of constants such that*

$$A^{(2)} = \sum_{j=1}^{\infty} |\alpha_j a_j|^2 < \infty.$$

Then

(i) *the series $T = \sum_{k=1}^{\infty} a_k X_k$ converges a.s.*;
(ii) *$\tau(T) \leqslant B^{(2)}$, where $B^{(2)}$ is the same as in Theorem* 1.

Corollary 4 can be compared with Theorem 1 of Ouy [17] which deals with the case of m-dependent random variables. The sequence of random variables $\{X_n,\ n \geqslant 1\}$ is said to be *m-dependent* if X_p and X_q are independent whenever $|p-q| \geqslant m$. But a careful analysis of the proof of Ouy's [17] result shows that in such a proof a stronger hypothesis is needed than usual m-dependence, namely, it must be assumed that for any $n \geqslant 2$ and any $i_1, \dots, i_n$ such that $|i_k - i_j| \geqslant m$ for all $1 \leqslant k \neq j \leqslant n$, the random variables $X_{i_1}, \dots, X_{i_n}$ are independent. The difference is the same as between pairwise independence and (global) independence.

Theorem 1 of Ouy [17] can be formulated in the following way.

Proposition 4. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of m-dependent classical subgaussian random variables and $\tau(X_n) \leqslant \alpha$. Let $\{a_n,\ n \geqslant 1\}$ be a sequence of constants such that*

$$A^{(2)} = \sum_{k=1}^{\infty} a_k^2 < \infty.$$

Then

(i) *the series $T = \sum_{k=1}^{\infty} a_k X_k$ converges a.s.*;
(ii) *$\tau(T) \leqslant 2^{(m-1)/2}\alpha (A^{(2)})^{1/2}$.*

We can point out the following differences. The estimation of the subgaussian standard provided in Corollary 4 is more accurate than the estimation in Proposition 4, the subgaussian standards are not necessarily uniformly bounded, and Corollary 4 deals with m-acceptable random variables, which is a weaker assumption than m-dependence (in Ouy's [17] understanding).

6. Other corollaries

In the case $p = 1$ Theorems 1 and 2 are true without the assumption of m-acceptability dependence. That is, we can formulate the following statements.

Corollary 5. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of φ-subgaussian random variables and $\alpha_n = \tau_\varphi(X_n)$. Let $\{a_n,\ n \geqslant 1\}$ be a sequence of constants such that*

$$A^{(1)} = \sum_{k=1}^{\infty} \alpha_k |a_k| < \infty.$$

Then

(i) *the series $T = \sum_{k=1}^{\infty} a_k X_k$ converges a.s.*;
(ii) *$\tau_\varphi(T) \leqslant A^{(1)}$.*

Corollary 6. *Let $\{X_{nk},\ k \geqslant 1,\ n \geqslant 1\}$ be an array of φ-subgaussian random variables with $\alpha_{nk} = \tau_\varphi(X_{nk})$. Let $\{a_{nk},\ k \geqslant 1,\ n \geqslant 1\}$ be an array of constants such that*

$$B_n^{(1)} = \sum_{k=1}^{\infty} \alpha_{nj} |a_{nj}| < \infty.$$

If for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \exp\left\{-\varphi^*\left(\frac{\epsilon}{B_n^{(1)}}\right)\right\} < \infty,$$

then the sequence of random variables $T_n = \sum_{k=1}^{\infty} a_{nk} X_{nk}$ (which is well defined by Theorem 1) converges to zero a.s.

The proofs of Corollaries 4 and 5 repeat those of Theorems 1 and 2 with the only difference that the estimation

$$\tau_\varphi^p\left(\sum_{k=1}^{L} U_k\right) \leqslant \sum_{k=1}^{L} \tau_\varphi^p(U_k)$$

is true without m-acceptability assumption for $p = 1$ by Lemma 3(a).

Remark 2. In the conclusions of Theorem 2 and Corollaries 2 and 6 we not only obtain that $T_n \to 0$ a.s., but also that $\tau_\varphi(T_n) \to 0$ as $n \to \infty$.

The next corollary gives the Marcinkiewicz–Zygmund law of large numbers for φ-subgaussian random variables.

Corollary 7. *Let $\varphi(t) = |t|^p/p$, $t \in \mathbf{R}$, $1 \leqslant p \leqslant 2$ and $\{X_n,\ n \geqslant 1\}$ be a sequence of φ-subgaussian m-acceptable random variables with $\sup_{n \geqslant 1} \tau_\varphi(X_n) < \infty$. Then*

$$\frac{1}{n^{1/r}} \sum_{k=1}^{n} X_k \to 0 \quad \textit{a.s. for any } 0 < r < p.$$

Proof. Denote $X_{nk} = X_k$ for all $n \geqslant 1, k \geqslant 1$ and let

$$a_{nk} = \begin{cases} 1/n^{1/r} & \text{if } k \leqslant n, \\ 0 & \text{if } k > n. \end{cases}$$

By the assumption we have $\alpha_n = \tau_\varphi(X_n) \leqslant \sup_{n \geqslant 1} \tau_\varphi(X_n) = \alpha < \infty$. Let $B_n^{(p)}$ be as in Theorem 2. Then

$$B_n^{(p)} \leqslant C\alpha m^{1-\frac{1}{p}} n^{\frac{1}{p}-\frac{1}{r}}$$

and $\varphi^*(t) = |t|^q/q$ for all t, where $\frac{1}{p} + \frac{1}{q} = 1$. For any $\epsilon > 0$, we have

$$\sum_{n=1}^{\infty} \exp\left\{-\varphi^*\left(\frac{\epsilon}{B_n^{(p)}}\right)\right\} \leqslant \sum_{n=1}^{\infty} \exp\left\{-\frac{\epsilon^q}{qC\alpha^q m^{q-q/p}} n^{q(\frac{1}{r}-\frac{1}{p})}\right\} < \infty$$

since $r < p$. By Theorem 2

$$\frac{1}{n^{1/r}} \sum_{k=1}^{n} X_k \to 0 \quad \text{a.s. for any } 0 < r < p. \qquad \square$$

Remark 3. Corollary 7 gives much more than almost sure convergence. It gives the exponential rate of convergence to zero for φ-subgaussian acceptable random variables

$$P\left\{\left|\sum_{k=1}^{n} X_k\right| > \epsilon n^{1/r}\right\} \leqslant \exp\left\{-C\epsilon^q n^{q(1/r-1/p)}\right\}.$$

The interested reader could compare this result with those of Tomkins [21] and Taylor and Hu [20].

7. Application to Fourier analysis

The following theorem is a generalization of the famous Salem and Zygmund theorem, cf. Salem and Zygmund [19]. A closely related theorem for semigaussian random variables has been given by Kahane [11, p. 78], and for classical subgaussian random variables by Chow [5, Theorem 4].

Theorem 3. *Let $\{X_n,\ n \geqslant 1\}$ and $\{D_n,\ n \geqslant 1\}$ be two independent sequences of random variables such that for each t, $\{Y_n(t) = X_n \cos(nt + D_n),\ n \geqslant 1\}$ is a sequence of m-acceptable random variables. Let moreover X_n be φ-subgaussian with $\tau_\varphi(X_n) = \alpha_n$. Assume that the function φ belong to $N(r,q)$ for some $r, q \in [1,2]$. Let $p \leqslant \min(r,q)$ and let $\{a_k,\ k \geqslant 1\}$ be a sequence of constants; $\{b_k,\ k \geqslant 1\}$ an increasing sequence of constants such that*

$$C = \sum_{k=1}^{\infty} \left(\sum_{j=(k-1)m+1}^{km} \left(\alpha_j |a_j|\right)^p b_k \right)^{1/p} < \infty.$$

Let moreover $\{n(j),\ j \geqslant 1\}$ be an increasing sequence of positive integers and $\{k(j),\ j \geqslant 1\}$ be a sequence of positive constants such that

$$\sum_{j=1}^{\infty} k(j) < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} n(j+1) \exp\left\{ -\varphi^* \left(\frac{k(j) b_{n(j)}}{C} \right) \right\} < \infty.$$

Then for each t, the series $\sum_{n=1}^{\infty} a_n X_n \cos(nt + D_n)$ converges a.s. to a stochastic process $f(t)$ that has a sample continuous stochastic modification.

Proof. Note that for any λ,

$$\begin{aligned} E \exp\{\lambda Y_n(t)\} &\leqslant E\big(E\big(\exp\{\lambda X_n \cos(nt + D_n)\} \mid D_1, \ldots, D_n\big)\big) \\ &\leqslant E\big(E\big(\exp\{\varphi\big(\lambda \alpha_n \cos(nt + D_n)\big)\} \mid D_1, \ldots, D_n\big)\big) \quad \text{since } X_n \text{ is } \varphi\text{-subgaussian} \\ &\leqslant \exp\{\varphi(\lambda \alpha_n)\} \quad \text{since } \varphi \text{ is even and increasing.} \end{aligned}$$

Hence, $Y_n(t)$ is a φ-subgaussian with $\tau_\varphi(Y_n(t)) \leqslant \alpha_n$ for each t. Since $b_n \uparrow$, we have that

$$B = \sum_{k=1}^{\infty} \left(\sum_{j \in I_k} \left(\alpha_j |a_j|\right)^p \right)^{1/p} + \left(\sum_{j \in J_k} \left(\alpha_j |a_j|\right)^p \right)^{1/p} < \infty,$$

where I_k and J_k are as in Lemma 4. Hence by Theorem 1 $\sum_{n=1}^{\infty} a_n Y_n(t) \to f(t)$ a.s. for every t.

In order to prove that the stochastic process $f(t)$ has a sample continuous modification, we introduce the following notations. For $j \geqslant 1$ let

$$\begin{aligned} Q_j(t) &= \sum_{n=2mn(j)+1}^{2mn(j+1)} a_n Y_n(t), \\ M_j &= \|Q_j\|_\infty, \\ B_j &= \sum_{k=n(j)+1}^{n(j+1)} \left(\sum_{j \in I_k} \left(\alpha_j |a_j|\right)^p \right)^{1/p} + \left(\sum_{j \in J_k} \left(\alpha_j |a_j|\right)^p \right)^{1/p}, \\ C_j &= \sum_{k=n(j)+1}^{n(j+1)} \left(\sum_{j \in I_k} \left(\alpha_j |a_j|\right)^p b_k \right)^{1/p} + \left(\sum_{j \in J_k} \left(\alpha_j |a_j|\right)^p b_k \right)^{1/p}. \end{aligned}$$

By Lemma 4 and monotonicity of the sequence $\{b_n,\ n \geqslant 1\}$, we have that $Q_j(t)$ is φ-subgaussian with

$$\tau_\varphi\big(Q_j(t)\big) \leqslant B_j \leqslant \frac{C_j}{b_{n(j)}}.$$

By Lemma 2 for any t,

$$P\{|Q_j(t)| > k(j)\} \leqslant 2\exp\left\{-\varphi^*\left(\frac{k(j)}{\tau_\varphi(Q_j(t))}\right)\right\} \leqslant 2\exp\left\{-\varphi^*\left(\frac{k(j)b_{n(j)}}{C_j}\right)\right\}.$$

By Lemma 6

$$\begin{aligned} P\{M_j \geqslant 2k(j)\} &\leqslant \sum_{l=0}^{\lfloor 2\pi m(j+1)\rfloor} P\{|Q_j(l/m(j+1))| \geqslant k(j)\} \\ &\leqslant 2(4\pi mn(j+1)+1)\exp\left\{-\varphi^*\left(\frac{k(j)b_{n(j)}}{C_j}\right)\right\} \\ &\leqslant 2(4\pi mn(j+1)+1)\exp\left\{-\varphi^*\left(\frac{k(j)b_{n(j)}}{C}\right)\right\}. \end{aligned}$$

Hence $\sum_{j=1}^{\infty} P\{M_j \geqslant 2k(j)\} < \infty$. By the Borel–Cantelli lemma for almost all ω there exists $j_0(\omega) < \infty$ such that $M_j(\omega) \leqslant 2k(j)$ for $j \geqslant j_0(\omega)$. Since $\sum_{j=1}^{\infty} k(j) < \infty$, almost surely $\sum_{j=1}^{\infty} M_j < \infty$ and $\sum_{j=1}^{\infty} Q_j(t)$ converges uniformly in t to a stochastic process $g(t)$. Since for each fixed ω, $Q_j(t)$ are continuous in t, $g(t)$ is sample continuous. Finally, by the first part of the proof $\sum_{j=1}^{\infty} Q_j(t) = f(t)$ a.s., hence $P\{f(t) = g(t)\} = 1$. □

Corollary 8. *Let $\varphi(t) = t^p/p$, $1 < p \leqslant 2$ for all $t \in \mathbf{R}$ and let $\{X_n,\ n \geqslant 1\}$ and $\{D_n,\ n \geqslant 1\}$ be two sequences of random variables such that for each t, $\{Y_n(t) = X_n\cos(nt + D_n),\ n \geqslant 1\}$ is a sequence of m-acceptable random variables. Let moreover X_n be φ-subgaussian with $\sup_{n\geqslant 1}\tau_\varphi(X_n) = \alpha < \infty$. Let $\{a_k,\ k \geqslant 1\}$ be a sequence of constants such that*

$$\sum_{k=1}^{\infty} |a_k|^p \log^{1+\delta-1/p}(k) < \infty$$

for some $\delta > 0$.

Then for each t, the series $\sum_{n=1}^{\infty} a_n X_n \cos(nt + D_n)$ converges a.s. to a stochastic process $f(t)$ that has a sample continuous stochastic modification.

Proof. In this case $\varphi^*(t) = t^q/q$ for all t, where $\frac{1}{p} + \frac{1}{q} = 1$. In Theorem 3 take $b_k = \log^{1+\delta-1/p}(k) = \log^{\delta+1/q}(k)$, $n(j) = 2^{2^j}$ and $k(j) = j^{-2}$. All assumptions of Theorem 3 are satisfied, the only one that requires some work is the convergence of the series $\sum_{j=1}^{\infty} n(j+1)\exp\{-\varphi^*(\frac{k(j)b_{n(j)}}{C})\}$, where

$$C = \left(\sum_{k=1}^{\infty}\left(\sum_{j=(k-1)m+1}^{2km}|a_j|\right)^p \log^{1+\delta-1/p}(k)\right)^{1/p} < \infty$$

by the assumption.

This can be done in the following way:

$$\begin{aligned} \sum_{j=1}^{\infty} n(j+1)\exp\left\{-\varphi^*\left(\frac{k(j)b_{n(j)}}{C}\right)\right\} &= \sum_{j=1}^{\infty} 2^{2^{j+1}}\exp\left\{-\left(\frac{j^{-2}\log^{\delta+1/q}(2^{2^j})}{C}\right)^q\right\} \\ &= \sum_{j=1}^{\infty}\exp\left\{2^j 2\log(2) - \frac{2^{j(1+\delta q)}\log^{1+\delta q}(2)}{j^{2q}C^q}\right\} \\ &\leqslant C\sum_{j=1}^{\infty}\exp\{-2^{j(1+\delta q/2)}\} < \infty. \qquad \square \end{aligned}$$

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