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# LIMITING BEHAVIOR FOR ARRAYS OF ROWWISE $\rho^*$-MIXING RANDOM VARIABLES*

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**Abstract.** We study the limiting behavior of maximal partial sums for arrays of rowwise $\rho^*$-mixing random variables and obtain some new results that improve the corresponding theorem of Zhu [M.H. Zhu, Strong laws of large numbers for arrays of rowwise $\rho^*$-mixing random variables, *Discrete Dyn. Nat. Soc.*, 2007, Article ID 74296, 6 pp., 2007].

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## 1 INTRODUCTION

A triangular array of random variables $\{X_{nk},\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ is said to be rowwise $\rho^*$-mixing if, for every $n \geqslant 1$, $\{X_{nk}, 1 \leqslant k \leqslant n\}$ is a $\rho^*$-mixing sequence of random variables. The concept of the coefficient $\rho^*$ was introduced by Moore [4], and Bradley [1] was the first who introduced the concept of $\rho^*$-mixing random variables to limit theorems.

Throughout this paper, we assume that the array of $\{X_{nk},\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ is rowwise $\rho^*$-mixing and the following condition is satisfied: $\rho_n^*(h) \leqslant a < 1$ for all arrays/rows with a fixed positive integer $h$.

Let $\{Z_n,\, n \geqslant 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n \mathbf{E}\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \quad \text{for all } \varepsilon > 0,$$

then the above result was called the complete moment convergence by Chow [2].

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A sequence of random variables $\{U_n,\ n \geqslant 1\}$ is said to converge completely to a constant $a$ if, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}\bigl(|U_n - a| > \varepsilon\bigr) < \infty.$$

In this case, we say that $U_n \to a$ completely. This notion was given firstly by Hsu and Robbins [3].

Let $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be an array of rowwise $\rho^*$-mixing random variables, $\{a_n,\ n \geqslant 1\}$ be a sequence of positive real numbers such that $a_n \uparrow \infty$, and let $\{\Psi(t)\}$ be a positive even function such that

$$\frac{\Psi(|t|)}{|t|^q}\uparrow \quad \text{and} \quad \frac{\Psi(|t|)}{|t|^p}\downarrow \quad \text{as } |t|\uparrow \tag{1.1}$$

for some $1 \leqslant q < p$. We introduce the following conditions:

$$\mathbf{E}X_{nk} = 0, \quad 1 \leqslant k \leqslant n,\ n \geqslant 1, \tag{1.2}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{\mathbf{E}\Psi(X_{nk})}{\Psi(a_n)} < \infty, \tag{1.3}$$

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} \mathbf{E}\left(\frac{X_{nk}}{a_n}\right)^2\right)^{v/2} < \infty, \tag{1.4}$$

where $v \geqslant p$ is a positive integer.

*Remark 1.* We can mention the following examples of function $\Psi(t)$ that satisfies assumption (1.1): $\Psi(t) = |t|^\beta$ for some $q < \beta < p$ or $\Psi(t) = |t|^q \log(1 + |t|^{p-q})$ for $t \in (-\infty, +\infty)$. Note that these functions are nonmonotone on $t \in (-\infty, +\infty)$, while it is simple to show that, under condition (1.1), the function $\Psi(t)$ is an increasing function for $t > 0$. Otherwise, let $0 < t_1 < t_2 < \infty$ be such that $\Psi(t_1) \geqslant \Psi(t_2)$. Then we have

$$\frac{\Psi(t_1)}{t_1^q} \geqslant \frac{\Psi(t_2)}{t_2^q},$$

which contradicts with $\frac{\Psi(|t|)}{|t|^q}\uparrow$ as $|t|\uparrow$.

The following complete convergence result by Zhu [6] was the starting point for our investigation.

**Theorem A.** *Let $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be an array of rowwise $\rho^*$-mixing random variables, and $\{a_n,\ n \geqslant 1\}$ be a sequence of positive real numbers such that $a_n \uparrow \infty$. Also, let $\Psi(t)$ be a positive even function satisfying* (1.1) *for $q = 1$ and some nonnegative integer $p \geqslant 2$. Then, under conditions* (1.2)–(1.4)*, we have*

$$\frac{1}{a_n}\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} X_{nk}\right| \to 0 \quad \textit{completely}. \tag{1.5}$$

In this work, we extend Theorem A to the complete moment convergence, which is a more general version of the complete convergence. In addition, compared with Zhu [6], we study the $L^q$ convergence for arrays of rowwise $\rho^*$-mixing random variables, which was not considered in his paper.

In this paper, the symbol $C$ always stands for a generic positive constant, which may vary from one place to another.

## 2 MAIN RESULTS

Now we present the main results of the paper. The proofs will be given in the next section.

**Theorem 1.** *Let* $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ *be an array of rowwise* $\rho^*$*-mixing random variables, and let* $\{a_n,\ n \geqslant 1\}$ *be a sequence of positive real numbers such that* $a_n \uparrow \infty$. *Also, let* $\Psi(t)$ *be a positive even function satisfying* (1.1) *for* $1 \leqslant q < p \leqslant 2$. *Then, under conditions* (1.2) *and* (1.3), *we have*

$$\sum_{n=1}^{\infty} a_n^{-q} \mathbf{E}\left\{ \max_{1\leqslant j\leqslant n} \left| \sum_{k=1}^{j} X_{nk} \right| - \varepsilon a_n \right\}_+^q < \infty \quad \forall \varepsilon > 0. \tag{2.1}$$

**Theorem 2.** *Let* $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ *be an array of rowwise* $\rho^*$*-mixing random variables, and let* $\{a_n,\ n \geqslant 1\}$ *be a sequence of positive real numbers such that* $a_n \uparrow \infty$. *Also, let* $\Psi(t)$ *be a positive even function satisfying* (1.1) *for* $1 \leqslant q < p$ *and* $p > 2$. *Then conditions* (1.2)–(1.4) *imply* (2.1).

**Theorem 3.** *Let* $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ *be an array of rowwise* $\rho^*$*-mixing random variables satisfying conditions* (1.2), *and let* $\{a_n,\ n \geqslant 1\}$ *be a sequence of positive real numbers such that* $a_n \uparrow \infty$. *Also, let* $\Psi(t)$ *be a positive even function satisfying* (1.1) *for* $1 \leqslant q < p$.

(1) *If* $1 < p \leqslant 2$ *and*

$$\sum_{k=1}^{n} \frac{\mathbf{E}\Psi(X_{nk})}{\Psi(a_n)} \to 0 \quad \textit{as } n \to \infty, \tag{2.2}$$

*then*

$$\frac{1}{a_n} \max_{1\leqslant j\leqslant n} \left| \sum_{k=1}^{j} X_{nk} \right| \xrightarrow{L^q} 0. \tag{2.3}$$

(2) *If* $p > 2$, (2.2) *is satisfied, and*

$$a_n^{-2} \sum_{k=1}^{n} \mathbf{E}X_{nk}^2 \to 0 \quad \textit{as } n \to \infty, \tag{2.4}$$

*then* (2.3) *holds.*

*Remark 2.* The proof of Theorem 3 immediately follows from the moment inequality applied to truncated variables. Therefore, we will omit the details.

## 3 PROOFS

To prove the results of this paper, we need the following two lemmas.

**Lemma 1.** (*See* [5].) *Let* $N$ *be a positive integer,* $0 \leqslant r < 1$, *and* $p \geqslant 2$. *Then there exists a positive constant* $C = C(N, r, p)$ *such that the following statement holds*:

*If* $\{X_i,\ i \geqslant 1\}$ *is a sequence of random variables such that* $\rho_N^* \leqslant r$ *and such that* $\mathbf{E}X_i = 0$ *and* $\mathbf{E}|X_i|^p < \infty$ *for every* $i \geqslant 1$, *then, for all* $n \geqslant 1$,

$$\mathbf{E} \max_{1\leqslant j\leqslant n} \left| \sum_{k=1}^{j} X_k \right|^p \leqslant C\left\{ \sum_{k=1}^{n} \mathbf{E}|X_k|^p + \left( \sum_{k=1}^{n} \mathbf{E}X_k^2 \right)^{p/2} \right\}. \tag{3.1}$$

**Lemma 2.** *Let $\{X_{nk},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ be an array of rowwise $\rho^*$-mixing random variables, and let $\{a_n,\ n \geqslant 1\}$ be a sequence of positive real numbers such that $a_n \uparrow \infty$. Also, let $\Psi(t)$ be a positive even function satisfying* (1.1) *for $1 \leqslant q < p$. Then* (1.3) *implies the following statements*:

(i) *for $r \geqslant 1$, $0 < u \leqslant q$,*

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{\mathbf{E}|X_{nk}|^u I(|X_{nk}| > a_n)}{a_n^u}\right)^r < \infty;$$

(ii) *for $v \geqslant p$,*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\mathbf{E}|X_{nk}|^v I(|X_{nk}| \leqslant a_n)}{a_n^v} < \infty.$$

*Proof.* From (1.1) and (1.3) we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{\mathbf{E}|X_{nk}|^u I(|X_{nk}| > a_n)}{a_n^u}\right)^r \leqslant \left(\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\mathbf{E}\Psi(X_{nk})}{\Psi(a_n)}\right)^r < \infty$$

and

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\mathbf{E}|X_{nk}|^v I(|X_{nk}| \leqslant a_n)}{a_n^v} \leqslant \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\mathbf{E}\Psi(X_{nk})}{\Psi(a_n)} < \infty,$$

where $r \geqslant 1$, $0 < u \leqslant q$, and $v \geqslant p$. The proof is complete. □

*Proof of Theorem 1.* Let $M_n(X) = \max_{1\leqslant j\leqslant n}|\sum_{k=1}^{j} X_{nk}|$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} a_n^{-q}\mathbf{E}\{M_n(X) - \varepsilon a_n\}_+^q \\
&\quad = \sum_{n=1}^{\infty} a_n^{-q}\int_0^{\infty}\mathbf{P}\{M_n(X) - \varepsilon a_n > t^{1/q}\}\,\mathrm{d}t \\
&\quad = \sum_{n=1}^{\infty} a_n^{-q}\left(\int_0^{a_n^q}\mathbf{P}\{M_n(X) > \varepsilon a_n + t^{1/q}\}\,\mathrm{d}t + \int_{a_n^q}^{\infty}\mathbf{P}\{M_n(X) > \varepsilon a_n + t^{1/q}\}\,\mathrm{d}t\right) \\
&\quad \leqslant \sum_{n=1}^{\infty}\mathbf{P}\{M_n(X) > \varepsilon a_n\} + \sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty}\mathbf{P}\{M_n(X) > t^{1/q}\}\,\mathrm{d}t \mathrel{\widehat=} I_1 + I_2.
\end{aligned}$$

To prove (2.1), it suffices to prove that $I_1 < \infty$ and $I_2 < \infty$. By a similar argument as in the proof of Zhu [6] we can prove that $I_1 < \infty$. We omit the details.

Let us prove that $I_2 < \infty$. Let $Y_{nk} = X_{nk}I(|X_{nk}| \leqslant t^{1/q})$, $Z_{nk} = X_{nk} - Y_{nk}$, and $M_n(Y) = \max_{1\leqslant j\leqslant n}|\sum_{k=1}^{j} Y_{nk}|$. Obviously,

$$\mathbf{P}\{M_n(X) > t^{1/q}\} \leqslant \sum_{k=1}^{n}\mathbf{P}\{|X_{nk}| > t^{1/q}\} + \mathbf{P}\{M_n(Y) > t^{1/q}\}.$$

Hence,

$$I_2 \leqslant \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} \mathbf{P}\{|X_{nk}| > t^{1/q}\}\,\mathrm{d}t + \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \mathbf{P}\{M_n(Y) > t^{1/q}\}\,\mathrm{d}t \mathrel{\widehat{=}} I_3 + I_4.$$

For $I_3$, by Lemma 2 we have

$$\begin{aligned} I_3 &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} \mathbf{P}\{|X_{nk}|I(|X_{nk}| > a_n) > t^{1/q}\}\,\mathrm{d}t \\ &\leqslant \sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q} \int_{0}^{\infty} \mathbf{P}\{|X_{nk}|I(|X_{nk}| > a_n) > t^{1/q}\}\,\mathrm{d}t \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{\mathbf{E}|X_{nk}|^q I(|X_{nk}| > a_n)}{a_n^q} < \infty. \end{aligned}$$

Now let us prove that $I_4 < \infty$. By (1.2) and Lemma 2 we have

$$\begin{aligned} &\max_{t\geqslant a_n^q}\max_{1\leqslant j\leqslant n} t^{-1/q}\left|\sum_{k=1}^{j}\mathbf{E}Y_{nk}\right| \\ &= \max_{t\geqslant a_n^q}\max_{1\leqslant j\leqslant n} t^{-1/q}\left|\sum_{k=1}^{j}\mathbf{E}Z_{nk}\right| \leqslant \max_{t\geqslant a_n^q} t^{-1/q}\sum_{k=1}^{n}\mathbf{E}|X_{nk}|I(|X_{nk}| > t^{1/q}) \\ &\leqslant \sum_{k=1}^{n}\frac{\mathbf{E}|X_{nk}|^q I(|X_{nk}| > a_n)}{a_n^q} \to 0. \end{aligned} \tag{3.2}$$

Therefore, for $n$ sufficiently large,

$$\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}\mathbf{E}Y_{nk}\right| \leqslant \frac{t^{1/q}}{2}$$

uniformly for $t \geqslant a_n^q$. Then

$$\mathbf{P}\{M_n(Y) > t^{1/q}\} \leqslant \mathbf{P}\left\{\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}(Y_{nk} - \mathbf{E}Y_{nk})\right| > \frac{t^{1/q}}{2}\right\}. \tag{3.3}$$

Let $d_n = [a_n] + 1$. By (3.3), Lemma 1, and $C_r$-inequality we have

$$\begin{aligned} I_4 &\leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} t^{-2/q}\mathbf{E}Y_{nk}^2\,\mathrm{d}t \\ &= C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q} \int_{a_n^q}^{\infty} t^{-2/q}\mathbf{E}X_{nk}^2 I(|X_{nk}| \leqslant d_n)\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+ C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\int_{d_n^q}^{\infty} t^{-2/q}\mathbf{E}X_{nk}^2 I\bigl(d_n < |X_{nk}| \leqslant t^{1/q}\bigr)\,\mathrm{d}t \\
&\mathrel{\widehat{=}} I_{41} + I_{42}.
\end{aligned}
$$

For $I_{41}$, since $q < 2$, we have

$$
\begin{aligned}
I_{41} &= C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\mathbf{E}X_{nk}^2 I\bigl(|X_{nk}| \leqslant d_n\bigr)\int_{a_n^q}^{\infty} t^{-2/q}\,\mathrm{d}t \leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\mathbf{E}X_{nk}^2 I(|X_{nk}| \leqslant d_n)}{a_n^2} \\
&= C\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\mathbf{E}X_{nk}^2 I(|X_{nk}| \leqslant a_n)}{a_n^2} + C\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\mathbf{E}X_{nk}^2 I(a_n < |X_{nk}| \leqslant d_n)}{a_n^2} \\
&\mathrel{\widehat{=}} I_{41}' + I_{41}''.
\end{aligned}
$$

Since $p \leqslant 2$, by Lemma 2 we get $I_{41}' < \infty$. Now we prove that $I_{41}'' < \infty$. Since $q < 2$ and $(a_n + 1)/a_n \to 1$ as $n \to \infty$, by Lemma 2 we have

$$
\begin{aligned}
I_{41}'' &\leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{d_n^{2-q}}{a_n^2}\mathbf{E}|X_{nk}|^q I\bigl(a_n < |X_{nk}| \leqslant d_n\bigr) \\
&\leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n}\left(\frac{a_n + 1}{a_n}\right)^{2-q}\frac{\mathbf{E}|X_{nk}|^q I(|X_{nk}| > a_n)}{a_n^q} < \infty.
\end{aligned}
$$

Let $t = u^q$ in $I_{42}$. Note that, for $q < 2$,

$$
\int_{d_n}^{\infty} u^{q-3}\mathbf{E}X_{nk}^2 I\bigl(d_n < |X_{nk}| \leqslant u\bigr)\,\mathrm{d}u \leqslant C\mathbf{E}|X_{nk}|^q I\bigl(|X_{nk}| > d_n\bigr).
$$

Since $d_n > a_n$, by Lemma 2 we have

$$
\begin{aligned}
I_{42} &= C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\int_{d_n}^{\infty} u^{q-3}\mathbf{E}X_{nk}^2 I\bigl(d_n < |X_{nk}| \leqslant u\bigr)\,\mathrm{d}u \\
&\leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n} a_n^{-q}\mathbf{E}|X_{nk}|^q I\bigl(|X_{nk}| > a_n\bigr) < \infty.
\end{aligned}
$$

The proof is complete. □

*Proof of Theorem 2.* Following the notation, by a similar argument as in the proof of Theorem 1 we can easily prove that $I_1 < \infty$, $I_3 < \infty$, and that (3.2), and (3.3) hold. Therefore, we need only to prove that $I_4 < \infty$.

Let $\delta \geqslant p$ and $d_n = [a_n] + 1$. By (3.3), the Markov inequality, Lemma 1, and the $C_r$-inequality we have

$$
I_4 \leqslant C\sum_{n=1}^{\infty} a_n^{-q}\int_{a_n^q}^{\infty} t^{-\delta/q}\mathbf{E}\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}(Y_{nk} - \mathbf{E}Y_{nk})\right|^{\delta}\mathrm{d}t
$$

$$\leqslant C\sum_{n=1}^{\infty}a_n^{-q}\int_{a_n^q}^{\infty}t^{-\delta/q}\left[\sum_{k=1}^{n}\mathbf{E}|Y_{nk}|^{\delta}+\left(\sum_{k=1}^{n}\mathbf{E}Y_{nk}^2\right)^{\delta/2}\right]\mathrm{d}t$$

$$\leqslant C\sum_{n=1}^{\infty}\sum_{k=1}^{n}a_n^{-q}\int_{a_n^q}^{\infty}t^{-\delta/q}\mathbf{E}|Y_{nk}|^{\delta}\,\mathrm{d}t+C\sum_{n=1}^{\infty}a_n^{-q}\int_{a_n^q}^{\infty}t^{-\delta/q}\left(\sum_{k=1}^{n}\mathbf{E}Y_{nk}^2\right)^{\delta/2}\mathrm{d}t$$

$$\widehat{=}\, I_{43}+I_{44}.$$

For $I_{43}$, we have

$$I_{43}=C\sum_{n=1}^{\infty}\sum_{k=1}^{n}a_n^{-q}\int_{a_n^q}^{\infty}t^{-\delta/q}\mathbf{E}|X_{nk}|^{\delta}I\big(|X_{nk}|\leqslant d_n\big)\,\mathrm{d}t$$

$$+C\sum_{n=1}^{\infty}\sum_{k=1}^{n}a_n^{-q}\int_{a_n^q}^{\infty}t^{-\delta/q}\mathbf{E}|X_{nk}|^{\delta}I\big(d_n<|X_{nk}|\leqslant t^{1/q}\big)\,\mathrm{d}t$$

$$\widehat{=}\, I'_{43}+I''_{43}.$$

By a similar argument as in the proof of $I_{41}<\infty$ and $I_{42}<\infty$ (replacing the exponent 2 by $\delta$), we can get $I'_{43}<\infty$ and $I''_{43}<\infty$.

For $I_{44}$, since $\delta>2$, we have

$$I_{44}=C\sum_{n=1}^{\infty}a_n^{-q}\int_{a_n^q}^{\infty}t^{-\delta/q}\left(\sum_{k=1}^{n}\mathbf{E}X_{nk}^2I\big(|X_{nk}|\leqslant a_n\big)+\sum_{k=1}^{n}\mathbf{E}X_{nk}^2I\big(a_n<|X_{nk}|\leqslant t^{1/q}\big)\right)^{\delta/2}\mathrm{d}t$$

$$\leqslant C\sum_{n=1}^{\infty}a_n^{-q}\int_{a_n^q}^{\infty}t^{-\delta/q}\left(\sum_{k=1}^{n}\mathbf{E}X_{nk}^2I\big(|X_{nk}|\leqslant a_n\big)\right)^{\delta/2}\mathrm{d}t$$

$$+C\sum_{n=1}^{\infty}a_n^{-q}\int_{a_n^q}^{\infty}\left(t^{-2/q}\sum_{k=1}^{n}\mathbf{E}X_{nk}^2I\big(a_n<|X_{nk}|\leqslant t^{1/q}\big)\right)^{\delta/2}\mathrm{d}t$$

$$\widehat{=}\, I'_{44}+I''_{44}.$$

Since $\delta\geqslant p>q$, from (1.4) we have

$$I'_{44}=C\sum_{n=1}^{\infty}a_n^{-q}\left(\sum_{k=1}^{n}\mathbf{E}X_{nk}^2I\big(|X_{nk}|\leqslant a_n\big)\right)^{\delta/2}\int_{a_n^q}^{\infty}t^{-\delta/q}\,\mathrm{d}t$$

$$\leqslant C\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{\mathbf{E}X_{nk}^2I(|X_{nk}|\leqslant a_n)}{a_n^2}\right)^{\delta/2}\leqslant C\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{\mathbf{E}X_{nk}^2}{a_n^2}\right)^{\delta/2}<\infty.$$

Now we prove that $I''_{44} < \infty$. To start with, we consider the case $1 \leqslant q \leqslant 2$. Since $\delta > 2$, by Lemma 2 we have

$$\begin{aligned} I''_{44} &\leqslant C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left( t^{-1} \sum_{k=1}^{n} \mathbf{E}|X_{nk}|^q I\big(a_n < |X_{nk}| \leqslant t^{1/q}\big) \right)^{\delta/2} \mathrm{d}t \\ &\leqslant C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left( t^{-1} \sum_{k=1}^{n} \mathbf{E}|X_{nk}|^q I\big(|X_{nk}| > a_n\big) \right)^{\delta/2} \mathrm{d}t \\ &= C \sum_{n=1}^{\infty} a_n^{-q} \left( \sum_{k=1}^{n} \mathbf{E}|X_{nk}|^q I\big(|X_{nk}| > a_n\big) \right)^{\delta/2} \int_{a_n^q}^{\infty} t^{-\delta/2}\, \mathrm{d}t \\ &\leqslant C \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \frac{\mathbf{E}|X_{nk}|^q I(|X_{nk}| > a_n)}{a_n^q} \right)^{\delta/2} < \infty. \end{aligned}$$

Finally, we prove that $I''_{44} < \infty$ in the case $2 < q < p$. Since $\delta > q$ and $\delta > 2$, by Lemma 2 we have

$$\begin{aligned} I''_{44} &\leqslant C \sum_{n=1}^{\infty} a_n^{-q} \int_{a_n^q}^{\infty} \left( t^{-2/q} \sum_{k=1}^{n} \mathbf{E}X_{nk}^2 I\big(|X_{nk}| > a_n\big) \right)^{\delta/2} \mathrm{d}t \\ &= C \sum_{n=1}^{\infty} a_n^{-q} \left( \sum_{k=1}^{n} \mathbf{E}X_{nk}^2 I\big(|X_{nk}| > a_n\big) \right)^{\delta/2} \int_{a_n^q}^{\infty} t^{-\delta/q}\, \mathrm{d}t \\ &\leqslant C \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \frac{\mathbf{E}X_{nk}^2 I(|X_{nk}| > a_n)}{a_n^2} \right)^{\delta/2} < \infty. \end{aligned}$$

The proof is complete. □

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## REFERENCES

1. R.C. Bradley, Equivalent mixing conditions for random fields, Technical report No. 336, Center for Stochastic Processes, Dept. Statistics, Univ. North Carolina, Chapel Hill, 1990.
2. Y.S. Chow, On the rate of moment complete convergence of sample sums and extremes, *Bull. Inst. Math., Acad. Sin.*, **16**:177–201, 1988.
3. P.L. Hsu and H. Robbins, Complete convergence and the law of large numbers, *Proc. Natl. Acad. Sci. USA*, **33**(2):25–31, 1947.
4. C.C. Moore, The degree of randomness in a stationary time series, *Ann. Math. Stat.*, **34**:1253–1258, 1963.
5. S. Utev and M. Peligrad, Maximal inequalities and an invariance principle for a class of weakly dependent random variables, *J. Theor. Probab.*, **16**(1):101–115, 2003.
6. M.H. Zhu, Strong laws of large numbers for arrays of rowwise $\rho^*$-mixing random variables, *Discrete Dyn. Nat. Soc.*, **2007**, Article ID 74296, 6 pp., 2007.