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Complete convergence and complete moment convergence for weighted sums of *m*-NA random variables

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Abstract
The authors study the complete convergence and the complete moment convergence for weighted sums of *m*-negatively associated (*m*-NA) random variables and obtain some new results. These results extend and improve the corresponding theorems of Sung (Stat. Pap. 52:447-454, 2011). In addition, we point out that an open problem presented in Sung (Stat. Pap. 54:773-781, 2013) can be solved by means of the method used in this paper.

MSC: 60F15; 62G05

Keywords: complete convergence; complete moment convergence; weighted sums; *m*-negatively associated random variable

1 Introduction

Let $\{X, X_n, n \geq 1\}$ be a sequence of random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants. Because the weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ play important roles in some useful linear statistics, many authors studied the strong convergence for the weighted sums. We refer the reader to Cuzick [1], Wu [2], Bai and Cheng [3], Sung [4], Chen and Gan [5], Cai [6], Wu [7], Zarei and Jabbari [8], Sung [9], Sung [10], Shen [11], Chen and Sung [12].

The concept of the complete convergence was introduced by Hsu and Robbins [13]. A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant θ if

$$\sum_{n=1}^{\infty} P\bigl(|U_n - \theta| > \varepsilon\bigr) < \infty \quad \text{for all } \varepsilon > 0.$$

Chow [14] presented the following more general concept of the complete moment convergence. Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\bigl\{b_n^{-1}|Z_n| - \varepsilon\bigr\}_+^q < \infty \quad \text{for some or all } \varepsilon > 0,$$

then the above result was called the complete moment convergence.

The following concept was introduced by Joag-Dev and Proschan [15].

Definition 1.1 A finite family of random variables $\{X_k, 1 \leq k \leq n\}$ is said to be negatively associated (abbreviated to NA) if for any disjoint subsets A and B of $\{1, 2, \ldots, n\}$ and any real coordinate-wise nondecreasing functions f on R^A and g on R^B,

$$\operatorname{Cov}\bigl(f(X_i, i \in A), g(Y_j, j \in B)\bigr) \leq 0$$

whenever the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

Definition 1.2 Let $m \geq 1$ be a fixed integer. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be m-negatively associated (abbreviated to m-NA) if for any $n \geq 2$ and any $i_1, \ldots, i_n$ such that $|i_k - i_j| \geq m$ for all $1 \leq k \neq j \leq n$, $X_{i_1}, \ldots, X_{i_n}$ are NA.

The concept of m-NA random variables was introduced by Hu *et al.* [16]. It is easily seen that this concept is a natural extension from NA random variables (wherein $m = 1$).

It is well known that the properties of NA random variables have been applied to the reliability theory, multivariate statistical analysis and percolation theory. Sequences of NA random variables have been an attractive research topic in the recent literature. For example, Matula [17], Su *et al.* [18], Shao [19], Gan and Chen [20], Fu and Zhang [21], Baek *et al.* [22], Chen *et al.* [23], Cai [6], Xing [24], Sung [10], Qin and Li [25], Wu [26]. Since NA implies m-NA, it is very significant to study the convergence properties of this wider m-NA class. However, to the best of our knowledge, besides Hu *et al.* [16] and Hu *et al.* [27], few authors discuss the convergence properties for sequences of m-NA random variables.

Cai [6] studied the complete convergence for weighted sums of identically distributed NA random variables. He obtained the following theorem.

Theorem A *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NA random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying*

$$A_\alpha = \lim_{n\to\infty} \sup A_{\alpha,n} < \infty, \qquad A_{\alpha,n}^\alpha = \sum_{i=1}^{n} |a_{ni}|^\alpha / n \tag{1.1}$$

for some $0 < \alpha \leq 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, suppose that $EX = 0$ when $1 < \alpha \leq 2$. If $E\exp(h|X|^\gamma) < \infty$ for some $h > 0$, then

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq m\leq n} \left|\sum_{i=1}^{m} a_{ni} X_i\right| > b_n \varepsilon\right) < \infty \quad \textit{for all } \varepsilon > 0. \tag{1.2}$$

Sung [10] improved Theorem A by replacing some much weaker moment conditions.

Theorem B *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NA random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying* (1.1) *for some $0 < \alpha \leq 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, suppose that $EX = 0$ when $1 < \alpha \leq 2$. Then the following statements hold*:

(i) *If $\alpha > \gamma$, then $E|X|^\alpha < \infty$ implies* (1.2).

(ii) *If $\alpha = \gamma$, then $E|X|^\alpha \log|X| < \infty$ implies* (1.2).

(iii) *If $\alpha < \gamma$, then $E|X|^\gamma < \infty$ implies* (1.2).

The main purpose of this article is to discuss the complete convergence and the complete moment convergence for weighted sums of m-NA random variables. We shall extend Theorem B to m-NA random variables. In addition, we shall extend and improve Theorem B by obtaining a much stronger conclusion under the same conditions (see Remark 3.2).

It is worthy to point out that the open problem presented in Sung [9], see Remark 2.2, can be solved by means of the method used in this article (see Remark 3.4).

Throughout this paper, the symbol C represents positive constants whose values may change from one place to another. For a finite set A the symbol $\sharp(A)$ denotes the number of elements in the set A.

2 Preliminaries

We first recall the following concept of stochastic domination, which is a slight generalization of identical distribution. An sequence of random variables $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X (write $\{X_n\} \prec X$) if there exists a constant $C > 0$ such that

$$\sup_{n\geq 1} P\bigl(|X_n| > x\bigr) \leq CP\bigl(|X| > x\bigr), \quad \forall x > 0.$$

The following exponential inequality for m-NA random variables can be proved by means of Theorem 3 in Shao [19] and the proof of Lemma 2 in Hu *et al.* [27]. Here we omit the details.

Lemma 2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of m-NA random variables with zero means and finite second moments. Let $S_j = \sum_{k=1}^{j} X_k$ and $B_n = \sum_{k=1}^{n} EX_k^2$. Then for all $n \geq m$, $x > 0$ and $a > 0$,*

$$P\Bigl(\max_{1\leq j\leq n} |S_j| \geq x\Bigr) \leq 2mP\Bigl(\max_{1\leq j\leq n} |X_j| > a\Bigr) + 4m\exp\left\{-\frac{x^2}{8m^2B_n}\right\} + 4m\left\{\frac{mB_n}{4(xa + mB_n)}\right\}^{x/(12ma)}. \tag{2.1}$$

Remark 2.1 Since $\mathrm{e}^{-x} \leq (1 + x)^{-1}$ for $x > 0$, we get, for $x > 0$ and $a > 0$,

$$\exp\left\{-\frac{x^2}{8m^2B_n}\right\} = \exp\left\{-\frac{3xa}{2mB_n}\right\}^{x/(12ma)} \leq \left(1 + \frac{3xa}{2mB_n}\right)^{-x/(12ma)}.$$

Noting that

$$\left\{\frac{mB_n}{4(xa + mB_n)}\right\}^{x/(12ma)} = \left(1 + \frac{4xa}{mB_n}\right)^{-x/(12ma)} \leq \left(1 + \frac{3xa}{2mB_n}\right)^{-x/(12ma)}.$$

Therefore, it follows by (2.1) that

$$P\Bigl(\max_{1\leq j\leq n} |S_j| \geq x\Bigr) \leq 2m\sum_{k=1}^{n} P\bigl(|X_k| > a\bigr) + 8m\left(1 + \frac{3xa}{2mB_n}\right)^{-x/(12ma)}. \tag{2.2}$$

Now we present a Rosenthal-type inequality for maximum partial sums of m-NA random variables, which is the crucial tool in the proof of our main results.

Lemma 2.2 *Let $\{X_n, n \geq 1\}$ be a sequence of m-NA random variables with mean zero and $E|X_k|^q < \infty$ for every $1 \leq k \leq n$. Let $S_j = \sum_{k=1}^{j} X_k$, $1 \leq j \leq n$. Then for $q \geq 2$, there exists a positive constant C depending only on q such that*

$$E \max_{1\leq j\leq n} |S_j|^q \leq C\left\{\sum_{k=1}^{n} E|X_k|^q + \left(\sum_{k=1}^{n} EX_k^2\right)^{q/2}\right\}. \tag{2.3}$$

Proof Let $B_n = \sum_{k=1}^{n} EX_k^2$. Noting that

$$E|Y|^q = q\int_0^\infty P(|Y| \geq x)x^{q-1}\,dx \qquad (E|Y|^q < \infty). \tag{2.4}$$

By taking $a = x/(12mq)$ in (2.2), we have

$$\begin{aligned} E \max_{1\leq j\leq n} |S_j|^q &= q\int_0^\infty P\Big(\max_{1\leq j\leq n} |S_j| \geq x\Big)x^{q-1}\,dx \\ &\leq 2mq\sum_{k=1}^{n}\int_0^\infty P\big(|X_k| \geq x/(12mq)\big)x^{q-1}\,dx \\ &\quad + 8mq\int_0^\infty \left(1 + \frac{x^2}{8m^2qB_n}\right)^{-q} x^{q-1}\,dx \\ &=: A + B. \end{aligned}$$

By (2.4), we have $A = 2^{2q+1}3^q m^{q+1}q^q \sum_{k=1}^{n} E|X_k|^q$. Letting $t = x^2/(8m^2qB_n)$, then

$$\begin{aligned} B &= 2^{2+3q/2}m^{1+q}q^{1+q/2}(B_n)^{q/2}\int_0^\infty (1+t)^{-q}t^{q/2-1}\,dt \\ &= 2^{2+3q/2}m^{1+q}q^{1+q/2}B(q/2, q/2)\left(\sum_{k=1}^{n} EX_k^2\right)^{q/2}, \end{aligned}$$

where

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \int_0^\infty t^{\alpha-1}(1+t)^{-(\alpha+\beta)}\,dt.$$

Letting $C = \max\{2^{2q+1}3^q m^{q+1}q^q, 2^{2+3q/2}m^{1+q}q^{1+q/2}B(q/2, q/2)\}$, we can get (2.3). The proof is complete. □

Lemma 2.3 (Wang *et al.* [28]) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables with $\{X_n\} \prec X$. Then there exists a constant C such that, for all $q > 0$ and $x > 0$,*

(i) $E|X_k|^q I(|X_k| \leq x) \leq C\{E|X|^q I(|X| \leq x) + x^q P(|X| > x)\}$,

(ii) $E|X_k|^q I(|X_k| > x) \leq CE|X|^q I(|X| > x)$.

The following lemma is very important in the proof of our result, which improves Lemma 2.2 and Lemma 2.3 of Sung [10].

Lemma 2.4 *Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^\alpha \leq n$ for some $\alpha > 0$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Then*

$$I =: \sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X|^{\alpha} I\bigl(|a_{ni}X| > b_n\bigr) \le \begin{cases} CE|X|^{\alpha} & \textit{for } \alpha > \gamma, \\ CE|X|^{\alpha} \log|X| & \textit{for } \alpha = \gamma, \\ CE|X|^{\gamma} & \textit{for } \alpha < \gamma. \end{cases}$$

Proof From $\sum_{i=1}^{n} |a_{ni}|^{\alpha} \le n$, we have

$$\begin{aligned} I &= \sum_{n=2}^{\infty} n^{-2} (\log n)^{-\alpha/\gamma} \sum_{i=1}^{n} E|a_{ni}X|^{\alpha} I\bigl(|X|^{\alpha} > n(\log n)^{\alpha/\gamma} |a_{ni}|^{-\alpha}\bigr) \\ &\le \sum_{n=2}^{\infty} n^{-2} (\log n)^{-\alpha/\gamma} \sum_{i=1}^{n} E|a_{ni}X|^{\alpha} I\left(|X|^{\alpha} > n(\log n)^{\alpha/\gamma} \left(\sum_{i=1}^{n} |a_{ni}|^{\alpha}\right)^{-1}\right) \\ &\le \sum_{n=2}^{\infty} n^{-2} (\log n)^{-\alpha/\gamma} \sum_{i=1}^{n} E|a_{ni}X|^{\alpha} I\bigl(|X| > (\log n)^{1/\gamma}\bigr) \\ &\le \sum_{n=2}^{\infty} n^{-1} (\log n)^{-\alpha/\gamma} E|X|^{\alpha} I\bigl(|X| > (\log n)^{1/\gamma}\bigr) \\ &= \sum_{n=2}^{\infty} n^{-1} (\log n)^{-\alpha/\gamma} \sum_{m=n}^{\infty} E|X|^{\alpha} I\bigl(\log m < |X|^{\gamma} \le \log(m+1)\bigr) \\ &= \sum_{m=2}^{\infty} E|X|^{\alpha} I\bigl(\log m < |X|^{\gamma} \le \log(m+1)\bigr) \sum_{n=2}^{m} n^{-1} (\log n)^{-\alpha/\gamma}. \end{aligned}$$

Observing that

$$\sum_{n=2}^{m} n^{-1} (\log n)^{-\alpha/\gamma} \le \begin{cases} C & \text{for } \alpha > \gamma, \\ C \log\log m & \text{for } \alpha = \gamma, \\ C(\log m)^{1-\alpha/\gamma} & \text{for } \alpha < \gamma, \end{cases}$$

we can get

$$I \le \begin{cases} CE|X|^{\alpha} & \text{for } \alpha > \gamma, \\ CE|X|^{\alpha} \log|X| & \text{for } \alpha = \gamma, \\ CE|X|^{\gamma} & \text{for } \alpha < \gamma. \end{cases}$$

The proof of Lemma 2.4 is completed. □

Remark 2.2 Noting that

$$\sum_{n=2}^{\infty} n^{-1} \sum_{i=1}^{n} P\bigl(|a_{ni}X| > b_n\bigr) \le \sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X|^{\alpha} I\bigl(|a_{ni}X| > b_n\bigr),$$

we know that Lemma 2.4 improves Lemma 2.2 and Lemma 2.3 of Sung [10]. In addition, the method used in this paper is novel and much simpler than that in Sung [10].

3 Main result

In this section, we state our main results and their proofs.

Theorem 3.1 *Let* $\{X_n, n \geq 1\}$ *be a sequence of m-NA random variables with* $\{X_n\} \prec X$, *and let* $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ *be an array of constants satisfying* (1.1) *for some* $0 < \alpha \leq 2$. *Let* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ *for some* $\gamma > 0$. *Furthermore, suppose that* $EX_i = 0$ *when* $1 < \alpha \leq 2$. *Then the following statements hold*:

(i) *If* $\alpha > \gamma$, *then* $E|X|^\alpha < \infty$ *implies* (1.2).

(ii) *If* $\alpha = \gamma$, *then* $E|X|^\alpha \log|X| < \infty$ *implies* (1.2).

(iii) *If* $\alpha < \gamma$, *then* $E|X|^\gamma < \infty$ *implies* (1.2).

Remark 3.1 Since NA implies m-NA, Theorem 3.1 extends Theorem B. Compared with Sung [10], the proof of Theorem 3.1 is different from that of Theorem 2.1 in Sung [10].

Corollary 3.1 *Let* $\{X_n, n \geq 1\}$ *be a sequence of m-NA random variables with* $\{X_n\} \prec X$, *and let* $\{a_i, 1 \leq i \leq n\}$ *be a sequence of constants satisfying*

$$A_\alpha = \lim_{n\to\infty} \sup A_{\alpha,n} < \infty, \qquad A_{\alpha,n}^\alpha = \sum_{i=1}^n |a_i|^\alpha / n$$

for some $0 < \alpha \leq 2$. *Let* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ *for some* $\gamma > 0$. *Furthermore, suppose that* $EX_i = 0$ *when* $1 < \alpha \leq 2$. *Then*

$$b_n^{-1} \sum_{i=1}^n a_i X_i \to 0 \quad a.s.$$

By a similar argument as the proof of Corollary 2.1 in Cai [6], we can prove this corollary. Here we omit the details.

Theorem 3.2 *Assume that the conditions of Theorem* 3.1 *hold, then the following statements hold*:

(i) *If* $\alpha > \gamma$, *then* $E|X|^\alpha < \infty$ *implies*

$$\sum_{n=2}^\infty n^{-1} E\left\{ b_n^{-1} \max_{1\leq m\leq n} \left| \sum_{i=1}^m a_{ni} X_i \right| - \varepsilon \right\}_+^\alpha < \infty \quad \text{for all } \varepsilon > 0. \tag{3.1}$$

(ii) *If* $\alpha = \gamma$, *then* $E|X|^\alpha \log|X| < \infty$ *implies* (3.1).

(iii) *If* $\alpha < \gamma$, *then* $E|X|^\gamma < \infty$ *implies* (3.1).

Remark 3.2 Noting that

$$\begin{aligned} &\sum_{n=2}^\infty n^{-1} E\left\{ b_n^{-1} \max_{1\leq m\leq n} \left| \sum_{i=1}^m a_{ni} X_i \right| - \varepsilon \right\}_+^\alpha \\ &\quad = \sum_{n=2}^\infty n^{-1} \int_0^\infty P\left(b_n^{-1} \max_{1\leq m\leq n} \left| \sum_{i=1}^m a_{ni} X_i \right| > \varepsilon + t^{1/\alpha} \right) \mathrm{d}t \\ &\quad = \int_0^\infty \sum_{n=2}^\infty n^{-1} P\left(b_n^{-1} \max_{1\leq m\leq n} \left| \sum_{i=1}^m a_{ni} X_i \right| > \varepsilon + t^{1/\alpha} \right) \mathrm{d}t. \end{aligned}$$

Therefore, Theorem 3.2 extends and improves Theorem B.

Proof of Theorem 3.1 Without loss of generality, we may assume that $a_{ni} \geq 0$. For fixed $n \geq 1$, let

$$Y_{ni} = -b_n I(a_{ni}X_i < -b_n) + a_{ni}X_i I\big(a_{ni}|X_i| \leq b_n\big) + b_n I(a_{ni}X_i > b_n),$$
$$Z_{ni} = (a_{ni}X_i + b_n)I(a_{ni}X_i < -b_n) + (a_{ni}X_i - b_n)I(a_{ni}X_i > b_n).$$

Then $Y_{ni} + Z_{ni} = a_{ni}X_i$, and it follows by the definition of m-NA and Property 6 of Joag-Dev and Proschan [15] that $\{Y_{ni}, i \geq 1, n \geq 1\}$ is sequence of m-NA random variables. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| > b_n\varepsilon\right) \\
&\quad \leq 1 + \sum_{n=2}^{\infty} n^{-1} \sum_{i=1}^{n} P\big(a_{ni}|X_i| > b_n\big) + \sum_{n=2}^{\infty} n^{-1} P\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m} Y_{ni}\right| > b_n\varepsilon\right) \\
&\quad =: 1 + H_1 + H_2.
\end{aligned}$$

By $\{X_n\} \prec X$ and Lemma 2.4, we have

$$H_1 \leq C\sum_{n=2}^{\infty} n^{-1} \sum_{i=1}^{n} P\big(a_{ni}|X| > b_n\big) \leq C\sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} a_{ni}^{\alpha} E|X|^{\alpha} I\big(a_{ni}|X| > b_n\big) < \infty.$$

Then we prove $H_2 < \infty$. Noting that either $E|X|^{\alpha}\log|X| < \infty$ for $\alpha = \gamma$, or $E|X|^{\gamma} < \infty$ for $\alpha < \gamma$ implies $E|X|^{\alpha} < \infty$. From (1.1), without loss of generality, we may assume that $\sum_{i=1}^{n} a_{ni}^{\alpha} \leq n$. We first prove

$$L =: b_n^{-1} \max_{1\leq m\leq n}\left|\sum_{i=1}^{m} EY_{ni}\right| \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

For $0 < \alpha \leq 1$, by Lemma 2.3 and $\sum_{i=1}^{n} a_{ni}^{\alpha} \leq n$, we have

$$\begin{aligned}
L &\leq Cb_n^{-1} \sum_{i=1}^{n} a_{ni}E|X| I\big(a_{ni}|X| \leq b_n\big) + C\sum_{i=1}^{n} P\big(a_{ni}|X| > b_n\big) \\
&\leq Cb_n^{-\alpha} \sum_{i=1}^{n} a_{ni}^{\alpha}E|X|^{\alpha} I\big(a_{ni}|X| \leq b_n\big) + Cb_n^{-\alpha} \sum_{i=1}^{n} a_{ni}^{\alpha}E|X|^{\alpha} I\big(a_{ni}|X| > b_n\big) \\
&\leq C(\log n)^{-\alpha/\gamma} E|X|^{\alpha} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

For $1 < \alpha \leq 2$, by $EX_i = 0$, $|Z_{ni}| \leq a_{ni}|X_i| I(a_{ni}|X_i| > b_n)$, and Lemma 2.3, we have

$$\begin{aligned}
L &= b_n^{-1} \max_{1\leq m\leq n}\left|\sum_{i=1}^{m} EZ_{ni}\right| \leq b_n^{-1} \sum_{i=1}^{n} a_{ni}E|X_i| I\big(a_{ni}|X_i| > b_n\big) \\
&\leq Cb_n^{-1} \sum_{i=1}^{n} a_{ni}E|X| I\big(a_{ni}|X| > b_n\big) \leq Cb_n^{-\alpha} \sum_{i=1}^{n} a_{ni}^{\alpha}E|X|^{\alpha} I\big(a_{ni}|X| > b_n\big) \\
&\leq C(\log n)^{-\alpha/\gamma} E|X|^{\alpha} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence (3.2) holds for $0 < \alpha \le 2$. Then, while n is sufficiently large,

$$\max_{1\le m\le n}\left|\sum_{i=1}^{m} EY_{ni}\right| \le b_n\varepsilon/2. \tag{3.3}$$

Let $q > \max\{2, 2\gamma/\alpha\}$. Then by (3.3), the Markov inequality, and Lemma 2.2, we have

$$\begin{aligned}
H_2 &\le \sum_{n=2}^{\infty} n^{-1} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m}(Y_{ni}-EY_{ni})\right| > b_n\varepsilon/2\right) \\
&\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-q} E\max_{1\le m\le n}\left|\sum_{i=1}^{m}(Y_{ni}-EY_{ni})\right|^{q} \\
&\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-q}\left(\sum_{i=1}^{n} E|Y_{ni}|^2\right)^{q/2} + C\sum_{n=2}^{\infty} n^{-1} b_n^{-q}\sum_{i=1}^{n} E|Y_{ni}|^q \\
&=: H_3 + H_4.
\end{aligned}$$

Firstly, we prove $H_3 < \infty$. By Lemma 2.3, $\alpha \le 2$, $\sum_{i=1}^{n}|a_{ni}|^\alpha \le n$, and $q > 2\gamma/\alpha$, we have

$$\begin{aligned}
H_3 &\le C\sum_{n=2}^{\infty} n^{-1}\left(b_n^{-2}\sum_{i=1}^{n} a_{ni}^2 E|X|^2 I\big(a_{ni}|X| \le b_n\big) + \sum_{i=1}^{n} P\big(a_{ni}|X| > b_n\big)\right)^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{-1}\left(b_n^{-\alpha}\sum_{i=1}^{n} a_{ni}^\alpha E|X|^\alpha I\big(a_{ni}|X| \le b_n\big) + b_n^{-\alpha}\sum_{i=1}^{n} a_{ni}^\alpha E|X|^\alpha I\big(a_{ni}|X| > b_n\big)\right)^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\frac{\alpha q}{2\gamma}}\big(E|X|^\alpha\big)^{q/2} < \infty.
\end{aligned}$$

Next we consider H_4. By Lemma 2.3, we have

$$\begin{aligned}
H_4 &\le C\sum_{n=2}^{\infty} n^{-1} b_n^{-q}\sum_{i=1}^{n} a_{ni}^q E|X|^q I\big(a_{ni}|X| \le b_n\big) + C\sum_{n=2}^{\infty} n^{-1}\sum_{i=1}^{n} P\big(a_{ni}|X| > b_n\big) \\
&=: H_5 + H_6.
\end{aligned}$$

Similar to the proof of $H_1 < \infty$, we get directly $H_6 < \infty$. Then final work is to prove $H_5 < \infty$. For $j \ge 1$ and $n \ge 2$, let

$$I_{nj} = \left\{1 \le i \le n : n^{1/\alpha}(j+1)^{-1/\alpha} < |a_{ni}| \le n^{1/\alpha} j^{-1/\alpha}\right\}.$$

Then $\{I_{nj}, j \ge 1\}$ are disjoint, $\bigcup_{j\ge1} I_{nj} = N$ for all $n \ge 1$ from $\sum_{i=1}^{n}|a_{ni}|^\alpha \le n$, where N is the set of positive integers. Noting that for all $k \ge 1$, we have

$$\begin{aligned}
n &\ge \sum_{i=1}^{n}|a_{ni}|^\alpha = \sum_{j=1}^{\infty}\sum_{i\in I_{nj}}|a_{ni}|^\alpha \ge \sum_{j=1}^{\infty}\sharp(I_{nj})n(j+1)^{-1} \ge \sum_{j=k}^{\infty}\sharp(I_{nj})n(j+1)^{-1} \\
&= \sum_{j=k}^{\infty}\sharp(I_{nj})n(j+1)^{-q/\alpha}(j+1)^{q/\alpha-1} \ge \sum_{j=k}^{\infty}\sharp(I_{nj})n(j+1)^{-q/\alpha}(k+1)^{q/\alpha-1}.
\end{aligned}$$

Hence for all $k \geq 1$, we have

$$\sum_{j=k}^{\infty} \sharp(I_{nj}) j^{-q/\alpha} \leq C(k+1)^{1-q/\alpha}. \tag{3.4}$$

Then

$$\begin{aligned}
H_5 &= \sum_{n=2}^{\infty} n^{-1-q/\alpha} (\log n)^{-q/\gamma} \sum_{i=1}^{n} |a_{ni}|^q E|X|^q I\bigl(|a_{ni}X| \leq n^{1/\alpha} (\log n)^{1/\gamma}\bigr) \\
&= \sum_{n=2}^{\infty} n^{-1-q/\alpha} (\log n)^{-q/\gamma} \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} |a_{ni}|^q E|X|^q I\bigl(|a_{ni}X| \leq n^{1/\alpha} (\log n)^{1/\gamma}\bigr) \\
&\leq \sum_{n=2}^{\infty} n^{-1-q/\alpha} (\log n)^{-q/\gamma} \sum_{j=1}^{\infty} \sharp(I_{nj}) n^{q/\alpha} j^{-q/\alpha} E|X|^q I\bigl(|X| \leq (j+1)^{1/\alpha} (\log n)^{1/\gamma}\bigr) \\
&\leq \sum_{n=2}^{\infty} n^{-1} (\log n)^{-q/\gamma} \sum_{j=1}^{\infty} \sharp(I_{nj}) j^{-q/\alpha} E|X|^q I\bigl(|X| \leq (\log n)^{1/\gamma}\bigr) \\
&\quad + \sum_{n=2}^{\infty} n^{-1} (\log n)^{-q/\gamma} \sum_{j=1}^{\infty} \sharp(I_{nj}) j^{-q/\alpha} \\
&\quad \times \sum_{k=1}^{j} E|X|^q I\bigl(k^{1/\alpha} (\log n)^{1/\gamma} < |X| \leq (k+1)^{1/\alpha} (\log n)^{1/\gamma}\bigr) \\
&=: H_5^* + H_5^{**}.
\end{aligned}$$

By (3.4) and $q > 2\gamma/\alpha \geq \gamma$, we have

$$\begin{aligned}
H_5^* &\leq C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-q/\gamma} E|X|^q I\bigl(|X|^\gamma \leq \log n\bigr) \\
&= C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-q/\gamma} \sum_{m=2}^{n} E|X|^q I\bigl(\log(m-1) < |X|^\gamma \leq \log m\bigr) \\
&= C \sum_{m=2}^{\infty} E|X|^q I\bigl(\log(m-1) < |X|^\gamma \leq \log m\bigr) \sum_{n=m}^{\infty} n^{-1} (\log n)^{-q/\gamma} \\
&\leq C \sum_{m=2}^{\infty} (\log m)^{1-q/\gamma} E|X|^q I\bigl(\log(m-1) < |X|^\gamma \leq \log m\bigr) \\
&\leq CE|X|^\gamma < \infty.
\end{aligned}$$

By (3.4), we have

$$\begin{aligned}
H_5^{**} &= \sum_{n=2}^{\infty} n^{-1} (\log n)^{-q/\gamma} \\
&\quad \times \sum_{k=1}^{\infty} E|X|^q I\bigl(k^{1/\alpha} (\log n)^{1/\gamma} < |X| \leq (k+1)^{1/\alpha} (\log n)^{1/\gamma}\bigr) \sum_{j=k}^{\infty} \sharp(I_{nj}) j^{-q/\alpha}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma} \\
&\quad\times \sum_{k=1}^{\infty}(k+1)^{1-q/\alpha}E|X|^q I\bigl(k^{1/\alpha}(\log n)^{1/\gamma} < |X| \leq (k+1)^{1/\alpha}(\log n)^{1/\gamma}\bigr) \\
&\leq C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha/\gamma}\sum_{k=1}^{\infty}E|X|^\alpha I\bigl(k^{1/\alpha}(\log n)^{1/\gamma} < |X| \leq (k+1)^{1/\alpha}(\log n)^{1/\gamma}\bigr) \\
&= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha/\gamma}E|X|^\alpha I\bigl(|X| > (\log n)^{1/\gamma}\bigr).
\end{aligned}
$$

Noting that we obtain the following result in the proof of Lemma 2.4,

$$
\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha/\gamma}E|X|^\alpha I\bigl(|X| > (\log n)^{1/\gamma}\bigr) \leq \begin{cases} CE|X|^\alpha & \text{for } \alpha > \gamma, \\ CE|X|^\alpha \log|X| & \text{for } \alpha = \gamma, \\ CE|X|^\gamma & \text{for } \alpha < \gamma. \end{cases}
$$

Hence we get $H_5^{**} < \infty$ combining the assumptions of Theorem 3.1. The proof is completed. □

Remark 3.3 It is easily seen that the proof of $H_5 < \infty$ complements Lemma 2.3 of Sung [9]. In fact, in that lemma Sung only proved $H_5 < \infty$ for the case $\alpha = \gamma$. It is worthy to point out that $a_{ni} = 0$ or $|a_{ni}| > 1$ is required in Sung [9]. Here, we do not require the extra conditions.

Remark 3.4 Sung [9] proved Theorem 3.1 for the case $\alpha = \gamma$ when $\{X_n, n \geq 1\}$ is a sequence of ρ^*-mixing random variables. However, he posed an open problem, that is, whether Theorem 3.1 (*i.e.* Theorem 1.1 in Sung [9]) remains true for ρ^*-mixing random variables.

The crucial tool of the proof of Theorem 3.1 is the Rosenthal-type inequality for maximum partial sums of m-NA random variables. For ρ^*-mixing random variables, the Rosenthal-type inequality for maximum partial sums also holds (see Utev and Peligrad [29]). Therefore, it is easy to solve the above open problem by following the method used in the proof of Theorem 3.1.

Proof of Theorem 3.2 For any given $\varepsilon > 0$, we have

$$
\begin{aligned}
&\sum_{n=2}^{\infty} n^{-1}E\left\{b_n^{-1}\max_{1\leq m\leq n}\left|\sum_{i=1}^{m}a_{ni}X_i\right| - \varepsilon\right\}_+^\alpha \\
&\quad= \sum_{n=2}^{\infty} n^{-1}\int_0^\infty P\left(b_n^{-1}\max_{1\leq m\leq n}\left|\sum_{i=1}^{m}a_{ni}X_i\right| > \varepsilon + t^{1/\alpha}\right)\mathrm{d}t \\
&\quad\leq \sum_{n=2}^{\infty} n^{-1}P\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m}a_{ni}X_i\right| > b_n\varepsilon\right) + \sum_{n=2}^{\infty} n^{-1}\int_1^\infty P\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m}a_{ni}X_i\right| > b_n t^{1/\alpha}\right)\mathrm{d}t \\
&\quad=: I_1 + I_2.
\end{aligned}
$$

Therefore, to prove (3.1), one needs only to prove that $I_1 < \infty$ and $I_2 < \infty$. By Theorem 3.1, we get directly $I_1 < \infty$. For all $t \geq 1$, we denote

$$\begin{aligned} Y_{ni} &= -b_n t^{1/\alpha} I\left(a_{ni}X_i < -b_n t^{1/\alpha}\right) + a_{ni}X_i I\left(a_{ni}|X_i| \leq b_n t^{1/\alpha}\right) + b_n t^{1/\alpha} I\left(a_{ni}X_i > b_n t^{1/\alpha}\right), \\ Z_{ni} &= a_{ni}X_i - Y_{ni}. \end{aligned}$$

Then

$$\begin{aligned} I_2 \leq{}& \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\left(\max_{1\leq i\leq n} a_{ni}|X_i| > b_n t^{1/\alpha}\right) \mathrm{d}t \\ &+ \sum_{n=2}^{\infty} n^{-1} \int_1^{\infty} P\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m} Y_{ni}\right| > b_n t^{1/\alpha}\right) \mathrm{d}t \\ =:{}& I_3 + I_4. \end{aligned}$$

Noting that

$$\int_1^{\infty} P\left(a_{ni}|X| > b_n t^{1/\alpha}\right) \mathrm{d}t \leq b_n^{-\alpha} a_{ni}^{\alpha} E|X|^{\alpha} I\left(|a_{ni}X| > b_n\right),$$

by $\{X_i\} \prec X$, Lemma 2.4, and the assumptions of Theorem 3.2, we have

$$\begin{aligned} I_3 &\leq \sum_{n=2}^{\infty} n^{-1} \sum_{i=1}^{n} \int_1^{\infty} P\left(a_{ni}|X_i| > b_n t^{1/\alpha}\right) \mathrm{d}t \leq \sum_{n=2}^{\infty} n^{-1} \sum_{i=1}^{n} \int_1^{\infty} P\left(a_{ni}|X| > b_n t^{1/\alpha}\right) \mathrm{d}t \\ &\leq \sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} a_{ni}^{\alpha} E|X|^{\alpha} I\left(a_{ni}|X| > b_n\right) < \infty. \end{aligned}$$

Next we prove that $I_4 < \infty$. We first show

$$J = \sup_{t\geq 1} t^{-1/\alpha} b_n^{-1} \max_{1\leq m\leq n}\left|\sum_{i=1}^{m} EY_{ni}\right| \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

For $0 < \alpha \leq 1$, by Lemma 2.3 and $\sum_{i=1}^{n} a_{ni}^{\alpha} \leq n$, we have

$$\begin{aligned} J &\leq C \sup_{t\geq 1} t^{-1/\alpha} b_n^{-1} \sum_{i=1}^{n} a_{ni} E|X| I\left(a_{ni}|X| \leq b_n t^{1/\alpha}\right) + C \sup_{t\geq 1} \sum_{i=1}^{n} P\left(a_{ni}|X| > b_n t^{1/\alpha}\right) \\ &\leq C \sup_{t\geq 1} t^{-1} b_n^{-\alpha} \sum_{i=1}^{n} a_{ni}^{\alpha} E|X|^{\alpha} I\left(a_{ni}|X| \leq b_n t^{1/\alpha}\right) + C b_n^{-\alpha} \sum_{i=1}^{n} a_{ni}^{\alpha} E|X|^{\alpha} \\ &\leq C(\log n)^{-\alpha/\gamma} E|X|^{\alpha} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

For $1 < \alpha \leq 2$, by $EX_i = 0$, $|Z_{ni}| \leq a_{ni}|X_i| I(a_{ni}|X_i| > b_n t^{1/\alpha})$, and Lemma 2.3, we have

$$\begin{aligned} J &= \sup_{t\geq 1} t^{-1/\alpha} b_n^{-1} \max_{1\leq m\leq n}\left|\sum_{i=1}^{m} EZ_{ni}\right| \\ &\leq C \sup_{t\geq 1} t^{-1/\alpha} b_n^{-1} \sum_{i=1}^{n} a_{ni} E|X| I\left(a_{ni}|X| > b_n t^{1/\alpha}\right) \end{aligned}$$

$$
\begin{aligned}
&\leq C\sup_{t\geq 1} t^{-1}b_n^{-\alpha}\sum_{i=1}^{n} a_{ni}^{\alpha}E|X|^{\alpha}I\big(a_{ni}|X|>b_n t^{1/\alpha}\big)\\
&\leq C(\log n)^{-\alpha/\gamma}E|X|^{\alpha}\to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

From (3.5), we know that, while n is sufficiently large,

$$
\max_{1\leq m\leq n}\left|\sum_{i=1}^{m} EY_{ni}\right|\leq b_n t^{1/\alpha}/2 \tag{3.6}
$$

holds uniformly for $t\geq 1$.

Let $q>\max\{2,2\gamma/\alpha\}$. Then by (3.6) and Lemma 2.2, we have

$$
\begin{aligned}
I_4 &\leq \sum_{n=2}^{\infty} n^{-1}\int_1^{\infty} P\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m}(Y_{ni}-EY_{ni})\right|>b_n t^{1/\alpha}/2\right)\mathrm{d}t\\
&\leq C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\int_1^{\infty} t^{-q/\alpha}E\max_{1\leq m\leq n}\left|\sum_{i=1}^{m}(Y_{ni}-EY_{ni})\right|^{q}\mathrm{d}t\\
&\leq C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\int_1^{\infty} t^{-q/\alpha}\sum_{i=1}^{n}E|Y_{ni}|^{q}\,\mathrm{d}t + C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\int_1^{\infty} t^{-q/\alpha}\left(\sum_{i=1}^{n}E|Y_{ni}|^{2}\right)^{q/2}\mathrm{d}t\\
&=: I_5+I_6.
\end{aligned}
$$

By Lemma 2.3, $\alpha\leq 2$, and $q>2\gamma/\alpha$, we have

$$
\begin{aligned}
I_6 &\leq C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\int_1^{\infty} t^{-q/\alpha}\left(\sum_{i=1}^{n} a_{ni}^{2}E|X|^{2}I\big(a_{ni}|X|\leq b_n t^{1/\alpha}\big)\right.\\
&\quad\left. + b_n^{2}t^{2/\alpha}\sum_{i=1}^{n}P\big(a_{ni}|X|>b_n t^{1/\alpha}\big)\right)^{q/2}\mathrm{d}t\\
&\leq C\sum_{n=2}^{\infty} n^{-1}\int_1^{\infty}\left(b_n^{-\alpha}t^{-1}\sum_{i=1}^{n} a_{ni}^{\alpha}E|X|^{\alpha}I\big(a_{ni}|X|\leq b_n t^{1/\alpha}\big)\right.\\
&\quad\left. + b_n^{-\alpha}t^{-1}\sum_{i=1}^{n} a_{ni}^{\alpha}E|X|^{\alpha}I\big(a_{ni}|X|>b_n t^{1/\alpha}\big)\right)^{q/2}\mathrm{d}t\\
&\leq C\sum_{n=2}^{\infty} n^{-1}\int_1^{\infty} t^{-q/2}\left(b_n^{-\alpha}\sum_{i=1}^{n} a_{ni}^{\alpha}E|X|^{\alpha}\right)^{q/2}\mathrm{d}t\\
&\leq C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha q/(2\gamma)}\big(E|X|^{\alpha}\big)^{q/2}<\infty.
\end{aligned}
$$

For I_5, we have

$$
\begin{aligned}
I_5 &\leq C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\sum_{i=1}^{n}\int_1^{\infty} t^{-q/\alpha}a_{ni}^{q}E|X|^{q}I\big(a_{ni}|X|\leq b_n t^{1/\alpha}\big)\,\mathrm{d}t\\
&\quad + C\sum_{n=2}^{\infty} n^{-1}\sum_{i=1}^{n}\int_1^{\infty} P\big(a_{ni}|X|>b_n t^{1/\alpha}\big)\,\mathrm{d}t
\end{aligned}
$$

$$\begin{aligned}
&= C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\sum_{i=1}^{n}\int_1^{\infty} t^{-q/\alpha}a_{ni}^{q}E|X|^{q}I\big(a_{ni}|X|\le b_n\big)\,\mathrm{d}t \\
&\quad + C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\sum_{i=1}^{n}\int_1^{\infty} t^{-q/\alpha}a_{ni}^{q}E|X|^{q}I\big(b_n< a_{ni}|X|\le b_n t^{1/\alpha}\big)\,\mathrm{d}t \\
&\quad + C\sum_{n=2}^{\infty} n^{-1}\sum_{i=1}^{n}\int_1^{\infty} P\big(a_{ni}|X|> b_n t^{1/\alpha}\big)\,\mathrm{d}t \\
&=: I_7 + I_8 + I_9.
\end{aligned}$$

Similar to the proof of $I_3 < \infty$, we get $I_9 < \infty$. Similar to the proof of $H_5 < \infty$, we get $I_7 < \infty$. By $q > 2 \ge \alpha$ and the following standard arguments, we get

$$\begin{aligned}
&b_n^{-q}\int_1^{\infty} t^{-q/\alpha}a_{ni}^{q}E|X|^{q}I\big(b_n< a_{ni}|X|\le b_n t^{1/\alpha}\big)\,\mathrm{d}t \\
&\le b_n^{-q}\sum_{m=1}^{\infty}\int_m^{m+1} t^{-q/\alpha}a_{ni}^{q}E|X|^{q}I\big(b_n< a_{ni}|X|\le b_n t^{1/\alpha}\big)\,\mathrm{d}t \\
&\le b_n^{-q}\sum_{m=1}^{\infty} m^{-q/\alpha}a_{ni}^{q}E|X|^{q}I\big(b_n< a_{ni}|X|\le b_n (m+1)^{1/\alpha}\big) \\
&\le b_n^{-q}\sum_{m=1}^{\infty} m^{-q/\alpha}\sum_{s=1}^{m}a_{ni}^{q}E|X|^{q}I\big(b_n s^{1/\alpha}< a_{ni}|X|\le b_n (s+1)^{1/\alpha}\big) \\
&\le b_n^{-q}\sum_{s=1}^{\infty}a_{ni}^{q}E|X|^{q}I\big(b_n s^{1/\alpha}< a_{ni}|X|\le b_n (s+1)^{1/\alpha}\big)\sum_{m=s}^{\infty} m^{-q/\alpha} \\
&\le Cb_n^{-q}\sum_{s=1}^{\infty}s^{1-q/\alpha}a_{ni}^{q}E|X|^{q}I\big(b_n s^{1/\alpha}< a_{ni}|X|\le b_n (s+1)^{1/\alpha}\big) \\
&\le Cb_n^{-\alpha}\sum_{s=1}^{\infty}a_{ni}^{\alpha}E|X|^{\alpha}I\big(b_n s^{1/\alpha}< a_{ni}|X|\le b_n (s+1)^{1/\alpha}\big) \\
&\le Cb_n^{-\alpha}a_{ni}^{\alpha}E|X|^{\alpha}I\big(a_{ni}|X|> b_n\big).
\end{aligned}$$

Hence by Lemma 2.4, we have

$$I_8 \le C\sum_{n=2}^{\infty} n^{-1}b_n^{-\alpha}\sum_{i=1}^{n}a_{ni}^{\alpha}E|X|^{\alpha}I\big(a_{ni}|X|> b_n\big) < \infty.$$

The proof is completed. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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