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Wittmann Type Strong Laws of Large Numbers for Blockwise m-Negatively Associated Random Variables

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Abstract In this paper, Wittmann type strong laws of large numbers for blockwise m-negatively associated random variables are established which extend and improve the related known works in the literature.

Keywords strong laws of large numbers; blockwise m-negatively associated random variables

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1. Introduction

Let $\{\Omega, \Im, P\}$ be a probability space. In the following, all random variables are assumed to be defined on $\{\Omega, \Im, P\}$.

Definition 1.1 *A finite family $\{X_1, X_2, \ldots, X_n\}$ of random variables is said to be negatively associated (NA) if for any disjoint subsets $A, B \subset \{1, 2, \ldots, n\}$ and any real coordinatewise nondecreasing functions $f : \mathbb{R}^a \to \mathbb{R}$ and $g : \mathbb{R}^b \to \mathbb{R}$, where $a = \sharp(A)$ and $b = \sharp(B)$,*

$$\mathrm{Cov}(f(X_i, i \in A), g(X_j, j \in B)) \le 0,$$

whenever the covariance exists. An infinite family of random variables is said to be NA if every its finite subfamilies are NA.

We refer to the paper by Joag-Dev and Proschan [1] where many examples and properties of NA random variables are presented.

Definition 1.2 *Let m be a positive integer. A sequence of random variables $\{X_n, n \ge 1\}$ is said to be m-negatively associated (m-NA) if for any finite subset of index $A = \{i_1, i_2, \ldots, i_n\} \subset N = \{1, 2, 3, \ldots\}$, where $n \ge 2$, such that $|i_k - i_j| \ge m$ for all $1 \le k \ne j \le n$, we have that $\{X_{i_1}, \ldots, X_{i_n}\}$ is NA.*

The concept of m-NA random variables was introduced by Hu et al. [2] where the complete convergence for arrays of rowwise m-NA random variables is studied.

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Definition 1.3 *Let $\{n_i, i \geq 1\}$ be an arbitrary strictly increasing sequence of natural numbers. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be blockwise negatively associated (blockwise NA) with respect to $\{n_i, i \geq 1\}$ if the sequence $\{X_n, n_{i-1} \leq n < n_i\}$ of finite families of random variables is NA for each i.*

The concept of blockwise NA random variables was introduced by Nezakati [3] where Marcinkiewicz strong law and classical strong law of large numbers for a sequence of blockwise NA are obtained. The blockwise structure has been considered for several class of random variables in the literature, for blockwise independent and blockwise orthogonal random variables in Gaposhkin [4], for double arrays of blockwise m-dependent random variables in Stadtmüller and Thanh [5], for blockwise martingale difference sequences in Rosalsky and Thanh [6], for blockwise and pairwise m-dependent random variables in Le and Vu [7], and so on.

The following notion seems to be new.

Definition 1.4 *Let m be a positive integer, $\{n_i, i \geq 1\}$ be an arbitrary strictly increasing sequence of natural numbers. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be blockwise m-negatively associated (blockwise m-NA)with respect to $\{n_i, i \geq 1\}$ if the sequence $\{X_n, n_{i-1} \leq n < n_i\}$ is m-NA for each i.*

The concept of blockwise m-NA is a natural extension from m-NA and blockwise NA.

Wittmann Strong law Wittmann [8] proved the following strong law for a sequence of independent random variables $\{X_n, n \geq 1\}$.

Let $p \geq 1$ and $\{a_n, n \geq 1\}$ be an increasing sequence of strictly positive real numbers such that

(i) $\sum_{n=1}^{\infty} a_n^{-2p} E(|X_n|^{2p}) < \infty$,

(ii) $\sum_{n=1}^{\infty} a_n^{-2}(a_n^2 - a_{n-1}^2)^{1-p}(E(X_n^2))^p < \infty$.

Then we have

$$\lim_{n\to\infty} a_n^{-1} \sum_{i=1}^{n} (X_i - EX_i) = 0 \text{ a.s.}$$

In this paper, we obtain Wittmann type strong law of large numbers for a sequence of blockwise m-NA random variables which extend and improve the related results from Nezakati [3] for blockwise NA random variables, from Wittmann [8] for independent random variables, and from Liu and Wu [9] for NA random variables.

In the following section we present the main results and an important lemma necessary to prove our results.

Throughout this paper, C will represent positive constant which may change from one place to another, $I(A)$ will represent the indicator function of a set A and $\{n_i, i \geq 1\}$ will be an arbitrary strictly increasing sequence of natural numbers.

2. Main results

For any $\{n_i, i \geq 1\}$ as above, we introduce the following notations:

$$I_k = \{i : [2^k, 2^{k+1}) \cap [n_i, n_{i+1}) \neq \emptyset\}, \quad \nu_k = \sharp(I_k),$$
$$[l_{ki}, r_{ki}) = [2^k, 2^{k+1}) \cap [n_i, n_{i+1}) \ \text{ if } \ i \in I_k,$$
$$\psi(n) = \max_{0 \leq j \leq k} \nu_j \ \text{ if } \ n \in [2^k, 2^{k+1}).$$

It is easy to see, for any $\{n_i, i \geq 1\}$, that $\psi(n) = O(n)$ and that $\psi(n) = O(1)$ if

$$\liminf_i n_{i+1}/n_i > 1.$$

In order to prove our main results, we present the following lemma.

Lemma 2.1 *Let $p > 1, \{X_n, n \geq 1\}$ be a sequence of blockwise m-NA random variables with respect to $\{n_i, i \geq 1\}$ with $E|X_n|^p < \infty$ for all $n \geq 1$ and $EX_n = 0$ for all $n \geq 1$. Then, for every $\varepsilon > 0$ and $k \geq 1$,*

(i) If $1 < p \leq 2$,

$$P\Big\{\max_{n_k \leq n < n_{k+1}} \Big|\sum_{i=n_k}^{n} X_i\Big| > \varepsilon\Big\} \leq \varepsilon^{-p} 2^{3-p} m^{p-1} \sum_{i=n_k}^{n_{k+1}-1} E|X_i|^p.$$

(ii) If $p > 2$,

$$P\Big\{\max_{n_k \leq n < n_{k+1}} \Big|\sum_{i=n_k}^{n} X_i\Big| > \varepsilon\Big\} \leq 2\varepsilon^{-p} (\frac{15p}{\ln p})^p m^{p-1} \Big\{\sum_{i=n_k}^{n_{k+1}-1} E|X_i|^p + \Big(\sum_{i=n_k}^{n_{k+1}-1} EX_i^2\Big)^{p/2}\Big\}.$$

(iii) If $1 < p \leq 2$,

$$P\Big\{\max_{2^k \leq n < 2^{k+1}} \Big|\sum_{i=2^k}^{n} X_i\Big| > \varepsilon\Big\} \leq \varepsilon^{-p} 2^{3-p} (m v_k)^{p-1} \sum_{i=2^k}^{2^{k+1}-1} E|X_i|^p.$$

(iv) If $p > 2$,

$$P\Big\{\max_{2^k \leq n < 2^{k+1}} \Big|\sum_{i=2^k}^{n} X_i\Big| > \varepsilon\Big\} \leq 2\varepsilon^{-p} (\frac{15p}{\ln p})^p (m v_k)^{p-1} \Big\{\sum_{i=2^k}^{2^{k+1}-1} E|X_i|^p + \Big(\sum_{i=2^k}^{2^{k+1}-1} EX_i^2\Big)^{p/2}\Big\}.$$

Proof We prove (iii) and (iv). The proofs of (i) and (ii) are similar to those of (iii) and (iv), respectively. Therefore, the proofs of (i) and (ii) are omitted. By Markov's inequality, C_r-inequality, Theorem 2 of Shao [10], we have

$$P\Big\{\max_{2^k \leq n < 2^{k+1}} \Big|\sum_{i=2^k}^{n} X_i\Big| > \varepsilon\Big\} \leq \varepsilon^{-p} E\Big(\max_{2^k \leq n < 2^{k+1}} \Big|\sum_{i=2^k}^{n} X_i\Big|\Big)^p$$
$$\leq \varepsilon^{-p} E\Big(\sum_{j \in I_k} \max_{l_{kj} \leq n < r_{kj}} \Big|\sum_{i=l_{kj}}^{n} X_i\Big|\Big)^p$$
$$\leq \varepsilon^{-p} v_k^{p-1} \sum_{j \in I_k} E\Big(\max_{l_{kj} \leq n < r_{kj}} \Big|\sum_{i=l_{kj}}^{n} X_i\Big|\Big)^p$$
$$\leq \varepsilon^{-p} v_k^{p-1} \sum_{j \in I_k} E\Big(\sum_{l=0}^{m-1} \max_{l_{kj} \leq n < r_{kj}} \Big|\sum_{\substack{l_{kj} \leq i \leq n \\ i \bmod m = l}} X_i\Big|\Big)^p$$

$$\le \varepsilon^{-p}(mv_k)^{p-1}\sum_{j\in I_k}\sum_{l=0}^{m-1}E\Big(\max_{l_{kj}\le n<r_{kj}}\Big|\sum_{\substack{l_{kj}\le i\le n\\ i \bmod m=l}}X_i\Big|\Big)^p$$

$$\le\begin{cases}\varepsilon^{-p}(mv_k)^{p-1}\sum_{j\in I_k}\sum_{l=0}^{m-1}\Big(2^{3-p}\sum_{\substack{l_{kj}\le i\le n<r_{kj}\\ i \bmod m=l}}E|X_i|^p\Big),\ 1<p\le 2\\ \varepsilon^{-p}(mv_k)^{p-1}\sum_{j\in I_k}\sum_{l=0}^{m-1}2(\frac{15p}{\ln p})^p\Big(\sum_{\substack{l_{kj}\le i\le n<r_{kj}\\ i \bmod m=l}}E|X_i|^p+\Big(\sum_{\substack{l_{kj}\le i\le n<r_{kj}\\ i \bmod m=l}}EX_i^2\Big)^{\frac{p}{2}}\Big),\ p>2\end{cases}$$

$$\le\begin{cases}\varepsilon^{-p}2^{3-p}(mv_k)^{p-1}\sum_{i=2^k}^{2^{k+1}-1}E|X_i|^p, 1<p\le 2,\\ 2\varepsilon^{-p}(\frac{15p}{\ln p})^p(mv_k)^{p-1}\Big\{\sum_{i=2^k}^{2^{k+1}-1}E|X_i|^p+\Big(\sum_{i=2^k}^{2^{k+1}-1}EX_i^2\Big)^{p/2}\Big\},\ p>2.\ \square\end{cases}$$

Theorem 2.2 *Let $\{X_n, n\ge 1\}$ be a sequence of blockwise m-NA random variables with respect to $\{n_i, i\ge 1\}$ and $EX_n=0$ for all $n\ge 1$, and let $\{b_n, n\ge 1\}$ be a positive strictly increasing sequence such that*

$$\lim_{n\to\infty}b_n=\infty \ \ \text{and}\ \ 0<a=\inf_{i\ge 1}b_{n_i}b_{n_{i+1}}^{-1}\le\sup_{i\ge 1}b_{n_i}b_{n_{i+1}}^{-1}=c<1. \tag{2.1}$$

If there exist constants p, q such that $0<p\le 2, pq>2, E|X_n|^{pq}<\infty$ and

$$\sum_{n=1}^{\infty}b_n^{-pq}E|X_n|^{pq}<\infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty}b_n^{-p}(b_n^p-b_{n-1}^p)^{-(q-1)}(E|X_n|^2)^{pq/2}<\infty, \tag{2.3}$$

then

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{k=1}^{n}X_k=0 \ \ a.s. \tag{2.4}$$

Proof For any positive integer $n\ge n_2$, there exists a positive integer $k\ge 3$ such that $n_{k-1}\le n<n_k$. Note that $b_n\ge b_{n_{k-1}}\ge ab_{n_k}>0$ by (2.1) and $k\to\infty$ as $n\to\infty$, we have

$$\Big|\frac{1}{b_n}\sum_{j=1}^{n}X_j\Big|\le\Big|\frac{1}{b_n}\sum_{j=1}^{n_1-1}X_j\Big|+\Big|\sum_{j=2}^{k-1}\frac{b_{n_j}}{b_{n_{k-1}}}\Big(\frac{1}{b_{n_j}}\sum_{i=n_{j-1}}^{n_j-1}X_i\Big)\Big|+\frac{1}{ab_{n_k}}\max_{n_{k-1}\le n<n_k}\Big|\sum_{j=n_{k-1}}^{n}X_j\Big|. \tag{2.5}$$

Obviously the first term on the right-hand side of (2.5) converges to zero a.s.. Utilizing (2.1) and similar method in the proof of Theorem 1 of Nezakati [3], we have

$$\sup_{k\ge 3}\sum_{j=2}^{k-1}\frac{b_{n_j}}{b_{n_{k-1}}}<\infty \ \text{ and }\ \lim_{n\to\infty}\frac{b_{n_j}}{b_{n_{k-1}}}=0,\ \text{ for every } j.$$

Then by Toeplitz Lemma we have that the second term on the right-hand side of (2.5) almost sure converges to zero if we prove that

$$\lim_{n\to\infty}\frac{1}{b_{n_k}}\sum_{j=n_{k-1}}^{n_k-1}X_j=0 \ \text{ a.s.}$$

Therefore, in order to prove (2.4), we only need to prove that

$$\lim_{n\to\infty}\frac{1}{b_{n_k}}\max_{n_{k-1}\le n<n_k}\Big|\sum_{j=n_{k-1}}^{n}X_j\Big|=0 \ \text{ a.s.} \tag{2.6}$$

From Lemma 2.1 (ii), we have for any $\varepsilon>0$

$$\begin{aligned}&\sum_{k=2}^{\infty}P\Big(\frac{1}{b_{n_k}}\max_{n_{k-1}\le n<n_k}\Big|\sum_{j=n_{k-1}}^{n}X_j\Big|>\varepsilon\Big)\\&\quad\le\sum_{k=2}^{\infty}Cb_{n_k}^{-pq}\Big\{\sum_{j=n_{k-1}}^{n_k-1}E|X_j|^{pq}+\Big(\sum_{j=n_{k-1}}^{n_k-1}E|X_j|^2\Big)^{pq/2}\Big\}.\end{aligned} \tag{2.7}$$

Let $\tau_i=\inf\{k\in N, i\le n_k\}$. By (2.1) and (2.2) we have,

$$\begin{aligned}&\sum_{k=2}^{\infty}b_{n_k}^{-pq}\sum_{j=n_{k-1}}^{n_k-1}E|X_j|^{pq}\le\sum_{k=1}^{\infty}b_{n_k}^{-pq}\sum_{j=1}^{n_k}E|X_j|^{pq}\\&\quad=\sum_{j=1}^{\infty}E|X_j|^{pq}\sum_{k=\tau_j}^{\infty}b_{n_k}^{-pq}\le\sum_{j=1}^{\infty}E|X_j|^{pq}\sum_{k=\tau_j}^{\infty}b_{n_{\tau_j}}^{-pq}c^{pq(k-\tau_j)}\\&\quad\le C\sum_{j=1}^{\infty}b_j^{-pq}E|X_j|^{pq}<\infty.\end{aligned} \tag{2.8}$$

Since $0<p\le 2$, $pq>2$, we deduce that $0<p/2\le 1$ and $q>1$. Set $b_0=0$. By C_r-inequality and Hölder inequality and (2.3), we have

$$\begin{aligned}&\sum_{k=2}^{\infty}b_{n_k}^{-pq}\Big(\sum_{j=n_{k-1}}^{n_k-1}E|X_j|^2\Big)^{pq/2}\le\sum_{k=1}^{\infty}b_{n_k}^{-pq}\Big(\sum_{j=1}^{n_k}E|X_j|^2\Big)^{pq/2}\\&\quad=\sum_{k=1}^{\infty}b_{n_k}^{-pq}\Big\{\sum_{j=1}^{n_k}(b_j^p-b_{j-1}^p)^{(q-1)/q}(b_j^p-b_{j-1}^p)^{-(q-1)/q}(E|X_j|^2)^{p/2}\Big\}^q\\&\quad\le\sum_{k=1}^{\infty}b_{n_k}^{-pq}\Big(\sum_{j=1}^{n_k}(b_j^p-b_{j-1}^p)^{-(q-1)}(EX_j^2)^{pq/2}\Big)\Big(\sum_{j=1}^{n_k}(b_j^p-b_{j-1}^p)\Big)^{(q-1)}\\&\quad=\sum_{k=1}^{\infty}b_{n_k}^{-p}\Big(\sum_{j=1}^{n_k}(b_j^p-b_{j-1}^p)^{-(q-1)}(EX_j^2)^{pq/2}\Big)\\&\quad\le\sum_{j=1}^{\infty}\Big((b_j^p-b_{j-1}^p)^{-(q-1)}(EX_j^2)^{pq/2}\sum_{k=\tau_j}^{\infty}b_{n_k}^{-p}\Big)\\&\quad\le C\sum_{j=1}^{\infty}b_j^{-p}(b_j^p-b_{j-1}^p)^{-(q-1)}(EX_j^2)^{pq/2}<\infty.\end{aligned} \tag{2.9}$$

Thus, (2.6) holds by virtue of Borel-Cantelli lemma and (2.7)~(2.9). □

When $1<pq\le 2$, we have the following theorem.

Theorem 2.3 *Let $\{X_n, n \ge 1\}$ be a sequence of blockwise m-NA random variables with respect to $\{n_i, i \ge 1\}$ and $EX_n = 0$ for all $n \ge 1$, and let $\{b_n, n \ge 1\}$ be a positive strictly increasing sequence satisfying (2.1). If there exists a constant p such that $1 < p \le 2, E|X_n|^p < \infty$, and*

$$\sum_{n=1}^{\infty} b_n^{-p} E|X_n|^p < \infty, \tag{2.10}$$

then (2.4) holds.

Proof In the same way as used in Theorem 2.2, we only need to prove that (2.6) holds. Utilizing Lemma 2.1(i), (2.10) and similar method in the proof of (2.8), we have for any $\varepsilon > 0$ that

$$\sum_{k=2}^{\infty} P\Big(\frac{1}{b_{n_k}} \max_{n_{k-1} \le n < n_k} \Big| \sum_{j=n_{k-1}}^{n} X_j \Big| > \varepsilon\Big) \le \sum_{k=2}^{\infty} C b_{n_k}^{-p} \sum_{j=n_{k-1}}^{n_k - 1} E|X_j|^p < \infty.$$

By virtue of Borel-Cantelli lemma, (2.6) holds. □

Corollary 2.4 *Let $\{X_n, n \ge 1\}$ be a sequence of blockwise m-NA random variables with respect to $\{n_i, i \ge 1\}$ and $EX_n = 0$ for all $n \ge 1$, and let $\{b_n, n \ge 1\}$ be a positive strictly increasing sequence satisfying (2.1). If there exist constants p, q such that $0 < p \le 2, q \ge 1, E|X_n|^{pq} < \infty$ for all $n \ge 1$, and*

$$\sum_{n=1}^{\infty} b_n^{-p} (b_n^p - b_{n-1}^p)^{-(q-1)} E|X_n|^{pq} < \infty, \tag{2.11}$$

then (2.4) holds.

Proof For $0 < p \le 2, q \ge 1$, we have

$$b_n^{-pq} \le b_n^{-p} (b_n^p - b_{n-1}^p)^{-(q-1)}. \tag{2.12}$$

If $pq > 2$, note that $(EX_n^2)^{pq/2} \le E|X_n|^{pq}$. By (2.11) and (2.12), (2.2) and (2.3) hold, therefore, (2.4) holds from Theorem 2.2. If $0 < pq \le 2$, by (2.11) and (2.12), (2.10) holds, therefore, (2.4) holds from Theorem 2.3. □

If we take $b_n = n^\alpha$ with $\alpha > 0$. By Theorem 2.1, Corollary 2.4, and

$$\lim_{n \to \infty} \{n^{\alpha p} - (n-1)^{\alpha p}\}^{-1} n^{\alpha p - 1} = 1/(\alpha p),$$

we have the following two corollaries.

Corollary 2.5 *Let $\{X_n, n \ge 1\}$ be a sequence of blockwise m-NA random variables with respect to $\{n_i, i \ge 1\}$ with $EX_n = 0$ for all $n \ge 1$ and α a real number with $\alpha > 0$. Suppose that*

$$0 < a = \inf_{i \ge 1} n_i^\alpha n_{i+1}^{-\alpha} \le \sup_{i \ge 1} n_i^\alpha n_{i+1}^{-\alpha} = c < 1. \tag{2.13}$$

If there exist some p, q such that $0 < p \le 2, pq > 2, E|X_n|^{pq} < \infty$,

$$\sum_{n=1}^{\infty} n^{-\alpha pq} E|X_n|^{pq} < \infty$$

and

$$\sum_{n=1}^{\infty} n^{-\{1+q(\alpha p-1)\}} \left(E|X_n|^2\right)^{pq/2} < \infty,$$

then

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} \sum_{k=1}^{n} X_k = 0 \ \text{ a.s.} \tag{2.14}$$

Corollary 2.6 *Let $\{X_n, n \geq 1\}$ be a sequence of blockwise m-NA random variables with respect to $\{n_i, i \geq 1\}$ and $EX_n = 0$ for all $n \geq 1$ and α a real number with $\alpha > 0$.*

Suppose that (2.13) holds and there exist some p, q, with $0 < p \leq 2, q \geq 1$, such that $E|X_n|^{pq} < \infty$ and

$$\sum_{n=1}^{\infty} n^{-\{1+q(\alpha p-1)\}} E|X_n|^{pq} < \infty,$$

then (2.14) holds.

By using (iv) of Lemma 2.1 and the same way as used in Theorem 2.2, we can prove the following two theorems.

Theorem 2.7 *Let $\{X_n, n \geq 1\}$ be a sequence of blockwise m-NA random variables with respect to $\{n_i, i \geq 1\}$ and $EX_n = 0$ for all $n \geq 1$, and let $\{b_n, n \geq 1\}$ be a positive strictly increasing sequence such that*

$$\lim_{n\to\infty} b_n = \infty \ \text{ and } \ 0 < a = \inf_{i\geq 1} b_{2^i} b_{2^{i+1}}^{-1} \leq \sup_{i\geq 1} b_{2^i} b_{2^{i+1}}^{-1} = c < 1. \tag{2.15}$$

Suppose that, for some p, q such that $0 < p \leq 2, pq > 2, E|X_n|^{pq} < \infty$, and

$$\sum_{n=1}^{\infty} b_n^{-pq} E|X_n|^{pq} < \infty,$$

$$\sum_{n=1}^{\infty} b_n^{-p} \left(b_n^p - b_{n-1}^p\right)^{-(q-1)} \left(E|X_n|^2\right)^{pq/2} < \infty.$$

Then

$$\lim_{n\to\infty} \frac{1}{b_n (\psi(n))^{(pq-1)/(pq)}} \sum_{k=1}^{n} X_k = 0 \ \text{ a.s.}$$

Theorem 2.8 *Let $\{X_n, n \geq 1\}$ be a sequence of blockwise m-NA random variables with respect to $\{n_i, i \geq 1\}$ and $EX_n = 0$ for all $n \geq 1$, and let $\{b_n, n \geq 1\}$ be a positive strictly increasing sequence satisfying (2.15).*

Suppose that $E|X_n|^p < \infty$ for some p $(1 < p \leq 2)$, and

$$\sum_{n=1}^{\infty} b_n^{-p} E|X_n|^p < \infty.$$

Then

$$\lim_{n\to\infty} \frac{1}{b_n (\psi(n))^{(p-1)/p}} \sum_{k=1}^{n} X_k = 0 \ \text{ a.s.}$$

Corollary 2.9 *Let $\{X_n, n \geq 1\}$ be a sequence of blockwise m-NA random variables with respect to $\{n_i, i \geq 1\}$ such that $n_{k+1}/n_k > 1$ and $EX_n = 0$ for all $n \geq 1$.*

Suppose that, for some $p, 1 < p \leq 2$, such that $E|X_n|^p = O(1), \forall\, n \geq 1$. Then, for all $\delta > 1$,

$$\lim_{n\to\infty} \frac{1}{(n \log n (\log\log n)^{\delta})^{1/p}} \sum_{k=1}^{n} X_k = 0 \quad a.s.$$

Remark 2.10 (i) Theorems 2.2 and 2.3 extend and improve the related known works in Wittmann [8] for independent random variables.

(ii) Theorems 2.2 and 2.3, Corollaries 2.4–2.6 extend the related known works in Liu and Wu [9] for NA random variables.

(iii) Let $p = 2, m = 1$, from Theorems 2.3, 2.7 and Corollary 2.9, we get Theorems 1, 2 and Corollary 1 in Nezakati [3], respectively.

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