Том 65 **ТЕОРИЯ ВЕРОЯТНОСТЕЙ И ЕЕ ПРИМЕНЕНИЯ 2020** **Выпуск 3**

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ON THE SUB-GAUSSIANITY OF THE r-CORRELOGRAMS[1)]

Устанавливается, что центрированная реле-коррелограмма (relay correlogram function) является субгауссовской случайной величиной. Доказательство осуществляется посредством анализа преобразования Лапласа и оценки субгауссовского штандарта r-коррелограмм.

Ключевые слова и фразы: реле-коррелограмма, субгауссовская случайная величина, субгауссовский штандарт, преобразование Лапласа, стационарные гауссовские процессы.

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1. Introduction. The topic of this paper concerns the classical problem of establishing properties of estimators for the correlation function of a stationary Gaussian process. Stationary Gaussian processes are the main object and bases for a large variety of probability and statistical models (see, for example, [1] and [2]). It is a well-known fact that a stationary Gaussian process is characterized by its autocorrelation function (see definitions in what follows). This is the reason why an estimation of the autocorrelation function from the observations of some realisation of the stationary Gaussian process plays a crucial role in the construction of an appropriate model for data.

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To be more precise, consider a real-valued standard stationary Gaussian process $Z = \{Z(t),\, t \geqslant 0\}$ such that $\mathbf{E}Z(t) = 0$ and $\operatorname{Var} Z(t) = 1$ for all $t \geqslant 0$. This process is characterized by its autocorrelation function

$$\rho(h) = \frac{\mathbf{E}[(Z(t) - \mathbf{E}Z(t))(Z(t+h) - \mathbf{E}Z(t+h))]}{\sqrt{\operatorname{Var} Z(t) \operatorname{Var} Z(t+h)}}, \qquad h \geqslant 0, \quad t \geqslant 0.$$

Note that in our case $\rho(h) = \mathbf{E}Z(t)Z(t+h)$, $h \geqslant 0$, $t \geqslant 0$.

Note that in the time series analysis, a discrete time $i \geqslant 0$ is considered (see, for example, [2]) and the following estimator of $\rho(h)$ is usually applied:

$$\widehat{\rho}(h) = n^{-1} \sum_{i=0}^{n-h} Z_i Z_{i+h}, \qquad h = 0, 1, 2, \dots . \tag{1.1}$$

In [3] a continuous time estimator of $\rho(h)$ is introduced. More precisely, let $T > 0$ be the time horizon, then the estimator studied in [3] is the following autocorrelation process $R = \{R(h),\, h \in [0, T]\}$:

$$R(h) = \frac{1}{T} \int_0^T Z_s Z_{s+h}\, ds. \tag{1.2}$$

We refer to [3] for properties of the autocorrelation process and some sharp exponential bounds for its deviation probabilities.

In this paper we do not consider an autocorrelation function. Instead, we consider the relay correlation function (defined in what follows), which is a modification of the autocorrelation function. It has been introduced mainly for computational advantages. The relay correlation function is mostly used in engineering application (see, for example, [4]).

Now we define the relay correlogram function. This is a process similar in some sense to the autocorrelation process (1.2).

Definition 1.1. The *relay correlation function* of a standard stationary Gaussian process $\{Z(t),\, t \geqslant 0\}$ is the function

$$\rho^r(h) = \mathbf{E}[Z(0) \operatorname{sign} Z(h)],$$

where $\operatorname{sign} Z(h)$ is defined as

$$\operatorname{sign} Z(h) = \mathbf{1}_{\{Z(h) > 0\}} - \mathbf{1}_{\{Z(h) \leqslant 0\}}$$

and $\mathbf{1}_A$ denotes the indicator function of the event A.

Clearly, $\rho^r(h) = \mathbf{E}[Z(s) \operatorname{sign} Z(s+h)]$ for any $s \geqslant 0$, since the process Z is stationary.

Notice that $\rho^r(h)$ takes in account only the sign, not the whole value of the second random variable. This is the main advantage of the relay correlation function over usual autocorrelation function. For instance, consider a statistical problem of plug-in estimation of the autocorrelation function based on (1.1) having a dataset of the process realizations $\{z_i,\ 1 \leqslant i \leqslant n\}$ in hand. Obviously, the calculation of $\sum_i^{n-h} z_i z_{i+h}$ involves a product for each term of the sum and the sum itself. But the calculation for the relay correlation function, that is, $\sum_i^{n-h} z_i \operatorname{sign} z_{i+h}$ involves only a control of the sign for each term and the sum. From the computational point of view a product is more expensive than a sign control, therefore the second calculation is cheaper than the first one.

Also, we do not loose any information if we consider the relay correlation function over the autocorrelation function. Namely, the following identity is true:

$$\begin{aligned}\rho^r(h) &= \mathbf{E}[Z(0) \operatorname{sign} Z(h)] = \mathbf{E}\big[\operatorname{sign} Z(h)\mathbf{E}[Z(0) \mid Z(h)]\big] \\ &= \mathbf{E}[\rho(h)Z(h) \operatorname{sign} Z(h)] = \rho(h)\mathbf{E}|Z(h)| = \sqrt{\frac{2}{\pi}}\rho(h).\end{aligned}$$

Here we used the fact that the random variables $Z(0)$ and $Z(h)$ are standard normal variables and jointly normally distributed, hence $\mathbf{E}[Z(0) \mid Z(h)] = \rho(h)Z(h)$.

In order to estimate the relay correlation function in a similar way as (1.2) estimates the autocorrelation function, we introduce the following definition.

Definition 1.2. The process $\{\widehat{R}^r(h),\ h \geqslant 0\}$ is called an *r-correlogram process* (or *a sample relay correlation process*) of $(Z(t))$ if

$$\widehat{R}^r(h) = \frac{1}{T}\int_0^T Z(t) \operatorname{sign} Z(t+h)\, dt, \qquad h > 0, \tag{1.3}$$

and the integral is interpreted as a mean square Riemann integral.

Since for every h we have $\mathbf{E}\widehat{R}^r(h) = \rho^r(h)$, the r-correlogram process is an unbiased estimator for the relay correlation function.

In the following we consider the centered r-correlogram process

$$\widehat{U}^r(h) = \widehat{R}^r(h) - \mathbf{E}\widehat{R}^r(h) = \widehat{R}^r(h) - \rho^r(h), \qquad h \geqslant 0.$$

The main result of this paper shows that the centered relay correlation process is a sub-Gaussian random process.

Now we review the notion of a sub-gaussianity. Up to our knowledge, the notion of sub-Gaussian random variable was introduced by Kahane [5] as *sous-gaussienne variables aléatoires*. A detailed discussion and proofs of all results we present in what follows may be found in [3, Section 1.1]. We collect only the most important definitions and results here.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space on which all random variables we consider are defined. A random variable X is called *sub-Gaussian* if its moment generating function (Laplace transform) $\mathbf{E}e^{tX}$ is defined for all t and there exists a positive constant a such that for every t

$$\mathbf{E}e^{tX} \leqslant e^{a^2t^2/2}.$$

The smallest a that satisfies the previous inequality is called the *sub-Gaussian standard* of the random variable X and is denoted by $\tau(X)$, that is,

$$\tau(X) = \inf\{a > 0\colon \mathbf{E}e^{tX} \leqslant e^{a^2t^2/2} \text{ for all } t\}.$$

The space of sub-Gaussian random variables is denoted by $\mathrm{Sub}(\Omega)$. It can be proved that $\mathrm{Sub}(\Omega)$ is a Banach space with norm $\tau(\,\cdot\,)$. Moreover, if $\{X_n,\, n \geqslant 1\}$ is a sequence of random variables in $\mathrm{Sub}(\Omega)$, which converges in probability to a random variable X and such that $\sup_{n\geqslant 1}\tau(X_n) < \infty$, then X is a sub-Gaussian random variable and $\tau(X) \leqslant \sup_{n\geqslant 1}\tau(X_n)$.

Sub-Gaussian random variables are interesting because they possess the following exponential upper bound for the tail of the distribution. If X is a sub-Gaussian random variable, then

$$\mathbf{P}(|X| \geqslant x) \leqslant 2e^{-x^2/(2\tau^2(X))}. \tag{1.4}$$

The problem of finding sub-Gaussian standard for an r-correlogram process was first considered in PhD thesis (unpublished, but available online) by A. Castellucci [6], a former PhD student of Dr. R. Giuliano, the co-author of this paper. In [6] the following result has been proved.

Theorem 1.1. *Let $\{\widehat{U}^r(h),\, h \geqslant 0\}$ be a centered r-correlogram process. For any $h \geqslant 0$ the random variable $\widehat{U}^r(h)$ is sub-Gaussian with the sub-Gaussian standard*

$$\tau(\widehat{U}^r(h)) \leqslant \sqrt[4]{3.1}\left(3 + \frac{\rho^2(h)}{\pi}\right). \tag{1.5}$$

The calculations of the sub-Gaussian standard for the centered r-correlogram process $\widehat{U}^r(h)$ presented in [6] are quite cumbersome and not straightforward. In this paper we were able to obtain much more precise estimation of the sub-Gaussian standard using a much simpler technique.

2. Main result. In order to formulate our main results, we introduce the following notation. Let X and Y be two standard normal random variables with correlation coefficient ρ. That is, $X \sim \mathbf{N}(0,1)$, $Y \sim \mathbf{N}(0,1)$, $\mathrm{cor}(X,Y) = \rho$. Further, we consider the random variables $\widetilde{U} = X\mathbf{1}_{\{Y>0\}}$ and $U = \widetilde{U} - \mathbf{E}\widetilde{U}$. It appears that the main “technical” question we need to solve in order to estimate the sub-Gaussian standard of the centered r-correlogram process $\widehat{U}^r(h)$ is in the finding of the sub-Gaussian standard τ of the random variable U. Namely, the following result can be presented.

Proposition 2.1. *Let $\{\widehat{U}^r(h),\, h \geqslant 0\}$ be a centered r-correlogram process. For any $h \geqslant 0$, the random variable $\widehat{U}^r(h)$ is sub-Gaussian with the sub-Gaussian standard*

$$\tau(\widehat{U}^r(h)) \leqslant 2\tau(U). \tag{2.1}$$

Proof. Fix any $h > 0$. We consider the left Riemann sums of the integral in (1.3).

Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of the interval $[0, T]$ and $\Delta t_k = t_k - t_{k-1}$, $1 \leqslant k \leqslant n$. Further, let

$$S = \sum_{k=1}^{n} Z(t_k) \operatorname{sign} Z(t_k + h) \Delta t_k$$

be a left Reimann sum of the integral in (1.3), and consider the centered variable

$$S - \mathbf{E}S = \sum_{k=1}^{n} \{Z(t_k) \operatorname{sign} Z(t_k + h) - \mathbf{E}[Z(t_k) \operatorname{sign} Z(t_k + h)]\} \Delta t_k.$$

The random variable $Z(t_k) \operatorname{sign} Z(t_k + h)$ can be written in the form

$$Z(t_k)(\mathbf{1}_{\{Z(t_k+h)>0\}} - \mathbf{1}_{\{Z(t_k+h)<0\}}),$$

and hence

$$\begin{aligned} &Z(t_k) \operatorname{sign} Z(t_k + h) - \mathbf{E}[Z(t_k) \operatorname{sign} Z(t_k + h)] \\ &\quad = \big(Z(t_k)\mathbf{1}_{\{Z(t_k+h)>0\}} - \mathbf{E}[Z(t_k)\mathbf{1}_{\{Z(t_k+h)>0\}}]\big) \\ &\qquad - \big(Z(t_k)\mathbf{1}_{\{Z(t_k+h)\leqslant 0\}} - \mathbf{E}[Z(t_k)\mathbf{1}_{\{Z(t_k+h)\leqslant 0\}}]\big). \end{aligned}$$

Consider two sequences of random variables

$$A_k^+(h) = Z(t_k)\mathbf{1}_{\{Z(t_k+h)>0\}} - \mathbf{E}[Z(t_k)\mathbf{1}_{\{Z(t_k+h)>0\}}]$$

and

$$A_k^-(h) = -Z(t_k)\mathbf{1}_{\{Z(t_k+h)<0\}} - \mathbf{E}[Z(t_k)\mathbf{1}_{\{Z(t_k+h)<0\}}].$$

Then

$$S = \sum_{k=1}^{n} \big(A_k^+(h) + A_k^-(h)\big) \Delta t_k.$$

The underlying process $\{Z(t),\, t \geqslant 0\}$ is stationary and hence the random variables $A_k^+(h)$, $k \geqslant 1$, have the same distribution as $U = Z(0)\mathbf{1}_{\{Z(h)>0\}} - \mathbf{E}[Z(0)\mathbf{1}_{\{Z(h)>0\}}]$. Similarly, the random variables $A_k^-(h)$, $k \geqslant 1$, have the same distribution as $V = -(Z(0)\mathbf{1}_{\{Z(h)<0\}} - \mathbf{E}[Z(0)\mathbf{1}_{\{Z(h)>0\}}])$.

Moreover, using symmetry arguments we prove in the next section, Proposition 3.1, that the random variables U and V are identically distributed.

In Proposition 4.1 in what follows, we shall prove that the random variable U is sub-Gaussian. Hence, since $\sum_{k=1}^n \Delta t_k = T$ and τ is a norm, we have

$$\tau\left(\frac{1}{T}(S - \mathbf{E}S)\right) = \frac{1}{T}\tau\left(\sum_{k=1}^n \big(A_k^+(h) + A_k^-(h)\big)\Delta t_k\right)$$
$$\leqslant \frac{1}{T}\sum_{k=1}^n \Delta t_k\big(\tau(A_k^+(h)) + \tau(A_k^-(h))\big) = \frac{1}{T}\sum_{k=1}^n \Delta t_k 2\tau(U) = 2\tau(U).$$

Now as a Reimann integral, $\widehat{U}^r(h) = \lim_{\max \Delta t_k \to 0}(1/T)(S - \mathbf{E}S)$ in mean squares and hence in probability. Since the space $\mathrm{Sub}(\Omega)$ is closed under convergence in probability (see above), we have

$$\tau(\widehat{U}^r(h)) \leqslant 2\tau(U).$$

Proposition 2.1 is proved.

Moreover, in Proposition 4.1 it is shown that $\tau(U) = 1$. Hence, the following result is true.

Theorem 2.1. *Let $\{\widehat{U}^r(h), h \geqslant 0\}$ be a centered r-correlogram process. For any $h \geqslant 0$ the random variable $\widehat{U}^r(h)$ is sub-Gaussian with the sub-Gaussian standard*

$$\tau(\widehat{U}^r(h)) \leqslant 2. \tag{2.2}$$

A simple comparison shows that estimate (2.2) is much sharper than (1.5).

Theorem 2.1 allows us to construct a pointwise confidence interval for $\widehat{R}^r(h)$.

Corollary 2.1. *For every h and for every positive x we have the inequality*

$$\mathbf{P}\big(|\widehat{R}^r(h) - \rho^r(h)| > x\big) \leqslant 2e^{-\pi^2 x^2/8}.$$

Proof. By (1.4) and since $|\rho(h)| \leqslant 1$, we easily get

$$\mathbf{P}\big(|\widehat{R}^r(h) - \rho^r(h)| > x\big) \leqslant 2e^{-x^2/(2\tau^2(U))} \leqslant 2e^{-\pi^2 x^2/8}.$$

Corollary 2.1 is proved.

3. Technicalities. Let X and Y be two standard normal random variables with the correlation coefficient ρ. That is, $X \sim \mathbf{N}(0,1)$, $Y \sim \mathbf{N}(0,1)$, $\mathrm{cor}(X,Y) = \rho$. It is simple to write the joint density function of X and Y:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\}, \qquad -\infty < x, y < \infty.$$

Consider the random variable $\widetilde{U} = X\mathbf{1}_{\{Y>0\}}$. Note that the random variable X is continuous and the random variable $\mathbf{1}_{\{Y>0\}}$ is discrete.

Observe that the random variable $U = Z(0)\mathbf{1}_{\{Zh>0\}} - \mathbf{E}[Z(0)\mathbf{1}_{\{Z(h)>0\}}]$ from the proof of Proposition 2.1 is of the form

$$U = X\mathbf{1}_{\{Y>0\}} - \mathbf{E}[X\mathbf{1}_{\{Y>0\}}] \quad \text{and} \quad V = -X\mathbf{1}_{\{Y\leqslant 0\}},$$

where clearly we have put $X = Z(0)$, $Y = Z(h)$, $\rho = \rho(h)$, $|\rho| \leqslant 1$.

In what follows, in Proposition 3.1 we show that the random variables U and V are identically distributed and hence it is enough to consider only the random variable U.

We can derive the distribution function of the random variable $\widetilde{U}$. The random variable $\widetilde{U}$ takes value zero with probability 1/2 and the remaining 1/2 is spread over by the normal distribution:

$$F_{\widetilde{U}}(t) = \mathbf{P}(\widetilde{U} < t) = \begin{cases} \displaystyle\int_{-\infty}^{t} dx \int_{0}^{+\infty} f_{X,Y}(x,y)\,dy, & \text{for } t \leqslant 0, \\ \displaystyle\frac{1}{2} + \int_{-\infty}^{t} dx \int_{0}^{+\infty} f_{X,Y}(x,y)\,dy, & \text{for } t > 0. \end{cases}$$

Therefore, the distribution function of $\widetilde{U}$ has a gap 1/2 at $t = 0$. For example, for $\rho = 0$ the distribution function $F_{\widetilde{U}}(0-) = 1/4$ and $F_{\widetilde{U}}(0+) = 3/4$.

First we prove that random variables $U = X\mathbf{1}_{\{Y\geqslant 0\}}$ and $V = -X\mathbf{1}_{\{Y\leqslant 0\}}$ are identically distributed. For this we need the following lemma.

Lemma 3.1. *Let (X,Y) be a bivariate random vector having a density f such that $f(x,y) = f(-x,-y)$. Let $\phi(x,y)$ be a measurable function and denote $U = \phi(X,Y)$ and $V = \phi(-X,-Y)$. Then U and V have the same distribution.*

The proof of this lemma is straightforward and omitted.

Applying the preceding lemma to a bivariate Gaussian law with mean 0 and to the function $\phi(x,y) = x\mathbf{1}_{[0,+\infty)}(y)$, we get the following statement.

Proposition 3.1. *Let (X,Y) be a bivariate random vector having Gaussian law with mean 0. Then the two random variables $U = X\mathbf{1}_{\{Y\geqslant 0\}}$; $V = -X\mathbf{1}_{\{Y\leqslant 0\}}$ have the same distribution.*

Proof. Since

$$\{Y \neq 0\} \subseteq \{-X\mathbf{1}_{\{Y\leqslant 0\}} = -X\mathbf{1}_{\{Y<0\}}\},$$

we have

$$1 = \mathbf{P}(Y \neq 0) \leqslant \mathbf{P}(-X\mathbf{1}_{\{Y\leqslant 0\}} = -X\mathbf{1}_{\{Y<0\}}) \leqslant 1.$$

Hence

$$\mathbf{P}(-X\mathbf{1}_{\{Y\leqslant 0\}} = -X\mathbf{1}_{\{Y<0\}}) = 1.$$

This implies that U and $V = -X\mathbf{1}_{\{Y<0\}}$ have the same distribution. Proposition 3.1 is proved.

In order to prove the sub-Gaussianity of the random variable U we need the knowledge of the moment generating function (Laplace transforms) of $\widetilde{U}$ and U.

Proposition 3.2. *The moment generating function of $\widetilde{U}$ is given by the formula*

$$\widetilde{\mathcal{M}}(t) = e^{t^2/2}\Phi(\rho t) + \frac{1}{2}, \tag{3.1}$$

where $\Phi(t) = (2\pi)^{-1/2}\int_{-\infty}^{t} e^{x^2/2}\,dx$ denotes the distribution function of the standard Gaussian law $\mathbf{N}(0,1)$.

Proof. The moment generating function of the random variable $\widetilde{U}$ can be evaluated by the following way:

$$\begin{aligned}\mathbf{E}e^{t\widetilde{U}} &= \int_{-\infty}^{+\infty} e^{tx}\,dx \int_0^{+\infty} f_{X,Y}(x,y)\,dy + \int_{-\infty}^{+\infty} dx \int_{-\infty}^{0} f_{X,Y}(x,y)\,dy \\ &= \int_{-\infty}^{+\infty} e^{tx}\,dx \int_0^{+\infty} f_{X,Y}(x,y)\,dy + \frac{1}{2},\end{aligned}$$

however the following approach is probably more elegant.

Since

$$e^{t\widetilde{U}} = e^{tX}\mathbf{1}_{\{Y>0\}} + \mathbf{1}_{\{Y\leqslant 0\}},$$

we can write

$$\mathbf{E}e^{t\widetilde{U}} = \mathbf{P}(Y \leqslant 0) + \mathbf{E}[e^{tX}\mathbf{1}_{\{Y>0\}}] = \frac{1}{2} + \mathbf{E}\big[\mathbf{1}_{\{Y>0\}}\mathbf{E}[e^{tX}\,|\,Y]\big].$$

Now X, conditioned to Y, has the normal distribution with mean ρY and variance $1-\rho^2$. Hence

$$\mathbf{E}[e^{tX}\,|\,Y] = \exp\Big\{t\rho Y + (1-\rho^2)\frac{t^2}{2}\Big\}.$$

From this we obtain

$$\begin{aligned}\mathbf{E}\big[\mathbf{1}_{\{Y>0\}}\mathbf{E}[e^{tX}\,|\,Y]\big] &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{y^2}{2} + t\rho y + (1-\rho^2)\frac{t^2}{2}\Big\}\,dy \\ &= e^{t^2/2}\int_0^{+\infty} \frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{(y-t\rho)^2}{2}\Big\}\,dy = e^{t^2/2}\Phi(\rho t).\end{aligned}$$

Proposition 3.2 is proved.

Proposition 3.3. *The moment generating function of U is given by the formula*

$$\mathcal{M}(t) = e^{-t\rho/\sqrt{2\pi}}\Big(e^{t^2/2}\Phi(\rho t) + \frac{1}{2}\Big). \tag{3.2}$$

Proof. We can write the direct formula for expectation of the random variable $\widetilde{U}$:

$$\mathbf{E}\widetilde{U} = \int_{-\infty}^{\infty} x\,dx \int_0^{\infty} f_{X,Y}(x,y)\,dy,$$

however, probably it is simpler to differentiate at zero the moment generating function.

As an immediate consequence of formula (3.1), we get

$$\mathbf{E}\widetilde{U} = \widetilde{\mathcal{M}}'(0) = \frac{\rho}{\sqrt{2\pi}}.$$

Hence the moment generating function $\mathcal{M}$ of $U = \widetilde{U} - \mathbf{E}\widetilde{U}$ is $e^{-\rho t/(\sqrt{2\pi})}\widetilde{\mathcal{M}}(t)$. Proposition 3.3 is proved.

Remark 3.1. Knowing the moment generating function $\widetilde{\mathcal{M}}(t)$ and taking second derivative at zero, we obtain that $\mathbf{E}\widetilde{U}^2 = 1/2$. Hence the variance of the random variable U is

$$\operatorname{Var} U = \frac{1}{2}\left(1 - \frac{\rho^2}{\pi}\right). \tag{3.3}$$

Note that from here we can state that the sub-Gaussian standard $\tau(U) \geqslant (1-\rho^2/\pi)/2$. It follows from the fact that for any sub-Gaussian random variable X, the sub-Gaussian standard $\tau(X) \geqslant \operatorname{Var} X$ (see [3, Lemma 1.2]).

4. Proof of the main result.

Proposition 4.1. *The random variable U is sub-Gaussian and $\tau(U) = 1$.*

Proof. First we show that $\tau(U) \leqslant 1$. By the definition of the sub-Gaussian standard, in order to show that $\tau(U) \leqslant 1$ we need to prove that the Laplace transform (moment generating function) of the random variable U satisfies the inequality $\mathbf{E}e^{tU} \leqslant e^{t^2/2}$. According to the formula (3.2), this is equivalent to the statement

$$e^{-\rho t/\sqrt{2\pi}}\left(e^{t^2/2}\Phi(\rho t) + \frac{1}{2}\right) \leqslant e^{t^2/2}, \tag{4.1}$$

for all $-\infty < t < \infty$ and all $-1 \leqslant \rho \leqslant 1$.

Simple calculations show that (4.1) is equivalent to the statement $f(\rho,t) \geqslant 0$, where

$$f(\rho,t) = \exp\left\{\frac{t^2}{2} + \frac{\rho}{\sqrt{2\pi}}t\right\} - \exp\left\{\frac{t^2}{2}\right\}\Phi(\rho t) - 0.5$$

and $-\infty < t < \infty$ and $-1 \leqslant \rho \leqslant 1$.

Note that the function $f(\rho,t)$ is "even", that is, $f(\rho,t) = f(-\rho,-t)$ and hence it is enough to consider only nonnegative values of t. So in the following we assume that $t \geqslant 0$.

Now we investigate the behavior of the function $f(\rho,t)$ in the region $-1 \leqslant \rho \leqslant 1$, $t \geqslant 0$ and show that it is always nonnegative.

The derivation of the function $f(\rho,t)$ by t is quite cumbersome, so we take derivative by ρ:

$$\frac{\partial f(\rho,t)}{\partial\rho} = \frac{e^{t^2/2}t}{\sqrt{2\pi}}\big(e^{\rho t/\sqrt{2\pi}} - e^{-\rho^2t^2/2}\big).$$

Letting $\partial f(\rho,t)/\partial\rho = 0$, we obtain the following three solutions.

1. The first solution is $t=0$. It is boundary point (we consider only $t\geqslant 0$) and gives the global minimum. Note that $f(\rho,0)=0$ for all ρ.

2. The second solution is $\rho=0$. This is a local minimum because the second derivative:

$$\frac{\partial^2 f(\rho,t)}{\partial\rho^2} = \frac{e^{t^2/2}t^2}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{2\pi}}e^{\rho t/\sqrt{2\pi}} + \rho t e^{-\rho^2t^2/2}\right)$$

and

$$\frac{\partial^2 f(0,t)}{\partial\rho^2} = \frac{e^{t^2/2}t^2}{2\pi}$$

is positive at this point (the case $t=0$ is considered above).

Note that $f(0,t) = (1/2)(e^{t^2/2}-1)\geqslant 0$ for all t.

3. The third solution is $\rho = -\sqrt{2/\pi}(1/t)$. This is a local maximum because the second derivative

$$\frac{\partial^2 f(-\sqrt{2/\pi}(1/t),t)}{\partial\rho^2} = -\frac{e^{t^2/2}t^2}{2\pi}e^{-1/\pi},$$

is negative (again, the case $t=0$ is considered above).

The typical behavior of the function $f(\rho,t)$ when t is fixed is presented in Fig. 1.

From our investigation of the derivative and Fig. 1 we see that the function $f(p,t)$ is positive at the point of the local minimum $\rho=0$. In the interval $\rho\in(0,1]$ the function is increasing, so it is nonnegative.

Further, the function $f(p,t)$ is decreasing between the local maximum $\rho = -\sqrt{2/\pi}(1/t)$ and local minimum $\rho=0$ and because it is positive at the point of the local minimum $\rho=0$, it is nonnegative in the interval $\rho\in[-\sqrt{2/\pi}(1/t),0]$ too.

The function is increasing in the interval $\rho\in[-1,-\sqrt{2/\pi}(1/t)]$ and hence it is left to show that at the point $\rho=-1$ the value is nonnegative.

Notice that

$$\begin{aligned} f(-1,t) &= \exp\left\{\frac{t^2}{2} - \frac{t}{\sqrt{2\pi}}\right\} - \exp\left\{\frac{t^2}{2}\right\}\Phi(-t) - 0.5 \\ &= \exp\left\{\frac{t^2}{2}\right\}\left(\Phi(t) + \exp\left\{-\frac{t}{\sqrt{2\pi}}\right\} - \frac{1}{2}\exp\left\{-\frac{t^2}{2}\right\} - 1\right) \end{aligned} \tag{4.2}$$

where $t>0$.

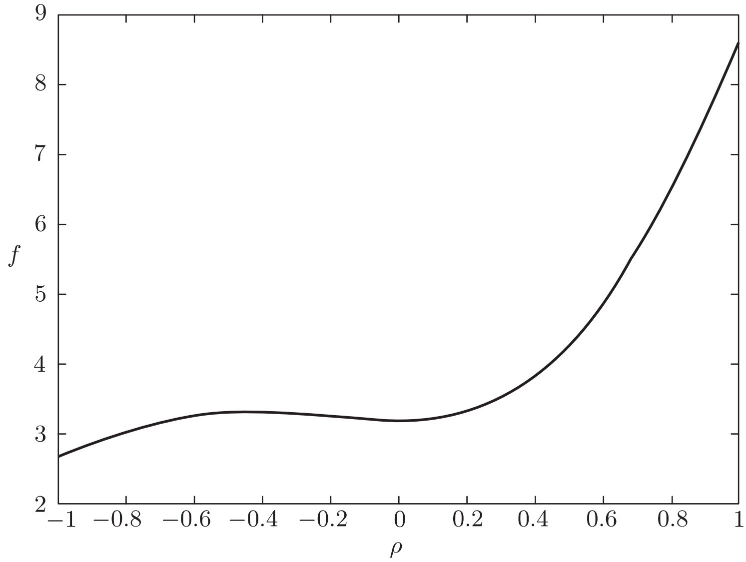

Fig. 1. Graph of the function $f(\rho, t), -1 \leqslant \rho \leqslant 1$, when t is fixed.

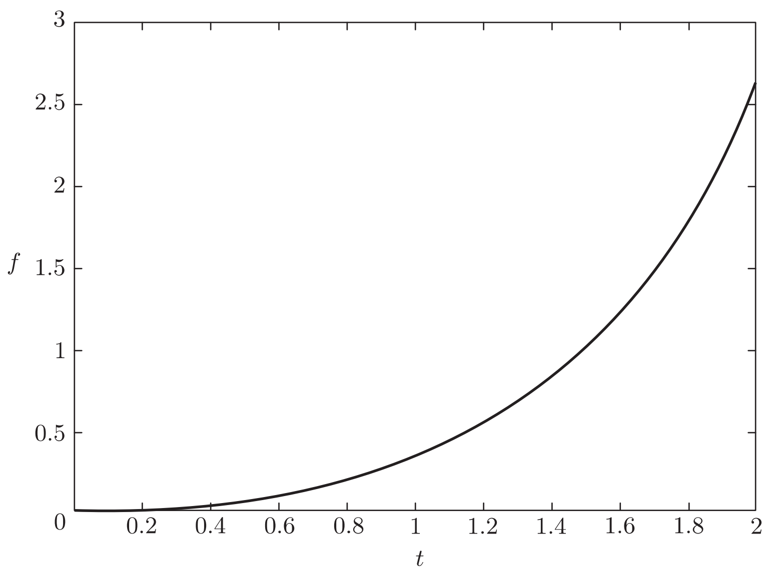

Fig. 2. Graph of the function $f(-1, t)$, $1 \leqslant t \leqslant 2$.

Fig. 2 shows the behavior of the function $f(-1, t)$, $0 \leqslant t \leqslant 2$.

Our goal is to show that $f(-1, t) \geqslant 0$ for $t > 0$. In the expression (4.2), cancelling out $\exp\{t^2/2\}$, it is needed to show that

$$G(t) = \Phi(t) + \exp\left\{-\frac{t}{\sqrt{2\pi}}\right\} - \frac{1}{2}\exp\left\{-\frac{t^2}{2}\right\} - 1 \geqslant 0. \tag{4.3}$$

Fig. 3 shows the behavior of the function $G(t)$, $0 \leqslant t \leqslant 6$.

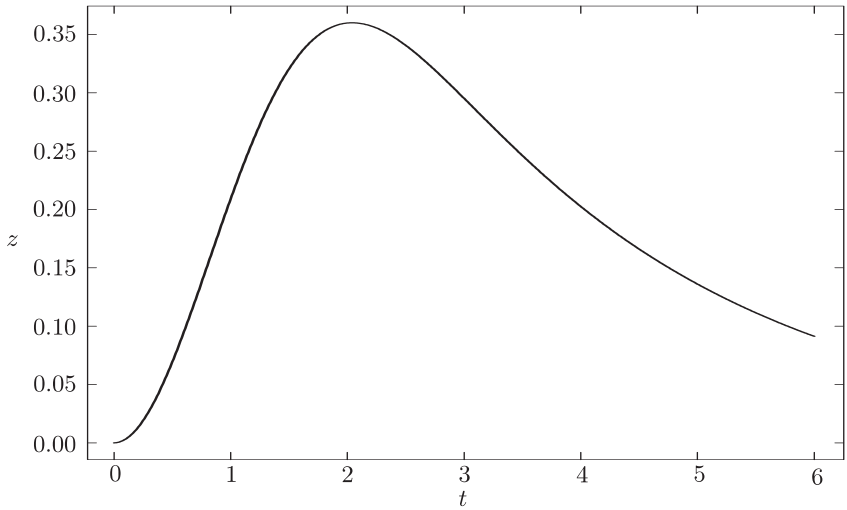

Fig. 3. Graph of the function $G(t)$, $0 \leqslant t \leqslant 6$.

First notice that

$$\lim_{t\to+\infty} G(t) = 1 + 0 - 0 - 1 = 0.$$

Taking the derivative we get

$$\begin{aligned} G'(t) &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\} - \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t}{\sqrt{2\pi}}\right\} + \frac{t}{2} \exp\left\{-\frac{t^2}{2}\right\} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\} \left[1 - \exp\left\{\frac{t^2}{2} - \frac{t}{\sqrt{2\pi}}\right\} + \sqrt{\frac{\pi}{2}}\, t\right]. \end{aligned}$$

Hence $G'(t) \geqslant 0$ if and only if

$$1 - \exp\left\{\frac{t^2}{2} - \frac{t}{\sqrt{2\pi}}\right\} + \sqrt{\frac{\pi}{2}}\, t \geqslant 0,$$

which is equivalent to

$$\exp\left\{\frac{t^2}{2} - \frac{t}{\sqrt{2\pi}}\right\} \leqslant 1 + \sqrt{\frac{\pi}{2}}\, t$$

and, in turn, to

$$\frac{t^2}{2} - \frac{t}{\sqrt{2\pi}} - \ln\left(1 + \sqrt{\frac{\pi}{2}}\, t\right) \leqslant 0.$$

We study the function

$$h(t) = \frac{t^2}{2} - \frac{t}{\sqrt{2\pi}} - \ln\left(1 + \sqrt{\frac{\pi}{2}}\, t\right).$$

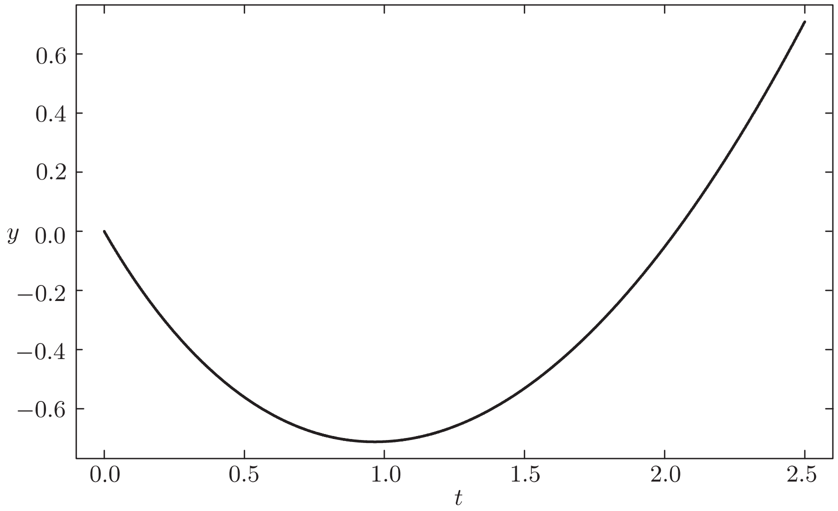

Fig. 4. Graph of the function $h(t)$, $0 \leqslant t \leqslant 2.5$.

Fig. 4 shows the behavior of the function $h(t)$, $0 \leqslant t \leqslant 2.5$.

We have $h(0) = 0$ and h is negative in the neighborhood of 0 since, as $t \to 0^+$, the principal part of h is

$$-\frac{t}{\sqrt{2\pi}} - \sqrt{\frac{\pi}{2}}\, t \leqslant 0.$$

On the other hand,

$$\lim_{t\to+\infty} h(t) = +\infty.$$

Hence there exists t_0 such that $h(t_0) = 0$. We prove that t_0 is unique.

We have

$$h'(t) = \frac{\sqrt{\pi/2}\, t^2 + t/2 - (1/\sqrt{2\pi} + \sqrt{\pi/2})}{1 + \sqrt{\pi/2}\, t}.$$

Letting $h'(t) = 0$, we find two solutions

$$t_{1,2} = \frac{-1/2 \pm \sqrt{9/4 + 2\pi}}{\sqrt{2\pi}}.$$

One of these solutions is negative, so we consider only the positive solution

$$t^* = \frac{-1/2 + \sqrt{9/4 + 2\pi}}{\sqrt{2\pi}} \approx 0.965903745.$$

Obviously, $h'(t) > 0$ for $t > t^*$ and $h'(t) < 0$ for $t < t^*$. Hence t^* is the only minimum point for h in the region $t \geqslant 0$ and this implies that t_0 is unique. Therefore, $G'(t) > 0$ for $t < t_0$ and $G'(t) < 0$ for $t > t_0$, i.e., t_0 is a unique maximum point for $G(t)$, $t \geqslant 0$.

Rough evaluation of t_0 can be done numerically. We found that $t_0 \approx 2.0414$.

Finally, we show that $\tau(U) \geqslant 1$. From Proposition 3.3 we know that the moment generating function of U is given by formula (3.2), that is

$$\mathbf{E}e^{tU} = e^{-t\rho/\sqrt{2\pi}}\left(e^{t^2/2}\Phi(\rho t) + \frac{1}{2}\right).$$

Obviously, if we drop $1/2$ in (3.2), we obtain the inequality

$$\mathbf{E}e^{tU} \geqslant e^{-t\rho/\sqrt{2\pi}}e^{t^2/2}\Phi(\rho t).$$

Let t be such that $\rho t \geqslant 0$, then $\Phi(\rho t) \geqslant 1/2$ and

$$\mathbf{E}e^{tU} \geqslant \frac{1}{2}e^{-t\rho/\sqrt{2\pi}}e^{t^2/2} = \frac{1}{2}\exp\left\{\frac{t^2}{2} - t\frac{\rho}{\sqrt{2\pi}}\right\}.$$

Now, for any $\varepsilon > 0$, choose t such that $\rho t \geqslant 0$ and

$$\frac{\varepsilon}{2}t^2 - \frac{\rho}{\sqrt{2\pi}} - \ln 2 \geqslant 0.$$

Then

$$\mathbf{E}e^{tU} \geqslant \frac{1}{2}\exp\left\{\frac{t^2}{2} - t\frac{\rho}{\sqrt{2\pi}}\right\} \geqslant \exp\left\{(1-\varepsilon)\frac{t^2}{2}\right\}.$$

By the definition of the sub-Gaussian standard and be arbitrariness of $\varepsilon > 0$ we obtain that $\tau(U) \geqslant 1$. Proposition 4.1 is proved.

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