Test (2012) 21:369–385
DOI 10.1007/s11749-011-0248-0

ORIGINAL PAPER

Some theorems on conditional mean convergence and conditional almost sure convergence for randomly weighted sums of dependent random variables

Manuel Ordóñez Cabrera · Andrew Rosalsky · Andrei Volodin

Received: 12 July 2010 / Accepted: 23 April 2011 / Published online: 15 May 2011

Abstract In (Ordóñez Cabrera and Volodin, J. Math. Anal. Appl. 305:644–658, 2005), the authors introduce the notion of h-integrability of an array of random variables with respect to an array of constants, and obtained some mean convergence theorems for weighted sums of random variables subject to some special kinds of dependence.

In view of the important role played by conditioning and dependence in the models used to describe many situations in the applied sciences, the concepts and results in the aforementioned paper are extended herein to the case of randomly weighted sums of dependent random variables when a sequence of conditioning sigma-algebras is given. The dependence conditions imposed on the random variables (conditional negative quadrant dependence and conditional strong mixing) as well as the convergence results obtained are conditional relative to the conditioning sequence of sigma-algebras.

In the last section, a strong conditional convergence theorem is also established by using a strong notion of conditional h-integrability.

The research of M. Ordóñez Cabrera has been partially supported by DGICYT grant BFM2003-03893-C02-01 and Junta de Andalucia FQM 127. The research of A. Volodin has been partially supported by the National Science and Engineering Research Council of Canada.

M. Ordóñez Cabrera
Department of Mathematical Analysis, University of Sevilla, Sevilla 41080, Spain
e-mail: cabrera@us.es

A. Rosalsky (✉)
Department of Statistics, University of Florida, Gainesville, FL 32611, USA
e-mail: rosalsky@stat.ufl.edu

A. Volodin
Department of Mathematics and Statistics, University of Regina, Regina, SK, S4S0A2, Canada
e-mail: andrei@math.uregina.ca

Keywords Conditional residual h-integrability · Randomly weighted sums · Conditional negative dependence · Conditional strong-mixing · Conditional strongly residual h-integrability

Mathematics Subject Classification (2000) 60F05 · 60F15 · 60G99

1 Introduction

The random nature of many problems arising in the applied sciences leads to mathematical models which concern the limiting behavior of weighted sums of random variables, where the weights are also random variables. Thus, let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed random variables, and let $\{\Theta_i, i \geq 1\}$ be a sequence of non-negative random variables which is independent of the sequence $\{X_i, i \geq 1\}$. Let us write $S_n = \sum_{i=1}^{n} \Theta_i X_i$, $n \geq 1$.

This model often appears in actuarial and economic situations, such as discrete time risk models for the activities of an insurance company (see Wang and Tang 2006 and Shen et al. 2009); each X_i can be understood as the net loss (the total claim amount minus total incoming premium) within the time period i, and each Θ_i as the discount factor from time i to time 0 (the present). Then S_n can be interpreted as the total discounted amount of the net loss of the company at time n.

If we write $M_n = \max_{1\leq j\leq n} S_j$, $n \geq 1$ and $M_\infty = \sup_{1\leq j<\infty} S_j$, then the tail probabilities $P[M_n > x]$ and $P[M_\infty > x]$ can be understood as the probabilities of ruin by time n and of ultimate ruin, respectively, where $x \geq 0$ is the initial surplus.

The assumption of independence of $\{X_i, i \geq 1\}$ does not necessarily answer to a real need, but rather to the fact of being able to simplify the mathematical treatment of models. In fact, several improvements of this model, by imposing various conditions of dependence among $\{X_i, i \geq 1\}$, have been considered lately (see, e.g., Weng et al. 2009). The consideration of conditions of dependence between the random variables $\{X_i, i \geq 1\}$ and the random weights $\{\Theta_i, i \geq 1\}$ could be a better approximation of models to real problems of more complexity.

At the same time, this question of dependence in nature often leads to mathematical models where conditioning is present. Thus, martingale sequences are well-known cases of stochastic processes defined through conditioning. Markov processes are another example of stochastic processes in which conditioning (specifically, conditional independence) is essential. See Aas et al. (2009) and Sheremet and Lucas (2009) for recent work on insurance models involving dependence and conditioning.

A typical example of statistical application of *conditional limit theorems* is in the study of statistical inference for some branching processes, such as the Galton–Watson process (see, e.g., Basawa and Prakasa Rao 1980). Let $\{Z_0 = 1, Z_n, n \geq 1\}$ be a Galton–Watson process with mean offspring Θ. This process can be studied by means of the following autoregressive type model:

$$Z_{n+1} = \Theta Z_n + Z_n^{1/2} U_{n+1}, \quad n \geq 0$$

where $\{U_k, k \geq 0\}$ is the sequence of error random variables.

In order to estimate the mean offspring Θ from a realization $\{Z_0 = 1, Z_1, \ldots, Z_n\}$, the maximum likelihood estimator of Θ is $\hat{\Theta}_n = (\sum_{k=1}^{n} Z_{k-1})^{-1}(\sum_{k=1}^{n} Z_k)$, which coincides with the "least-squares" estimator of Θ obtained by minimizing $\sum_{k=0}^{n} U_k^2$ with respect to Θ.

The study of asymptotic properties of $\hat{\Theta}_n$ leads to a conditional limit theorem since, as it is detailed in Basawa and Prakasa Rao (1980), these asymptotic properties of $\hat{\Theta}_n$ depend on the event of non-extinction of the process.

Asymptotic properties of estimators in conditional models involving high-dimensional genomic data have recently been studied by Leek (2011).

The interested reader is referred to Roussas (2008) for a more extensive enumeration of models in which conditioning plays a key role.

We are interested in two concepts of conditional dependence which generalize the concept of conditional independence, namely the concepts of conditional negative quadrant dependence and conditional strong mixing.

The concept of conditional negative quadrant dependence is an extension to the conditional case of the concept of negative quadrant dependence introduced by Lehmann (1966) as a measure of the degree of association between two random variables, and are applied, for example, to study tests of independence based on rank correlation, Kendall's τ-statistic, or normal scores. In that paper, Lehmann provided an extensive overview of various concepts of positive and negative dependence.

The origin of the concept of conditional strong mixing (Prakasa Rao 2009) is the concept of strong-mixing (or α-mixing) for sequences of random variables, introduced by Rosenblatt (1956) to study short range dependence, although the properties of conditional strong mixing and strong-mixing do not imply each other.

The aim of this paper is to extend the concepts and results of Ordóñez Cabrera and Volodin (2005) to a much wider setting in which conditional convergence and conditional dependence play a key role.

In Ordóñez Cabrera and Volodin (2005), the notion of h-integrability of an array $\{X_{nk}\}$ of random variables with respect to an array of constants $\{a_{nk}\}$ is introduced, starting from the notion of $\{a_{nk}\}$-uniform integrability introduced in Ordóñez Cabrera (1994), which is a weakening of classical notion of uniform integrability. This concept of h-integrability with respect to an array of constant weights, which is related to tail probabilities of random variables, is, in any case, more general and weaker than the concept of Cesàro α-integrability of Chandra and Goswami (2003). For a more detailed development of these notions and their relationships, the reader may consult Ordóñez Cabrera and Volodin (2005).

With this background, in the current work we extend the notion of h-integrability of $\{X_{nk}\}$ with respect to constant weights $\{a_{nk}\}$ to the corresponding conditional notion in the more general setting of randomly weighted sums of random variables (i.e., to the case in which the weights are also random variables $\{A_{nk}\}$) when a sequence of conditioning σ-algebras $\{\mathcal{B}_n\}$ is given. We then obtain some results on conditional convergence of these sums given the conditioning σ-algebras of events $\{\mathcal{B}_n\}$ that extend, in a substantial way, the main mean convergence theorems in Ordóñez Cabrera and Volodin (2005).

The notions and the results herein are of the greatest interest when $\mathcal{B}_n = \sigma(A_{nk}, u_n \le k \le v_n)$, i.e., when $\mathcal{B}_n$ is the σ-algebra generated by the nth row of the array $\{A_{nk}\}$.

In the last section, we introduce a strong concept of conditional h-integrability relative to a σ-algebra of events $\mathcal{B}$ in order to establish a strong version of the main result obtained in Sect. 3.

2 Definitions and basic results on conditioning

We present at first basic definitions and results concerning conditional independence and conditional negative dependence. The interested reader can find further results in Chow and Teicher (1997) and Roussas (2008). All events and random variables are defined on the same probability space $(\Omega, \mathcal{A}, P)$. Throughout, $\mathcal{B}$ is a sub-σ-algebra of $\mathcal{A}$. We denote by $E^{\mathcal{B}}(X)$ the conditional expectation of the random variable X relative to $\mathcal{B}$, and by $P^{\mathcal{B}}(A)$ the conditional probability of the event $A \in \mathcal{A}$ relative to $\mathcal{B}$.

Definition A sequence $\{\mathcal{G}_n, n \geq 1\}$ of classes of events is said to be *conditionally independent given* $\mathcal{B}$ (*$\mathcal{B}$-independent*, for short) if for all $n \geq 2$ and all choices of $k_1, \ldots, k_n \in \mathbf{N}$ where $k_i \neq k_j$ for $i \neq j$ and all choices of $A_i \in \mathcal{G}_{k_i}$, $1 \leq i \leq n$

$$P^{\mathcal{B}}\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P^{\mathcal{B}}(A_i) \quad \text{almost surely (a.s.).}$$

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be *conditionally independent given* $\mathcal{B}$ (*$\mathcal{B}$-independent*, for short) if the sequence of σ-algebras generated by them, $\{\sigma(X_n), n \geq 1\}$, is $\mathcal{B}$-independent.

It is easy to prove (see Roussas 2008, Theorem 2.1) that the random variables $\{X_n, n \geq 1\}$ are $\mathcal{B}$-independent if, and only if, for every $(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$:

$$P^{\mathcal{B}}(X_i \leq x_i, i = 1, 2, \ldots, n) = \prod_{i=1}^{n} P^{\mathcal{B}}(X_i \leq x_i) \quad \text{a.s.}$$

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be *pairwise $\mathcal{B}$-independent* if every pair of random variables in the sequence is $\mathcal{B}$-independent.

If $\mathcal{B} = \{\emptyset, \Omega\}$, then $\mathcal{B}$-independence become ordinary (unconditional) independence.

Prakasa Rao (2009) and Roussas (2008) illustrated by simple examples that conditioning may destroy independence, and dependence may be turned into independence by conditioning. See also Chow and Teicher (1997), p. 229.

The following results are basic:

Proposition 1 (Roussas 2008, Proposition 3.8) *If the integrable random variables X and Y are $\mathcal{B}$-independent, then*

$$E^{\mathcal{B}}(XY) = E^{\mathcal{B}}(X)E^{\mathcal{B}}(Y) \quad a.s.,$$

and similarly for any finite number of random variables.

Proposition 2 (Roussas 2008, Proposition 3.9) *Let the random variables X and Y be $\mathcal{B}$-independent, and let $EX^2 < \infty$ and $EY^2 < \infty$. Then*

$$\mathrm{Cov}^{\mathcal{B}}(X,Y) = E^{\mathcal{B}}\big[\big(X - E^{\mathcal{B}}X\big)\big(Y - E^{\mathcal{B}}Y\big)\big] = 0 \quad a.s.$$

Roussas (2008) provides a detailed proof of an integral representation of the covariance of two random variables, a brief proof of which is available in Lehmann (1966). By applying a conditional version of the Fubini theorem, Roussas (2008) obtains the following integral representation for the conditional covariance of two random variables:

Proposition 3 (Roussas 2008, Proposition 4.3) *Let X and Y be random variables with $EX^2 < \infty$ and $EY^2 < \infty$. Then*

$$\mathrm{Cov}^{\mathcal{B}}(X,Y) = \int_{\mathbf{R}^2} H^{\mathcal{B}}(x,y)\,dx\,dy \quad a.s.,$$

where $H^{\mathcal{B}}(x,y) = P^{\mathcal{B}}[X \le x, Y \le y] - P^{\mathcal{B}}[X \le x]P^{\mathcal{B}}[X \le y]$.

We now present the basic definitions and results concerning conditional negative dependence.

Definition Random variables X and Y are said to be *conditionally negative quadrant dependent relative to a σ-algebra $\mathcal{B}$* ($\mathcal{B}$-CNQD, for short) if

$$P^{\mathcal{B}}[X \le x, Y \le y] \le P^{\mathcal{B}}[X \le x]P^{\mathcal{B}}[X \le y] \quad \text{a.s. for all } x, y \in \mathbf{R}.$$

A sequence of random variables $\{X_n, n \ge 1\}$ is said to be *pairwise conditionally negative quadrant dependent relative to a σ-algebra $\mathcal{B}$* if every pair of random variables in the sequence is $\mathcal{B}$-CNQD.

An immediate consequence of Proposition 3 is the following lemma.

Lemma 1 *Let $\{X_n, n \ge 1\}$ be a sequence of pairwise $\mathcal{B}$-CNQD random variables with finite second moments. Then for all $i, j \ge 1$, $i \ne j$ we have*

$$E^{\mathcal{B}}(X_i X_j) \le E^{\mathcal{B}}(X_i)E^{\mathcal{B}}(X_j) \quad a.s.$$

Note that if $\mathcal{B} = \{\emptyset, \Omega\}$, then a sequence of pairwise $\mathcal{B}$-CNQD random variables is precisely a sequence of random variables which are negative quadrant dependent (NQD) in the unconditional case, and Lemma 1 becomes the well-known result that pairwise NQD random variables are non-positively correlated.

Another well-known result for NQD random variables is the fact that the technique of continuous truncation preserves the NQD property.

The next lemma establishes that the conditional property of being $\mathcal{B}$-CNQD is also preserved by this technique of truncation.

Lemma 2 *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise $\mathcal{B}$-CNQD random variables. Then, for all sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ of constants such that $a_n < b_n$ for all $n \in \mathbf{N}$, the sequence $\{Y_n, n \geq 1\}$ defined by*

$$Y_n = X_n I[a_n \leq X_n \leq b_n] + a_n I[X_n < a_n] + b_n I[X_n > b_n], \quad n \geq 1$$

is likewise a sequence of pairwise $\mathcal{B}$-CNQD random variables.

Proof For all $n \geq 1$, let $g_n : \mathbf{R} \to \mathbf{R}$ be defined by

$$g_n(x) = \begin{cases} a_n, & x < a_n, \\ x, & a_n \leq x \leq b_n, \\ b_n & x > b_n. \end{cases}$$

Then $Y_n = g_n(X_n)$ and g_n is a non-decreasing function, $n \geq 1$. By the same argument as that for a sequence of NQD random variables (see Lemma 1 of Lehmann 1966), we have for $m, n \in \mathbf{N}$ where $m \neq n$ and $y_1, y_2 \in \mathbf{R}$,

$$\begin{aligned} P^{\mathcal{B}}[Y_m \leq y_1, Y_n \leq y_2] &= P^{\mathcal{B}}\big[g_m(X_m) \leq y_1, g_n(X_n) \leq y_2\big] \\ &\leq P^{\mathcal{B}}\big[g_m(X_m) \leq y_1\big] \cdot P^{\mathcal{B}}\big[g_n(X_n) \leq y_2\big] \\ &= P^{\mathcal{B}}[Y_m \leq y_1] \cdot P^{\mathcal{B}}[Y_n \leq y_2]. \end{aligned}$$

Hence, $\{Y_n, n \geq 1\}$ is a sequence of pairwise $\mathcal{B}$-CNQD random variables. □

3 Conditional residual *h*-integrability

Recall that all random variables appearing are defined on the same probability space $(\Omega, \mathcal{A}, P)$ and we let $\mathcal{B}$ and $\mathcal{B}_n,\ n \geq 1$ be sub-σ-algebras of $\mathcal{A}$.

In the following, $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ will be two sequences of integers (not necessary positive or finite) such that $v_n > u_n$ for all $n \geq 1$ and $v_n - u_n \to \infty$ as $n \to \infty$. Moreover, $\{h(n),\ n \geq 1\}$ will be a sequence of positive constants with $h(n) \uparrow \infty$ as $n \to \infty$.

We introduce the notion of conditional residual h-integrability relative to the sequence $\{\mathcal{B}_n\}$ as follows:

Definition Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ and $\{A_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be two arrays of random variables. The array $\{X_{nk}\}$ is said to be *conditionally residually h-integrable relative to $\mathcal{B}_n$* ($\mathcal{B}_n$-CR-*h-integrable*, for short) *concerning the array* $\{A_{nk}\}$ if the following conditions hold:

(a)

$$\sup_{n\geq 1} \sum_{k=u_n}^{v_n} |A_{nk}| E^{\mathcal{B}_n} |X_{nk}| < \infty \quad \text{a.s.},$$

(b)

$$\lim_{n\to\infty}\sum_{k=u_n}^{v_n}|A_{nk}|E^{\mathcal{B}_n}\big(|X_{nk}|-h(n)\big)I\big[|X_{nk}|>h(n)\big]=0 \quad \text{a.s.}$$

Remark 1 This concept is a conditional extension to the more general setting of randomly weighted sums of random variables of (i) the concept of residual Cesàro α-integrability introduced by Chandra and Goswami (2006) and (ii) the concept of residual h-integrability concerning an array of (nonrandom) constants introduced by Yuan and Tao (2008). The work of Yuan and Tao (2008) extends many results of both Chandra and Goswami (2003, 2006) and Ordóñez Cabrera and Volodin (2005).

Remark 2 Let $\{h_1(n),\ n\ge 1\}$ and $\{h_2(n),\ n\ge 1\}$ be two positive monotonically increasing to infinity sequences such that $h_2(n)\ge h_1(n)$ for all sufficiently large n. Then $\mathcal{B}_n$-CR-h_1-integrability implies $\mathcal{B}_n$-CR-h_2-integrability.

Remark 3 If $A_{nk}\equiv a_{nk}$ are constants, and $\mathcal{B}_n=\{\emptyset,\Omega\}$ for all $n\in\mathbf{N}$, we have the concept of residual h-integrability concerning the array of constants $\{a_{nk}\}$ of Yuan and Tao (2008) which we referred to in Remark 1.

Definition Let $\{X_{nk}, u_n\le k\le v_n, n\ge 1\}$ be an array of random variables and $\{a_{nk}, u_n\le k\le v_n, n\ge 1\}$ an array of constants. The array $\{X_{nk}\}$ is said to be *residually h-integrable* (R-*h-integrable*, for short) concerning the array of constants $\{a_{nk}\}$ if the following conditions hold:

(a)

$$\sup_{n\ge 1}\sum_{k=u_n}^{v_n}|a_{nk}|E|X_{nk}|<\infty,$$

(b)

$$\lim_{n\to\infty}\sum_{k=u_n}^{v_n}|a_{nk}|E\big(|X_{nk}|-h(n)\big)I\big[|X_{nk}|>h(n)\big]=0.$$

Remark 4 The concept of R-h-integrability concerning an array of constants $\{a_{nk}\}$ with the additional condition $\sup_n\sum_{k=u_n}^{v_n}|a_{nk}|\le C$ for some constant $C>0$ is weaker than the concept of h-integrability in Ordóñez Cabrera and Volodin (2005) because

$$\big(|X_{nk}|-h(n)\big)I\big[|X_{nk}|>h(n)\big]\le |X_{nk}|I\big[|X_{nk}|>h(n)\big].$$

A very interesting example which reveals *inter alia* that R-h-integrability is strictly weaker than h-integrability was provided by Chandra and Goswami (2006) (see Example 2.1 in Chandra and Goswami 2006).

We will now obtain some conditional mean convergence theorems for randomly weighted sums of arrays of $\mathcal{B}$-CR-h-integrable random variables under some conditions of conditional dependence. Namely, we consider the following row-wise conditional dependence structures for an array: conditional negative quadrant dependence, non-positive conditional correlation, and conditional strong-mixing.

In the first theorem of this section, we will show that, for an array of row-wise pairwise conditionally negative quadrant dependent random variables, the technique of continuous truncation, which preserves the conditional negative quadrant dependence, can be used to obtain a conditional mean convergence theorem, that is, a limit theorem whose conclusion is $E^{\mathcal{B}_n}|S_n| \to 0$ a.s. as $n \to \infty$ where $\{S_n, n \geq 1\}$ is a sequence of random variables. Theorem 1 extends Theorem 1 of Ordóñez Cabrera and Volodin (2005), Theorem 2.2 of Chandra and Goswami (2006), and Theorem 2.2 of Yuan and Tao (2008).

Theorem 1 *Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of row-wise pairwise $\mathcal{B}_n$-CNQD random variables. Let $\{A_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of non-negative random variables such that, for each $n \in \mathbf{N}$, the $\{A_{nk}, u_n \leq k \leq v_n\}$ are $\mathcal{B}_n$-measurable. Suppose that*

(a) *$\{X_{nk}\}$ is $\mathcal{B}_n$-CR-h-integrable concerning the array $\{A_{nk}\}$,*
(b) *$h(n)(\sup_{u_n \leq k \leq v_n} A_{nk}) \to 0$ a.s. as $n \to \infty$.*

Let $S_n = \sum_{k=u_n}^{v_n} A_{nk}(X_{nk} - E^{\mathcal{B}_n} X_{nk})$, $n \geq 1$. Then $E^{\mathcal{B}_n}|S_n| \to 0$ a.s. as $n \to \infty$.

Proof For each $n \in \mathbf{N}$ and $u_n \leq k \leq v_n$, we define by using the method of continuous truncation:

$$Y_{nk} = X_{nk} I\big[|X_{nk}| \leq h(n)\big] - h(n) I\big[X_{nk} < -h(n)\big] + h(n) I\big[X_{nk} > h(n)\big],$$

$$S_{1n} = \sum_{k=u_n}^{v_n} A_{nk}(X_{nk} - Y_{nk}),$$

$$S_{2n} = \sum_{k=u_n}^{v_n} A_{nk}\big(Y_{nk} - E^{\mathcal{B}_n} Y_{nk}\big), \quad \text{and}$$

$$S_{3n} = \sum_{k=u_n}^{v_n} A_{nk} E^{\mathcal{B}_n}(Y_{nk} - X_{nk}).$$

It follows from the following that in the case of infinite u_n and/or v_n, the corresponding conditional expectations of series S_{1n}, S_{2n}, and S_{3n} converge absolutely a.s. Hence, we can write that

$$S_n = S_{1n} + S_{2n} + S_{3n}, \quad n \geq 1,$$

and we will estimate the conditional expectation of each of these terms separately. Note that for $n \geq 1$,

$$E^{\mathcal{B}_n}|S_{1n}| \leq \sum_{k=u_n}^{v_n} A_{nk} E^{\mathcal{B}_n}|X_{nk} - Y_{nk}| \quad \text{a.s.}$$

and

$$E^{\mathcal{B}_n}|S_{3n}| \leq \sum_{k=u_n}^{v_n} A_{nk} E^{\mathcal{B}_n}|X_{nk} - Y_{nk}| \quad \text{a.s.}$$

But since

$$|X_{nk} - Y_{nk}| = \big(|X_{nk}| - h(n)\big) I\big[|X_{nk}| > h(n)\big],$$

we get that

$$\sum_{k=u_n}^{v_n} A_{nk} E^{\mathcal{B}_n}|X_{nk} - Y_{nk}| = \sum_{k=u_n}^{v_n} A_{nk} E^{\mathcal{B}_n}\big(|X_{nk}| - h(n)\big) I\big[|X_{nk}| > h(n)\big] \to 0 \quad \text{a.s.}$$

as $n \to \infty$. Thus $E^{\mathcal{B}_n}|S_{1n}| \to 0$ a.s. and $E^{\mathcal{B}_n}|S_{3n}| \to 0$ a.s. as $n \to \infty$.

For S_{2n} we will initially prove that $E^{\mathcal{B}_n} S_{2n}^2 \to 0$ a.s. as $n \to \infty$. Note that for $n \geq 1$,

$$\begin{aligned}
0 &\leq E^{\mathcal{B}_n}\left[\sum_{k=u_n}^{v_n} A_{nk}\big(Y_{nk} - E^{\mathcal{B}_n} Y_{nk}\big)\right]^2 \\
&= \sum_{k=u_n}^{v_n} A_{nk}^2 E^{\mathcal{B}_n}\big(Y_{nk} - E^{\mathcal{B}_n} Y_{nk}\big)^2 + \sum_{j \neq k} A_{nj} A_{nk}\big[E^{\mathcal{B}_n}(Y_{nj} Y_{nk}) - E^{\mathcal{B}_n} Y_{nj} E^{\mathcal{B}_n} Y_{nk}\big] \\
&\leq \sum_{k=u_n}^{v_n} A_{nk}^2 E^{\mathcal{B}_n} Y_{nk}^2 + \sum_{j \neq k} A_{nj} A_{nk}\big[E^{\mathcal{B}_n}(Y_{nj} Y_{nk}) - E^{\mathcal{B}_n} Y_{nj} E^{\mathcal{B}_n} Y_{nk}\big] \\
&= B_{1n} + B_{2n}, \quad \text{say.}
\end{aligned}$$

But noting that $|Y_{nk}| = \min\{|X_{nk}|, h(n)\}$, $u_n \leq k \leq v_n$, $n \geq 1$, we have

$$\begin{aligned}
B_{1n} &\leq \sum_{k=u_n}^{v_n} A_{nk}^2 h(n) E^{\mathcal{B}_n}|X_{nk}| \\
&\leq h(n)\Big(\sup_{u_n \leq k \leq v_n} A_{nk}\Big) \sum_{k=u_n}^{v_n} A_{nk} E^{\mathcal{B}_n}|X_{nk}| \to 0 \quad \text{a.s. as } n \to \infty.
\end{aligned}$$

With regard to B_{2n}, taking into account that continuous truncation preserves $\mathcal{B}_n$-CNQD (Lemma 2), by applying Lemma 1 we get

$$E^{\mathcal{B}_n}(Y_{nj} Y_{nk}) - E^{\mathcal{B}_n} Y_{nj} E^{\mathcal{B}_n} Y_{nk} \leq 0, \quad j \neq k, \quad \text{a.s. for each } n \in \mathbf{N},$$

and hence

$$0 \le E^{\mathcal{B}_n} S_{2n}^2 \le B_{1n} \to 0 \quad \text{a.s. as } n \to \infty.$$

We have $E^{\mathcal{B}_n} S_{2n}^2 \to 0$ a.s. and so $E^{\mathcal{B}_n}|S_{2n}| \to 0$ a.s. as $n \to \infty$ since $(E^{\mathcal{B}_n}|S_{2n}|)^2 \le E^{\mathcal{B}_n} S_{2n}^2$ a.s., $n \ge 1$, by Jensen's inequality for conditional expectations (see, e.g., Chow and Teicher 1997, p. 217). Thus we have shown that $E^{\mathcal{B}_n}|S_n| \to 0$ a.s. as $n \to \infty$. □

Remark 5 (i) In the particular case $A_{nk} \equiv a_{nk}$ (constants) with $\sup_n \sum_{k=u_n}^{v_n} |a_{nk}| \le C$ for some constant $C > 0$, and $\mathcal{B}_n = \{\emptyset, \Omega\}$ for every $n \in \mathbf{N}$, the preceding theorem reduces to Theorem 2.2 of Yuan and Tao (2008), which is an improvement of Theorem 1 of Ordóñez Cabrera and Volodin (2005).

(ii) As conditional pairwise independence is a particular case of CNQD, specializing Theorem 1 to an array of row-wise pairwise $\mathcal{B}_n$-conditionally independent random variables extends Theorem 2.2 of Chandra and Goswami (2006) and extends and improves Corollary 1 in Ordóñez Cabrera and Volodin (2005) to this much wider scope.

In many theoretical and practical situations, the random variables $\{X_{nk}\}$ are restricted to be non-negative. In the next theorem, we prove that for non-negative $\{X_{nk}\}$ the condition of $\mathcal{B}_n$-CNQD can be replaced by the weaker condition of non-positive conditional correlation. Theorem 2 extends Theorems 1 and 2 of Ordóñez Cabrera and Volodin (2005).

Theorem 2 *Let $\{X_{nk}, u_n \le k \le v_n, n \ge 1\}$ be an array of non-negative random variables with* $\mathrm{Cov}^{\mathcal{B}_n}(X_{nj}, X_{nk}) \le 0$, $j \ne k$, *for each $n \ge 1$ and let $\{A_{nk}, u_n \le k \le v_n, n \ge 1\}$ be an array of non-negative random variables such that, for each $n \in \mathbf{N}$, the $\{A_{nk}, u_n \le k \le v_n\}$ are $\mathcal{B}_n$-measurable. Suppose that*

(a) *$\{X_{nk}\}$ is $\mathcal{B}_n$-CR-h-integrable concerning the array $\{A_{nk}\}$,*
(b) *$h(n)(\sup_{u_n \le k \le v_n} A_{nk}) \to 0$ a.s. as $n \to \infty$.*

Let $S_n = \sum_{k=u_n}^{v_n} A_{nk}(X_{nk} - E^{\mathcal{B}_n} X_{nk})$, $n \ge 1$. Then $E^{\mathcal{B}_n}|S_n| \to 0$ a.s. as $n \to \infty$.

Proof The proof is similar to that of Theorem 1. For each $n \in \mathbf{N}$ and $u_n \le k \le v_n$, let

$$Y_{nk} = X_{nk} I\big[X_{nk} \le h(n)\big] + h(n) I\big[X_{nk} > h(n)\big]$$

and let S_{1n}, S_{2n}, and S_{3n} be defined as in the proof in Theorem 1.

In this case, $X_{nk} - Y_{nk} = (X_{nk} - h(n)) I[X_{nk} > h(n)]$, and so

$$E^{\mathcal{B}_n}|S_{1n}| = E^{\mathcal{B}_n} S_{1n} = -S_{3n} \le \sum_{k=u_n}^{v_n} A_{nk} E^{\mathcal{B}_n}\big(X_{nk} - h(n)\big) I\big[X_{nk} > h(n)\big] \to 0 \quad \text{a.s.}$$

as $n \to \infty$.

For S_{2n} we will prove that $E^{\mathcal{B}_n} S_{2n}^2 \to 0$ a.s. as $n \to \infty$ which gives $E^{\mathcal{B}_n}|S_{2n}| \to 0$ a.s. as $n \to \infty$ as in the proof of Theorem 1. Note that $E^{\mathcal{B}_n} S_{2n}^2 = B_{1n} + B_{2n}$ as in Theorem 1, and $B_{1n} \to 0$ a.s. as $n \to \infty$ in the same way as previously.

Next, it suffices to show that $\limsup_{n\to\infty} B_{2n} \le 0$ a.e. Because of the non-negativity of the random variables X_{nk} and A_{nk} and the hypothesis of non-positive conditional correlation of X_{nj} and X_{nk}, $j \neq k$, we have

$$\begin{aligned}
B_{2n} &= \sum_{j\neq k} A_{nj}A_{nk}\big[E^{\mathcal{B}_n}(Y_{nj}Y_{nk}) - E^{\mathcal{B}_n}Y_{nj}E^{\mathcal{B}_n}Y_{nk}\big] \\
&\le \sum_{j\neq k} A_{nj}A_{nk}\big[E^{\mathcal{B}_n}(X_{nj}X_{nk}) - E^{\mathcal{B}_n}Y_{nj}E^{\mathcal{B}_n}Y_{nk}\big] \\
&\le \sum_{j\neq k} A_{nj}A_{nk}\big(E^{\mathcal{B}_n}X_{nj}E^{\mathcal{B}_n}X_{nk} - E^{\mathcal{B}_n}Y_{nj}E^{\mathcal{B}_n}Y_{nk}\big) \\
&\le \sum_{j,k=u_n}^{v_n} A_{nj}A_{nk}\big[\big(E^{\mathcal{B}_n}X_{nj} - E^{\mathcal{B}_n}Y_{nj}\big)E^{\mathcal{B}_n}X_{nk} + \big(E^{\mathcal{B}_n}X_{nk} - E^{\mathcal{B}_n}Y_{nk}\big)E^{\mathcal{B}_n}Y_{nj}\big] \\
&= \left(\sum_{j=u_n}^{v_n} A_{nj}E^{\mathcal{B}_n}\big(X_{nj}-h(n)\big)I\big[X_{nj}>h(n)\big]\right)\left(\sum_{k=u_n}^{v_n} A_{nk}E^{\mathcal{B}_n}X_{nk}\right) \\
&\quad + \left(\sum_{j=u_n}^{v_n} A_{nj}E^{\mathcal{B}_n}Y_{nj}\right)\left(\sum_{k=u_n}^{v_n} A_{nk}E^{\mathcal{B}_n}\big(X_{nk}-h(n)\big)I\big[X_{nk}>h(n)\big]\right) \\
&\le 2\left(\sum_{j=u_n}^{v_n} A_{nj}E^{\mathcal{B}_n}X_{nj}\right)\left(\sum_{k=u_n}^{v_n} A_{nk}E^{\mathcal{B}_n}\big(X_{nk}-h(n)\big)I\big[X_{nk}>h(n)\big]\right) \to 0
\end{aligned}$$

a.s. as $n \to \infty$. □

Remark 6 In the same way as we commented on in Remark 5, Theorem 2 is an extension and an improvement of Theorem 1 in Chandra and Goswami (2006) and Theorem 2 in Ordóñez Cabrera and Volodin (2005).

Perhaps the most fruitful concept in order to study short range dependence is the concept of strong-mixing which was introduced by Rosenblatt (1956) as follows:

Definition A sequence $\{X_n, n \ge 1\}$ of random variables is said to be *strong-mixing* if there exists a non-negative sequence $\{\alpha_i\}$ converging to 0 and such that $|P(A\cap B) - P(A)P(B)| \le \alpha_i$ for all $A \in \sigma(X_1, X_2, \ldots, X_k)$, $B \in \sigma(X_{k+i}, X_{k+i+1}, \ldots)$ and $k \ge 1, i \ge 1$.

The essence behind this definition is that $(X_1, X_2, \ldots, X_k)$ and $(X_{k+i}, X_{k+i+1}, \ldots)$ are approximately independent for all sufficiently large i and all $k \ge 1$.

Prakasa Rao (2009) extends this concept to the conditional case and introduces the concept of conditional strong-mixing for a sequence of random variables, which also generalizes the concept of conditional independence. Also Prakasa Rao (2009) constructs an example of a conditionally strong mixing sequence.

Definition Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{A}$. Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on $(\Omega, \mathcal{A}, P)$. The sequence $\{X_n, n \geq 1\}$ is said to be *conditionally strong-mixing* ($\mathcal{B}$*-strong-mixing*) if there exist non-negative $\mathcal{B}$-measurable random variables $\alpha_i^{\mathcal{B}}$ converging to 0 a.s. as $i \to \infty$ such that

$$\left|P^{\mathcal{B}}(A \cap B) - P^{\mathcal{B}}(A)P^{\mathcal{B}}(B)\right| \leq \alpha_i^{\mathcal{B}} \quad \text{a.s.}$$

for all $A \in \sigma(X_1, X_2, \ldots, X_k)$, $B \in \sigma(X_{k+i}, X_{k+i+1}, \ldots)$ and $k \geq 1, i \geq 1$.

The following covariance inequality holds for $\mathcal{B}$-strong-mixing sequences of random variables (see Prakasa Rao 2009):

Lemma 3 *Let $\{X_n, n \geq 1\}$ be a $\mathcal{B}$-strong-mixing sequence of random variables with mixing coefficient $\alpha_n^{\mathcal{B}}$ defined on a probability space $(\Omega, \mathcal{A}, P)$. Suppose that a random variable Y is measurable with respect to $\sigma(X_1, X_2, \ldots, X_k)$ and bounded by a $\mathcal{B}$-measurable function C, and let Z be a random variable measurable with respect to $\sigma(X_{k+i}, X_{k+i+1}, \ldots)$ and bounded by a $\mathcal{B}$-measurable function D. Then*

$$\left|E^{\mathcal{B}}(YZ) - E^{\mathcal{B}}(Y)E^{\mathcal{B}}(Z)\right| \leq 4CD\alpha_i^{\mathcal{B}} \quad a.s.$$

The next theorem is a conditional mean convergence theorem for randomly weighted sums of $\mathcal{B}_n$-strong-mixing sequences of random variables and it extends Theorem 3 in Ordóñez Cabrera and Volodin (2005) to this conditional case of dependence.

Theorem 3 *Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of random variables such that for each $n \geq 1$ the row $\{X_{nk}, u_n \leq k \leq v_n\}$ is a $\mathcal{B}_n$-strong-mixing sequence of random variables with*

$$\limsup_{n\to\infty} \sum_{i=1}^{v_n - u_n} \alpha_i^{\mathcal{B}_n} < \infty \quad a.s.$$

Let $\{A_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of non-negative random variables such that, for each $n \in \mathbf{N}$, the $\{A_{nk}, u_n \leq k \leq v_n\}$ are $\mathcal{B}_n$-measurable. Suppose that for each $n \in \mathbf{N}$ the array $\{A_{nk}\}$ is row-wise non-increasing a.s., i.e., $A_{nj} \leq A_{ni}$ a.s. if $i < j$. Suppose that

(a) *$\{X_{nk}\}$ is $\mathcal{B}_n$-CR-h-integrable concerning the array $\{A_{nk}\}$,*
(b) *$h^2(n)\sum_{u_n}^{v_n} {A_{nk}}^2 \to 0$ a.s. as $n \to \infty$.*

Let $S_n = \sum_{k=u_n}^{v_n} A_{nk}(X_{nk} - E^{\mathcal{B}_n} X_{nk})$, $n \geq 1$. Then $E^{\mathcal{B}_n}|S_n| \to 0$ a.s. as $n \to \infty$.

Proof The proof is the same as in Theorem 2 concerning S_{1n}, S_{3n}, and B_{1n}. Thus we only need to prove that

$$\limsup_{n\to\infty} \sum_{\substack{k,j=u_n \\ k<j}}^{v_n} A_{nj}A_{nk}\left[E^{\mathcal{B}_n}(Y_{nj}Y_{nk}) - E^{\mathcal{B}_n}Y_{nj}E^{\mathcal{B}_n}Y_{nk}\right] \leq 0 \quad \text{a.s.}$$

To this end, for all $n \geq 1$

$$\sum_{\substack{k,j=u_n \\ k<j}}^{v_n} A_{nj} A_{nk} \big[E^{\mathcal{B}_n}(Y_{nj} Y_{nk}) - E^{\mathcal{B}_n} Y_{nj} E^{\mathcal{B}_n} Y_{nk} \big]$$

$$= \sum_{i=1}^{v_n - u_n} \sum_{k=u_n}^{v_n - i} A_{nk} A_{n(k+i)} \big[E^{\mathcal{B}_n}(Y_{nk} Y_{n(k+i)}) - E^{\mathcal{B}_n} Y_{nk} E^{\mathcal{B}_n} Y_{n(k+i)} \big]$$

$$\leq 4h^2(n) \sum_{i=1}^{v_n - u_n} \sum_{k=u_n}^{v_n - i} A_{nk}^2 \alpha_i^{\mathcal{B}_n} \leq 4h^2(n) \sum_{k=u_n}^{v_n} A_{nk}^2 \sum_{i=1}^{v_n - u_n} \alpha_i^{\mathcal{B}_n},$$

(by Lemma 3 and $\{A_{nk}\}$ being row-wise non-increasing) and this last expression converges to 0 a.s. as $n \to \infty$. □

4 Conditional strongly residual *h*-integrability

In order to obtain a conditional strong convergence result, we introduce the concept of conditional strongly residual h-integrability relative to the sequence $\mathcal{B}_n$ as follows. Let $0 < h(n) \uparrow \infty$.

Definition Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ and $\{A_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be two arrays of random variables. The array $\{X_{nk}\}$ is said to be *conditionally strongly residually h-integrable relative to* $\mathcal{B}_n$ (*$\mathcal{B}_n$-CSR-h-integrable*, for short) *concerning the array* $\{A_{nk}\}$ if the following conditions hold:

(a)

$$\sup_{n\geq 1} \sum_{k=u_n}^{v_n} |A_{nk}| E^{\mathcal{B}_n} |X_{nk}| < \infty \quad \text{a.s.},$$

(b)

$$\sum_{n=1}^{\infty} \sum_{k=u_n}^{v_n} |A_{nk}| E^{\mathcal{B}_n} \big(|X_{nk}| - h(n) \big) I \big[|X_{nk}| > h(n) \big] < \infty \quad \text{a.s.}$$

Remark 7 If $A_{nk} \equiv a_{nk}$ are constants, and $\mathcal{B}_n = \{\emptyset, \Omega\}$ for all $n \in \mathbf{N}$, the preceding definition reduces to the following new concept of *strongly residual h-integrability concerning the array of constants* $\{a_{nk}\}$:

Definition Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of random variables and $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ an array of constants. The array $\{X_{nk}\}$ is said to be *strongly residually h-integrable* (*SR-h-integrable*, for short) *concerning the array of constants* $\{a_{nk}\}$ if the following conditions hold:

(a)

$$\sup_{n\geq 1}\sum_{k=u_n}^{v_n}|a_{nk}|E|X_{nk}|<\infty,$$

(b)

$$\sum_{n=1}^{\infty}\sum_{k=u_n}^{v_n}|a_{nk}|E\big(|X_{nk}|-h(n)\big)I\big[|X_{nk}|>h(n)\big]<\infty.$$

Remark 8 It is immediate that the concept of $\mathcal{B}_n$-CSR-h-integrability is stronger than the concept of $\mathcal{B}_n$-CR-h-integrability. Likewise the unconditional concept of SR-h-integrability is stronger than the concept of R-h-integrability.

We will now establish a strong version of Theorem 1 under the condition of $\mathcal{B}$-CSR-h-integrability (i.e., when $\mathcal{B}_n=\mathcal{B}$, a sub-$\sigma$-algebra of $\mathcal{A}$, for all $n\in\mathbf{N}$).

Theorem 4 *Let* $\{X_{nk}, u_n\leq k\leq v_n, n\geq 1\}$ *be an array of row-wise pairwise* $\mathcal{B}$-*CNQD random variables. Let* $\{A_{nk}, u_n\leq k\leq v_n, n\geq 1\}$ *be an array of non-negative* $\mathcal{B}$-*measurable random variables. Suppose that*

(a) $\{X_{nk}\}$ *is* $\mathcal{B}$-*CSR*-h-*integrable concerning the array* $\{A_{nk}\}$,
(b) $\sum_{n=1}^{\infty}h(n)(\sup_{u_n\leq k\leq v_n}A_{nk})<\infty$ *a.s.*

Then $S_n=\sum_{k=u_n}^{v_n}A_{nk}(X_{nk}-E^{\mathcal{B}}X_{nk})\to 0$ *a.s. as* $n\to\infty$.

Proof For each $n\in\mathbf{N}$, $u_n\leq k\leq v_n$, let Y_{nk}, S_{1n}, S_{2n}, and S_{3n} be as in the proof of Theorem 1 by putting $\mathcal{B}_n\equiv\mathcal{B}$. Then for each $n\in\mathbf{N}$, we can write $S_n=S_{1n}+S_{2n}+S_{3n}$, and we will estimate each of these terms separately.

Condition (a) implies via the non-negativity of every summand that

$$E^{\mathcal{B}}\left(\sum_{n=1}^{\infty}\sum_{k=u_n}^{v_n}|A_{nk}|\big(|X_{nk}|-h(n)\big)I\big[|X_{nk}|>h(n)\big]\right)<\infty\quad\text{a.s.},$$

which, in turn, implies that

$$\sum_{n=1}^{\infty}\sum_{k=u_n}^{v_n}|A_{nk}|\big(|X_{nk}|-h(n)\big)I\big[|X_{nk}|>h(n)\big]=\sum_{n=1}^{\infty}\sum_{k=u_n}^{v_n}A_{nk}|X_{nk}-Y_{nk}|<\infty\quad\text{a.s.}$$

Hence

$$|S_{1n}|\leq\sum_{k=u_n}^{v_n}A_{nk}|X_{nk}-Y_{nk}|\to 0\quad\text{a.s.}$$

Next, again by condition (a), we have

$$|S_{3n}|\leq\sum_{k=u_n}^{v_n}A_{nk}E^{\mathcal{B}}|Y_{nk}-X_{nk}|\to 0\quad\text{a.s. as }n\to\infty,$$

and so $S_{3n} \to 0 \quad$ a.s. as $n \to \infty$.

Now we will prove that $S_{2n} \to 0$ a.s. as $n \to \infty$. By the conditional Markov inequality, for all $\varepsilon > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} P^{\mathcal{B}}\big[|S_{2n}| > \varepsilon\big] &\le \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty} E^{\mathcal{B}}|S_{2n}|^2 \\ &= \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\Bigg(\sum_{k=u_n}^{v_n} A_{nk}^2 E^{\mathcal{B}}\big(Y_{nk} - E^{\mathcal{B}}Y_{nk}\big)^2 \\ &\quad + \sum_{j\neq k} A_{nj}A_{nk}\big[E^{\mathcal{B}}(Y_{nj}Y_{nk}) - E^{\mathcal{B}}Y_{nj}E^{\mathcal{B}}Y_{nk}\big]\Bigg) \quad \text{a.s.}\end{aligned}$$

But $E^{\mathcal{B}}(Y_{nj}Y_{nk}) - E^{\mathcal{B}}Y_{nj}EY_{nk} \le 0$, $j \neq k$, a.s. for each $n \ge 1$, according to Lemma 1.

As $\sum_{n=1}^{\infty} P^{\mathcal{B}}[|S_{2n}| > \varepsilon] \ge 0$ a.s., if we prove that

$$\sum_{n=1}^{\infty}\sum_{k=u_n}^{v_n} A_{nk}^2 E^{\mathcal{B}}\big(Y_{nk} - E^{\mathcal{B}}Y_{nk}\big)^2 < \infty \quad \text{a.s.}, \tag{4.1}$$

then $\sum_{n=1}^{\infty}\sum_{j\neq k} A_{nj}A_{nk}[E^{\mathcal{B}}(Y_{nj}Y_{nk}) - E^{\mathcal{B}}Y_{nj}E^{\mathcal{B}}Y_{nk}]$ will be an a.s. convergent series with non-positive terms.

To accomplish (4.1), note that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=u_n}^{v_n} A_{nk}^2 E^{\mathcal{B}}\big(Y_{nk} - E^{\mathcal{B}}Y_{nk}\big)^2 &\le \sum_{n=1}^{\infty}\sum_{k=u_n}^{v_n} A_{nk}^2 E^{\mathcal{B}}Y_{nk}^2 \\ &= \sum_{n=1}^{\infty}\sum_{k=u_n}^{v_n} A_{nk}^2 E^{\mathcal{B}}\big(X_{nk}^2 I\big[|X_{nk}| \le h(n)\big] + h^2(n)I\big[|X_{nk}| > h(n)\big]\big) \\ &\le \sum_{n=1}^{\infty}\Big(h(n)\sup_{u_n\le k\le v_n} A_{nk}\Big)\sum_{k=u_n}^{v_n} A_{nk}E^{\mathcal{B}}|X_{nk}| < \infty \quad \text{a.s.},\end{aligned}$$

proving (4.1). Therefore,

$$\sum_{n=1}^{\infty} P^{\mathcal{B}}\big[|S_{2n}| > \varepsilon\big] \le \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\sum_{k=u_n}^{v_n} A_{nk}^2 E^{\mathcal{B}}\big(Y_{nk} - E^{\mathcal{B}}Y_{nk}\big)^2 < \infty \quad \text{a.s.}$$

and so by the conditional Borel–Cantelli lemma,

$$P^{\mathcal{B}}\big[\limsup\big[|S_{2n}| > \varepsilon\big]\big] = 0 \quad \text{a.s.}$$

Consequently, $S_{2n} \to 0$ a.s. since the $P^{\mathcal{B}}$-null sets and the P-null sets coincide.

Thus, we have proved that $S_n = S_{1n} + S_{2n} + S_{3n} \to 0$ a.s. □

A particular case of pairwise $\mathcal{B}$-CNQD random variables is the case of pairwise $\mathcal{B}$-independent random variables. Thus we have the following corollary to Theorem 4.

Corollary 1 *Let $\{X_{nk}, u_n \le k \le v_n, n \ge 1\}$ be an array of row-wise pairwise $\mathcal{B}$-independent random variables. Let $\{A_{nk}, u_n \le k \le v_n, n \ge 1\}$ be an array of non-negative $\mathcal{B}$-measurable random variables. Suppose that*

(a) *$\{X_{nk}\}$ is $\mathcal{B}$-CSR-h-integrable concerning the array $\{A_{nk}\}$,*
(b) *$\sum_{n=1}^{\infty} h(n)(\sup_{u_n \le k \le v_n} A_{nk}) < \infty$ a.s.*

Then $S_n = \sum_{k=u_n}^{v_n} A_{nk}(X_{nk} - E^{\mathcal{B}} X_{nk}) \to 0$ a.s. as $n \to \infty$.

Remark 9 As was stated previously, Theorem 4 is a strong version of Theorem 1, but an attentive reading of its proof shows that the a.s. finiteness of

$$\sum_{n=1}^{\infty} \sum_{k=u_n}^{v_n} A_{nk}^2 E^{\mathcal{B}} \big(Y_{nk} - E^{\mathcal{B}} Y_{nk}\big)^2$$

can be proved by replacing condition (b) and the condition

$$\sup_{n\ge 1} \sum_{k=u_n}^{v_n} A_{nk} E^{\mathcal{B}} |X_{nk}| < \infty \quad \text{a.s.}$$

(in the definition of $\mathcal{B}$-CSR-h-integrability) by the single condition

$$\sum_{n=1}^{\infty} h(n) \sum_{k=u_n}^{v_n} A_{nk}^2 E^{\mathcal{B}} |X_{nk}| < \infty \quad \text{a.s.,}$$

which is weaker than both conditions together.

Thus, we can state a slightly stronger version of Theorem 4 as follows:

Theorem 5 *Let $\{X_{nk}, u_n \le k \le v_n, n \ge 1\}$ be an array of row-wise pairwise $\mathcal{B}$-CNQD random variables. Let $\{A_{nk}, u_n \le k \le v_n, n \ge 1\}$ be an array of non-negative $\mathcal{B}$-measurable random variables. Suppose that*

(a)

$$\sum_{n=1}^{\infty} \sum_{k=u_n}^{v_n} A_{nk} E^{\mathcal{B}} \big(|X_{nk}| - h(n)\big) I\big[|X_{nk}| > h(n)\big] < \infty \quad \text{a.s.,}$$

(b)

$$\sum_{n=1}^{\infty} h(n) \sum_{k=u_n}^{v_n} A_{nk}^2 E^{\mathcal{B}} |X_{nk}| < \infty \quad \text{a.s.}$$

Then $S_n = \sum_{k=u_n}^{v_n} A_{nk}(X_{nk} - E^{\mathcal{B}} X_{nk}) \to 0$ a.s. as $n \to \infty$.

An analogous version of Corollary 1 apropos of Theorem 5 also holds.

Acknowledgements The authors are grateful to the Associate Editor and the Referees for carefully reading the manuscript and for offering extremely valuable comments and suggestions which enabled the authors to substantially improve the paper.

References

Aas K, Czado C, Frigessi A, Bakken H (2009) Pair-copula constructions of multiple dependence. Insur Math Econ 44:182–198

Basawa IV, Prakasa Rao BLS (1980) Statistical inference for stochastic processes. Academic Press, London

Chandra TK, Goswami A (2003) Cesàro α-integrability and laws of large numbers I. J Theor Probab 16:655–669

Chandra TK, Goswami A (2006) Cesàro α-integrability and laws of large numbers-II. J Theor Probab 19:789–816

Chow YS, Teicher H (1997) Probability theory: independence, interchangeability, martingales, 3rd edn. Springer, New York. xxii+488 pp

Leek JT (2011) Asymptotic conditional singular value decomposition for high-dimensional genomic data. Biometrics. doi:10.1111/j.1541-0420.2010.01455.x

Lehmann EL (1966) Some concepts of dependence. Ann Math Stat 37:1137–1153

Ordóñez Cabrera M (1994) Convergence of weighted sums of random variables and uniform integrability concerning the weights. Collect Math 45:121–132

Ordóñez Cabrera M, Volodin AI (2005) Mean convergence theorems and weak laws of large numbers for weighted sums of random variables under a condition of weighted integrability. J Math Anal Appl 305:644–658

Prakasa Rao BLS (2009) Conditional independence, conditional mixing and conditional association. Ann Inst Stat Math 61:441–460

Rosenblatt M (1956) A central limit theorem and a strong mixing condition. Proc Natl Acad Sci USA 42:43–47

Roussas GG (2008) On conditional independence, mixing, and association. Stoch Anal Appl 26:1274–1309

Shen X, Lin Z, Zhang Y (2009) Uniform estimate for maximum of randomly weighted sums with applications to ruin theory. Methodol Comput Appl Probab 11:669–685

Sheremet O, Lucas A (2009) Global loss diversification in the insurance sector. Insur Math Econ 44:415–425

Wang D, Tang Q (2006) Tail probabilities of randomly weighted sums of random variables with dominated variation. Stoch Models 22:253–272

Weng C, Zhang Y, Tan KS (2009) Ruin probabilities in a discrete time risk model with dependent risks of heavy tail. Scand Actuar J 3:205–218

Yuan D, Tao B (2008) Mean convergence theorems for weighted sums of arrays of residually h-integrable random variables concerning the weights under dependence assumptions. Acta Appl Math 103:221–234