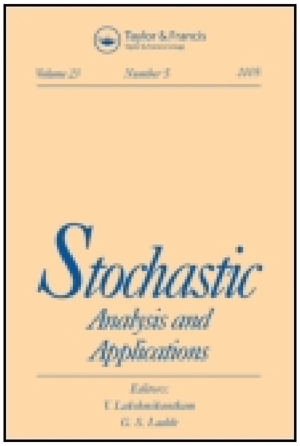

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A Note on the Rate of Complete Convergence for Weighted Sums of Arrays of Banach Space Valued Random Elements

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A Note on the Rate of Complete Convergence for Weighted Sums of Arrays of Banach Space Valued Random Elements

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A rate of complete convergence for weighted sums of arrays of rowwise independent Banach space valued random elements was obtained by Ahmed et al. [1]. Recently, Sung and Volodin [2], Chen et al. [3], and Kim and Ko [4] solved an open question posed by Ahmed et al. In this article, we improve and complement the result of Ahmed et al. The method used in this article is simpler than those in Ahmed et al., Sung and Volodin, Chen et al., and Kim and Ko.

Keywords Array of random elements; Complete convergence; Convergence in probability; Rowwise independence; Weighted sums.

AMS Subject Classification 60B12; 60F05; 60F15.

1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [5] as follows. A sequence $\{U_n, n \geq 1\}$ of random variables converges completely to the constant θ if $\sum_{n=1}^{\infty} P(|U_n - \theta| > \epsilon) < \infty$ for all $\epsilon > 0$. By the Borel–Cantelli lemma, this implies that $U_n \to \theta$ almost surely (a.s.). The converse is true if $\{U_n, n \geq 1\}$ are independent random variables. Hsu and Robbins proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite.

This result has been generalized and extended in several directions. Some of these generalizations are in a Banach space setting (e.g., see, [1, 6–10]). A sequence of Banach space valued random elements is said to converge completely to the

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0 element of the Banach space if the corresponding sequence of norms converges completely to 0.

Hu et al. [6] presented a general result establishing complete convergence for the row sums of an array of rowwise independent but not necessarily identically distributed Banach space valued random elements. Using this, Hu et al. [11] obtained the following complete convergence result. Theorem A generalizes results of [5, 7, 8, 10].

Rowwise independence means that the random elements within each row are independent but that no independence is assumed between rows.

In the following, we assume that $\{X_{ni}, i \geq 1, n \geq 1\}$ is an array of rowwise independent random elements in a real separable Banach space and $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of real numbers. We recall that the array $\{X_{ni}, i \geq 1, n \geq 1\}$ is said to be *stochastically dominated* by a random variable X if

$$P(\|X_{ni}\| > x) \leq CP(|X| > x) \quad \text{for all } x > 0 \text{ and for all } i \geq 1 \text{ and } n \geq 1,$$

where C is a positive constant.

Theorem A ([6]). *Suppose that the array $\{X_{ni}, i \geq 1, n \geq 1\}$ is stochastically dominated by a random variable X. Assume that*

$$\sup_{i\geq 1} |a_{ni}| = O(n^{-\gamma}) \quad \textit{for some } \gamma > 0 \tag{1.1}$$

and

$$\sum_{i=1}^{\infty} |a_{ni}| = O(n^{\alpha}) \quad \textit{for some } \alpha \in [0, \gamma).$$

If $E|X|^{1+(1+\alpha+\beta)/\gamma} < \infty$ for some $\beta \in (-1, \gamma - \alpha - 1]$ and $\sum_{i=1}^{\infty} a_{ni}X_{ni} \to 0$ in probability, then

$$\sum_{n=1}^{\infty} n^{\beta} P\left(\left\| \sum_{i=1}^{\infty} a_{ni}X_{ni} \right\| > \epsilon\right) < \infty \quad \textit{for all } \epsilon > 0. \tag{1.2}$$

It is assumed in Theorem A that $\sum_{i=1}^{\infty} a_{ni}X_{ni}$ converges a.s. for all $n \geq 1$, since the a.s. convergence is not automatic from the hypotheses. Ahmed et al. [1] established the following more general result than Theorem A.

Theorem B ([1]). *Suppose that the array $\{X_{ni}, i \geq 1, n \geq 1\}$ is stochastically dominated by a random variable X. Assume that* (1.1) *holds and*

$$\sum_{i=1}^{\infty} |a_{ni}| = O(n^{\alpha}) \quad \textit{for some } \alpha < \gamma. \tag{1.3}$$

Let β be such that $\alpha + \beta \neq -1$ and let $\delta > 1$ be such that $1 + \alpha/\gamma < \delta \leq 2$. If $E|X|^{v} < \infty$, where $v = \max\{1 + (1 + \alpha + \beta)/\gamma, \delta\}$, and assume $\sum_{i=1}^{\infty} a_{ni}X_{ni} \to 0$ in probability. Then (1.2) *holds.*

Note that there was a typographical error in Ahmed et al. [1] (the relation $\delta > 0$ should be $\delta > 1$). If $\beta < -1$, then the conclusions of Theorems A and B are immediate. Hence these theorems are of interest only for $\beta \geq -1$. Note that the condition for v in [1] is as follows:

$$v = \begin{cases} 1 + \dfrac{1+\alpha+\beta}{\gamma}, & \text{if } 1+\alpha+\beta > 0 \text{ and } \beta > -1, \\ \delta(\delta > 1+\alpha/\gamma), & \text{if } 1+\alpha+\beta > 0 \text{ and } \beta = -1, \\ \delta(\delta > 1), & \text{if } 1+\alpha+\beta < 0. \end{cases} \tag{1.4}$$

The case of $\alpha + \beta = 1$ was not treated by Ahmed et al. [1]. In particular, Ahmed et al. [1] conjectured that if $\beta = -1$, then the moment condition $E|X|^v < \infty$ can be replaced by the weaker condition $E|X|^{1+\alpha/\gamma} \log^\rho(|X|) < \infty$ for some $\rho > 0$, where $\log(x) = \max\{1, \ln(x)\}$ and $\ln(x)$ denotes the natural logarithm function. When $\alpha > 0$, Sung and Volodin [2] gave a positive answer to this problem as follows:

Theorem C ([2]). *Suppose that the array $\{X_{ni}, i \geq 1, n \geq 1\}$ is stochastically dominated by a random variable X. Assume that* (1.1) *holds and*

$$\sum_{i=1}^{\infty} |a_{ni}|^\theta = O(n^\alpha) \ \text{ for some } \alpha > 0 \text{ and } \theta > 0 \text{ such that } \theta + \alpha/\gamma < 2. \tag{1.5}$$

If $E|X|^{\theta+\alpha/\gamma} \log^\rho(|X|) < \infty$ for some $\rho > 0$ and $\sum_{i=1}^\infty a_{ni}X_{ni} \to 0$ in probability, then

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\left\| \sum_{i=1}^{\infty} a_{ni}X_{ni} \right\| > \epsilon \right) < \infty \ \text{ for all } \epsilon > 0.$$

Two years later, Kim and Ko [4] proved the same result as in Theorem C by using the same method. Chen et al. [3] improved the result of Sung and Volodin [2] by proving that the condition $E|X|^{\theta+\alpha/\gamma} \log^\rho(|X|) < \infty (\rho > 0)$ can be replaced by the weaker condition $E|X|^{\theta+\alpha/\gamma} < \infty$.

It is important to compare condition (1.3) with condition (1.5). If $\alpha > 0$, then (1.5) is more general than (1.3). However, condition (1.5) cannot be applied to the case of $\alpha \leq 0$. Thus, in general, two conditions (1.3) and (1.5) are not comparable.

In this article, we improve and complement the result of Ahmed et al. [1]. The method used in this article is simpler than those in [1–4]. The symbol C denotes a positive constant which is not necessarily the same one in each appearance.

2. Preliminaries

In this section, we present some inequalities and elementary results which will be useful in the proof of our main result.

Let B be a real separable Banach space with norm $\| \cdot \|$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A random element (or B-valued random element) is defined to be an $\mathcal{F}$-measurable mapping from Ω to B equipped with the Borel σ-algebra (the σ-algebra generated by the open sets determined by $\| \cdot \|$).

The following inequalities are Banach space analogues of the classical Marcinkiewicz-Zygmund and Rosenthal inequalities and are due to de Acosta [12].

Lemma 2.1. *Let $\{X_i, 1 \le i \le n\}$ be a sequence of independent random elements. Then there exists a positive constant C_p depending only on p such that*

(i) *for* $1 \le p \le 2$,

$$E\left|\left\|\sum_{i=1}^{n} X_i\right\| - E\left\|\sum_{i=1}^{n} X_i\right\|\right|^p \le C_p \sum_{i=1}^{n} E\|X_i\|^p,$$

(ii) *for* $p > 2$,

$$E\left|\left\|\sum_{i=1}^{n} X_i\right\| - E\left\|\sum_{i=1}^{n} X_i\right\|\right|^p \le C_p\left\{\left(\sum_{i=1}^{n} E\|X_i\|^2\right)^{p/2} + \sum_{i=1}^{n} E\|X_i\|^p\right\}.$$

The following lemma is precisely Lemma 2.2(ii) of Hu et al. [11].

Lemma 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of random elements. If $X_n \to 0$ in probability, then for all $x > 0$ and sufficiently large n*

$$P(\|X_n\| > x) \le 2P\left(\left\|X_n^s\right\| > \frac{x}{2}\right),$$

where X^s is a symmetrized version of X.

The next lemma is a modification of a result of Kuelbs and Zinn [13] concerning the relationship between convergence in probability and mean convergence for sums of independent bounded random variables. We refer to Lemma 2.1 of Hu et al. [11] for the proof.

Lemma 2.3. *Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise independent symmetric random elements. Suppose there exists $\delta > 0$ such that $\|X_{ni}\| \le \delta$ almost surely for all $i \ge 1$ and $n \ge 1$. Put $S_n = \sum_{i=1}^{\infty} X_{ni}$. If $S_n \to 0$ in probability, then $E\|S_n\| \to 0$ as $n \to \infty$.*

The following lemma shows that the symmetry assumption in Lemma 2.3 can be dropped without any additional conditions.

Lemma 2.4. *Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise independent random elements. Suppose there exists $\delta > 0$ such that $\|X_{ni}\| \le \delta$ a.s. for all $i \ge 1$ and $n \ge 1$. Put $S_n = \sum_{i=1}^{\infty} X_{ni}$. If $S_n \to 0$ in probability, then $E\|S_n\| \to 0$ as $n \to \infty$.*

Proof. Let X^s be a symmetrized version of X. Observe that, by Lemma 2.2, for sufficiently large n

$$\begin{aligned} E\|S_n\| &= \int_0^\infty P(\|S_n\| > x)dx \\ &\le 2\int_0^\infty P(\|S_n^s\| > x/2)dx \quad \text{(by Lemma 2.2)} \\ &= 4E\|S_n^s\| = 4E\left\|\sum_{i=1}^{\infty} X_{ni}^s\right\|. \end{aligned} \tag{2.1}$$

If $\sum_{i=1}^{\infty} X_{ni} \to 0$ in probability, then $\sum_{i=1}^{\infty} X_{ni}^s \to 0$ in probability. Since $\|X_{ni}^s\| \le 2\delta$, we have by Lemma 2.3 that $E\|\sum_{i=1}^{\infty} X_{ni}^s\| \to 0$. So the result follows by (2.1). □

3. Main Result

Throughout this section, let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise independent random elements. The following theorem is our main result.

Theorem 3.1. *Suppose $\beta \ge -1$. Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise independent random elements which are stochastically dominated by a random variable X. Let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of constants satisfying* (1.1) *and* (1.3). *Assume that $\sum_{i=1}^{\infty} a_{ni}X_{ni} \to 0$ in probability. Then the following statements hold*:

(i) *If $1+\alpha+\beta < 0$ and $E|X| < \infty$, then* (1.2) *holds.*
(ii) *If $1+\alpha+\beta = 0$ and $E|X|\log|X| < \infty$, then* (1.2) *holds.*
(iii) *If $1+\alpha+\beta > 0$ and $E|X|^{1+(1+\alpha+\beta)/\gamma} < \infty$, then* (1.2) *holds.*

Proof. If $1+\alpha+\beta < 0$, then the result can be easily proved by

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\beta} P\left(\left\|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right\| > \epsilon\right) &\le \frac{1}{\epsilon}\sum_{n=1}^{\infty} n^{\beta} E\left\|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right\| \\ &\le C\sum_{n=1}^{\infty} n^{\beta} n^{\alpha} E|X| \quad \text{(by (1.3) and stochastic domination)} \\ &\le CE|X| < \infty.\end{aligned}$$

We now prove the result when $1+\alpha+\beta \ge 0$. To do this, define for $i \ge 1$ and $n \ge 1$

$$X'_{ni} = X_{ni}I(\|X_{ni}\| \le n^{\gamma}), \quad X''_{ni} = X_{ni} - X'_{ni}.$$

First we prove that

$$E\left\|\sum_{i=1}^{\infty} a_{ni}X''_{ni}\right\| \to 0 \tag{3.1}$$

and

$$E\left\|\sum_{i=1}^{\infty} a_{ni}X'_{ni}\right\| \to 0. \tag{3.2}$$

From (1.3) and stochastic domination, we get

$$\begin{aligned}E\left\|\sum_{i=1}^{\infty} a_{ni}X''_{ni}\right\| &\le \sum_{i=1}^{\infty} |a_{ni}|E\|X''_{ni}\| \\ &\le Cn^{\alpha}E|X|I(|X| > n^{\gamma}) \\ &\le C\frac{1}{n^{1+\beta}}E|X|^{1+(1+\alpha+\beta)/\gamma}I(|X| > n^{\gamma}) \to 0,\end{aligned}$$

since $1+\beta \ge 0$ and $E|X|^{1+(1+\alpha+\beta)/\gamma}I(|X| > n^{\gamma}) \to 0$ as $n \to \infty$. Hence, (3.1) holds.

By (3.1) and the hypothesis $\sum_{i=1}^{\infty} a_{ni}X_{ni} \to 0$ in probability,

$$\sum_{i=1}^{\infty} a_{ni}X'_{ni} \to 0 \quad \text{in probability,}$$

which implies (3.2) by $\|a_{ni}X'_{ni}\| = O(1)$ (which follows from (1.1)) and by Lemma 2.4.

Note that, by (3.1) and (3.2), for all sufficiently large $n \geq 1$

$$\begin{aligned} P\left(\left\|\sum_{i=1}^{\infty} a_{ni}X_{ni}\right\| > \epsilon\right) &\leq P\left(\left\|\sum_{i=1}^{\infty} a_{ni}X'_{ni}\right\| > \epsilon/2\right) + P\left(\left\|\sum_{i=1}^{\infty} a_{ni}X''_{ni}\right\| > \epsilon/2\right) \\ &\leq P\left(\left|\left\|\sum_{i=1}^{\infty} a_{ni}X'_{ni}\right\| - E\left\|\sum_{i=1}^{\infty} a_{ni}X'_{ni}\right\|\right| > \epsilon/4\right) \\ &\quad + P\left(\left|\left\|\sum_{i=1}^{\infty} a_{ni}X''_{ni}\right\| - E\left\|\sum_{i=1}^{\infty} a_{ni}X''_{ni}\right\|\right| > \epsilon/4\right). \end{aligned}$$

Hence, it suffices to show that

$$I_1 =: \sum_{n=1}^{\infty} n^{\beta} P\left(\left|\left\|\sum_{i=1}^{\infty} a_{ni}X'_{ni}\right\| - E\left\|\sum_{i=1}^{\infty} a_{ni}X'_{ni}\right\|\right| > \epsilon/4\right) < \infty \tag{3.3}$$

and

$$I_2 =: \sum_{n=1}^{\infty} n^{\beta} P\left(\left|\left\|\sum_{i=1}^{\infty} a_{ni}X''_{ni}\right\| - E\left\|\sum_{i=1}^{\infty} a_{ni}X''_{ni}\right\|\right| > \epsilon/4\right) < \infty. \tag{3.4}$$

We will prove (3.3) and (3.4) with four cases.

Case 1. $1 + \frac{1+\alpha+\beta}{\gamma} = 1$ (i.e., $1 + \alpha + \beta = 0$)

For I_1, we take $t > 0$ such that $1 + (1+\alpha+\beta)/\gamma + t \leq 2$. Then we get by Markov's inequality and Lemma 2.1 that

$$\begin{aligned} I_1 &\leq \left(\frac{4}{\epsilon}\right)^{1+\frac{1+\alpha+\beta}{\gamma}+t} \sum_{n=1}^{\infty} n^{\beta} E\left|\left\|\sum_{i=1}^{\infty} a_{ni}X'_{ni}\right\| - E\left\|\sum_{i=1}^{\infty} a_{ni}X'_{ni}\right\|\right|^{1+\frac{1+\alpha+\beta}{\gamma}+t} \\ &\leq C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=1}^{\infty} E\|a_{ni}X'_{ni}\|^{1+\frac{1+\alpha+\beta}{\gamma}+t} \\ &\leq C\sum_{n=1}^{\infty} n^{\beta} \sup_{i\geq 1} |a_{ni}|^{\frac{1+\alpha+\beta}{\gamma}+t} \sum_{i=1}^{\infty} |a_{ni}| E\|X'_{ni}\|^{1+\frac{1+\alpha+\beta}{\gamma}+t} \\ &\leq C\sum_{n=1}^{\infty} n^{\beta} \sup_{i\geq 1} |a_{ni}|^{\frac{1+\alpha+\beta}{\gamma}+t} \sum_{i=1}^{\infty} |a_{ni}| \\ &\quad \times \left\{E|X|^{1+\frac{1+\alpha+\beta}{\gamma}+t} I(|X| \leq n^{\gamma}) + n^{\gamma(1+\frac{1+\alpha+\beta}{\gamma}+t)} P(|X| > n^{\gamma})\right\} \\ &\leq C\sum_{n=1}^{\infty} \frac{1}{n^{1+\gamma t}} \left\{E|X|^{1+\frac{1+\alpha+\beta}{\gamma}+t} I(|X| \leq n^{\gamma}) + n^{\gamma(1+\frac{1+\alpha+\beta}{\gamma}+t)} P(|X| > n^{\gamma})\right\} \\ &\leq CE|X|^{1+\frac{1+\alpha+\beta}{\gamma}} < \infty. \end{aligned}$$

Here, we used the fact that if a random variable X_i is stochastically dominated by a random variable X, then for all $s > 0$ and $b > 0$

$$E|X_i|^s I(|X_i| \leq b) \leq C\{E|X|^s I(|X| \leq b) + b^s P(|X| > b)\}.$$

For I_2, we get by Markov's inequality, (1.3), and stochastic domination that

$$\begin{aligned}
I_2 &\leq \frac{4}{\epsilon}\sum_{n=1}^{\infty} n^{\beta} E\left|\left\|\sum_{i=1}^{\infty} a_{ni}X''_{ni}\right\| - E\left\|\sum_{i=1}^{\infty} a_{ni}X''_{ni}\right\|\right| \\
&\leq \frac{8}{\epsilon}\sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{\infty}|a_{ni}|E\|X''_{ni}\| \\
&\leq C\sum_{n=1}^{\infty} n^{\beta}n^{\alpha}E|X|I(|X| > n^{\gamma}) \\
&= C\sum_{n=1}^{\infty} n^{\beta+\alpha}\sum_{i=n}^{\infty}E|X|I(i^{\gamma} < |X| \leq (i+1)^{\gamma}) \\
&= C\sum_{i=1}^{\infty}E|X|I(i^{\gamma} < |X| \leq (i+1)^{\gamma})\sum_{n=1}^{i} n^{\beta+\alpha} \\
&\leq CE|X|\log|X| < \infty,
\end{aligned}$$

since $\beta + \alpha = -1$.

Case 2. $1 < 1 + \frac{1+\alpha+\beta}{\gamma} < 2$

As in Case 1, we have that $I_1 \leq CE|X|^{1+\frac{1+\alpha+\beta}{\gamma}} < \infty$.

For I_2, we take $t > 0$ such that $1 + (1+\alpha+\beta)/\gamma - t > 1$. That is, $(1+\alpha+\beta)/\gamma > t > 0$. Then we get by Markov's inequality and Lemma 2.1 that

$$\begin{aligned}
I_2 &\leq \left(\frac{4}{\epsilon}\right)^{1+\frac{1+\alpha+\beta}{\gamma}-t}\sum_{n=1}^{\infty} n^{\beta} E\left|\left\|\sum_{i=1}^{\infty} a_{ni}X''_{ni}\right\| - E\left\|\sum_{i=1}^{\infty} a_{ni}X''_{ni}\right\|\right|^{1+\frac{1+\alpha+\beta}{\gamma}-t} \\
&\leq C\sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{\infty}E\|a_{ni}X''_{ni}\|^{1+\frac{1+\alpha+\beta}{\gamma}-t} \\
&\leq C\sum_{n=1}^{\infty} n^{\beta}\sup_{i\geq 1}|a_{ni}|^{\frac{1+\alpha+\beta}{\gamma}-t}\sum_{i=1}^{\infty}|a_{ni}|E\|X''_{ni}\|^{1+\frac{1+\alpha+\beta}{\gamma}-t} \\
&\leq C\sum_{n=1}^{\infty}\frac{1}{n^{1-\gamma t}}E|X|^{1+\frac{1+\alpha+\beta}{\gamma}-t}I(|X| > n^{\gamma}) \\
&\leq CE|X|^{1+\frac{1+\alpha+\beta}{\gamma}} < \infty.
\end{aligned}$$

Case 3. $1 + \frac{1+\alpha+\beta}{\gamma} = 2$

For I_1, we take $t > 0$ sufficiently large such that $(\gamma-\alpha)(1 + (1+\alpha+\beta)/\gamma + t)/2 > 1 + \beta$. Then we get by Markov's inequality and Lemma 2.1 that

$$I_1 \leq \left(\frac{4}{\epsilon}\right)^{1+\frac{1+\alpha+\beta}{\gamma}+t}\sum_{n=1}^{\infty} n^{\beta} E\left|\left\|\sum_{i=1}^{\infty} a_{ni}X'_{ni}\right\| - E\left\|\sum_{i=1}^{\infty} a_{ni}X'_{ni}\right\|\right|^{1+\frac{1+\alpha+\beta}{\gamma}+t}$$

$$\le C\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{i=1}^{\infty} E\|a_{ni}X'_{ni}\|^2\right)^{(1+(1+\alpha+\beta)/\gamma+t)/2} + C\sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{\infty} E\|a_{ni}X'_{ni}\|^{1+\frac{1+\alpha+\beta}{\gamma}+t}$$
$$=: I_3 + I_4.$$

From (1.1) and (1.3), we have

$$\begin{aligned} I_3 &\le C\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{i=1}^{\infty} a_{ni}^2 E\|X_{ni}\|^2\right)^{(1+(1+\alpha+\beta)/\gamma+t)/2} \\ &\le C\sum_{n=1}^{\infty} n^{\beta}\left(\sup_{i\ge 1}|a_{ni}|\sum_{i=1}^{\infty}|a_{ni}|E\|X_{ni}\|^2\right)^{(1+(1+\alpha+\beta)/\gamma+t)/2} \\ &\le C\sum_{n=1}^{\infty} n^{\beta}\left(\frac{CE|X|^2 n^{\alpha}}{n^{\gamma}}\right)^{(1+(1+\alpha+\beta)/\gamma+t)/2} < \infty, \end{aligned}$$

since $(\gamma-\alpha)(1+(1+\alpha+\beta)/\gamma+t)/2-\beta>1$. Similar to I_1 in Case 1, we have $I_4 \le CE|X|^{1+\frac{1+\alpha+\beta}{\gamma}} < \infty$. Hence, $I_1 < \infty$. Similar to I_2 in Case 2, we have $I_2 \le CE|X|^{1+\frac{1+\alpha+\beta}{\gamma}} < \infty$.

Case 4. $1+\frac{1+\alpha+\beta}{\gamma} > 2$

In this case, we have that $\beta > -1$, since $1+(1+\alpha+\beta)/\gamma < 2+(1+\beta)/\gamma$. As in Case 3, we have that $I_1 \le CE|X|^{1+\frac{1+\alpha+\beta}{\gamma}} < \infty$.

For I_2, we take $t>0$ sufficiently small such that $1+(1+\alpha+\beta)/\gamma - t > 2$. Then we get by Markov's inequality and Lemma 2.1 that

$$I_2 \le \left(\frac{4}{\epsilon}\right)^{1+\frac{1+\alpha+\beta}{\gamma}-t}\sum_{n=1}^{\infty} n^{\beta} E\left|\left\|\sum_{i=1}^{\infty} a_{ni}X''_{ni}\right\| - E\left\|\sum_{i=1}^{\infty} a_{ni}X''_{ni}\right\|\right|^{1+\frac{1+\alpha+\beta}{\gamma}-t}$$
$$\le C\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{i=1}^{\infty} E\|a_{ni}X''_{ni}\|^2\right)^{(1+(1+\alpha+\beta)/\gamma-t)/2} + C\sum_{n=1}^{\infty} n^{\beta}\sum_{i=1}^{\infty} E\|a_{ni}X''_{ni}\|^{1+\frac{1+\alpha+\beta}{\gamma}-t}$$
$$=: I_5 + I_6.$$

From (1.1) and (1.3), we have

$$\begin{aligned} I_5 &\le C\sum_{n=1}^{\infty} n^{\beta}\left(\sup_{i\ge 1}|a_{ni}|\sum_{i=1}^{\infty}|a_{ni}|E\|X_{ni}\|^2 I(\|X_{ni}\|>n^{\gamma})\right)^{(1+(1+\alpha+\beta)/\gamma-t)/2} \\ &\le C\sum_{n=1}^{\infty} n^{\beta}\left(\frac{n^{\alpha}}{n^{\gamma}}CE|X|^2 I(|X|>n^{\gamma})\right)^{(1+(1+\alpha+\beta)/\gamma-t)/2} \\ &\le C\sum_{n=1}^{\infty} n^{\beta}\left(\frac{1}{n^{1+\beta}}CE|X|^{1+(1+\alpha+\beta)/\gamma} I(|X|>n^{\gamma})\right)^{(1+(1+\alpha+\beta)/\gamma-t)/2} \\ &\le C\sum_{n=1}^{\infty} n^{\beta}\left(\frac{1}{n^{1+\beta}}CE|X|^{1+(1+\alpha+\beta)/\gamma}\right)^{(1+(1+\alpha+\beta)/\gamma-t)/2} < \infty, \end{aligned}$$

since $\beta > -1$ and $(1+\beta)(1+(1+\alpha+\beta)/\gamma - t)/2 - \beta > 1$. Similarly to I_2 in Case 2, we have $I_6 \le CE|X|^{1+\frac{1+\alpha+\beta}{\gamma}} < \infty$. Hence, $I_2 < \infty$. □

Remark 3.1. Ahmed et al. [1] have not treated the case of $1+\alpha+\beta=0$. When $1+\alpha+\beta\neq 0$, they proved Theorem 3.1 under the stronger moment condition $E|X|^v<\infty$, where v is as in (1.4). When $1+\alpha+\beta=0$, the log term in the moment condition cannot be removed (see [14]).

4. Complete Convergence of Moving Average Processes

In this section, we present one result about the convergence of moving average processes, which follows from Theorem 3.1. This result extends the corresponding result of Ahmed et al. [1].

Theorem 4.1. *Suppose $\beta\geq -1$. Let $\{Y_i, -\infty<i<\infty\}$ be a doubly infinite sequence of independent random elements which are stochastically dominated by a random variable X. Let $\{a_i, -\infty<i<\infty\}$ be an absolutely summable sequence of real numbers and set $X_k=\sum_{i=-\infty}^{\infty}a_{i+k}Y_i$, $k\geq 1$. Assume that $\sum_{k=1}^{n}X_k/n^{1/p}\to 0$ in probability, where $1\leq p<2$. Then the following statements hold*:

(i) *If $\beta>-1$, $1\leq p<2$, and $E|X|^{p(\beta+2)}<\infty$, then*

$$\sum_{n=1}^{\infty}n^{\beta}P\left(\left\|\sum_{k=1}^{n}X_k\right\|>n^{1/p}\epsilon\right)<\infty \quad \textit{for all } \epsilon>0.$$

(ii) *If $1<p<2$ and $E|X|^p<\infty$, then*

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\left\|\sum_{k=1}^{n}X_k\right\|>n^{1/p}\epsilon\right)<\infty \quad \textit{for all } \epsilon>0.$$

(iii) *If $E|X|\log|X|<\infty$, then*

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\left\|\sum_{k=1}^{n}X_k\right\|>n\epsilon\right)<\infty \quad \textit{for all } \epsilon>0.$$

Proof. Note that

$$\sum_{k=1}^{n}X_k=\sum_{i=-\infty}^{\infty}\sum_{k=1}^{n}a_{i+k}Y_i.$$

Set $a_{ni}=\sum_{k=1}^{n}a_{i+k}/n^{1/p}$ and $X_{ni}=Y_i$ for $-\infty<i<\infty$ and $n\geq 1$. Then $\{X_{ni}, -\infty<i<\infty, n\geq 1\}$ are rowwise independent random variables. Since $\{a_i, -\infty<i<\infty\}$ is absolutely summable, say $\sum_{i=-\infty}^{\infty}|a_i|=b$, we have that $|a_{ni}|\leq b/n^{1/p}$ and $\sum_{i=-\infty}^{\infty}|a_{ni}|\leq\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty}|a_{i+k}|/n^{1/p}=bn^{1-1/p}$. Take $\gamma=1/p$ and $\alpha=1-1/p$. Since $1\leq p<2$, conditions (1.1) and (1.3) are satisfied. Moreover, $1+\alpha+\beta=(1-1/p)+(1+\beta)\geq 0$ and $1+\alpha+\beta=0$ if and only if $p=1$ and $\beta=-1$. Thus, the result follows from Theorem3.1. □

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