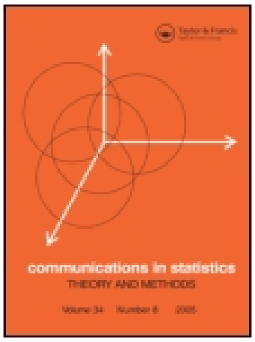

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Communications in Statistics - Theory and Methods

ISSN: 0361-0926 (Print) 1532-415X (Online) Journal homepage: http://www.tandfonline.com/loi/lsta20

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To cite this article: Aiting Shen, Ying Zhang & Andrei Volodin (2016) On the rate of convergence in the strong law of large numbers for negatively orthant-dependent random variables, Communications in Statistics - Theory and Methods, 45:21, 6209-6222, DOI: 10.1080/03610926.2014.957858

To link to this article: http://dx.doi.org/10.1080/03610926.2014.957858

Accepted author version posted online: 11 Jan 2016.
Published online: 11 Jan 2016.

COMMUNICATIONS IN STATISTICS—THEORY AND METHODS
2016, VOL. 45, NO. 21, 6209–6222
http://dx.doi.org/10.1080/03610926.2014.957858

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On the rate of convergence in the strong law of large numbers for negatively orthant-dependent random variables

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ABSTRACT
In this paper, we study the complete convergence and complete moment convergence for negatively orthant-dependent random variables. Especially, we obtain the Hsu–Robbins-type theorem for negatively orthant-dependent random variables. Our results generalize the corresponding ones for independent random variables.

ARTICLE HISTORY
Received 16 February 2014
Accepted 15 August 2014

KEYWORDS
Complete convergence; Hsu–Robbins-type theorem; NOD random variables.

MATHEMATICS SUBJECT CLASSIFICATION
60F15

1. Introduction

The concept of complete convergence was introduced by Hsu and Robbins (1947) as follows. A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant C if

$$\sum_{n=1}^{\infty} P\left(|U_n - C| > \varepsilon\right) < \infty \tag{1.1}$$

for all $\varepsilon > 0$. In view of the Borel–Cantelli lemma, this implies that $U_n \to C$ almost surely (a.s.). The converse is true if the $\{U_n, n \geq 1\}$ are independent. Hsu and Robbins (1947) proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös (1949) proved the converse. The result of Hsu–Robbins–Erdös is a fundamental theorem in probability theory and has been generalized and extended in several directions by many authors. One of the most important generalizations is Baum and Katz (1965) for the strong law of large numbers. The main purpose of this paper is to present the Hsu–Robbins-type theorem for negatively orthant-dependent random variables.

First, let us recall the definition of orthant-dependent random variables which was introduced by Joag-Dev and Proschan (1983).

Definition 1.1. A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be negatively orthant dependent (NOD, in short) if

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i) \tag{1.2}$$

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and

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i) \tag{1.3}$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$. An infinite sequence $\{X_n, n \geq 1\}$ is said to be NOD if every finite subcollection is NOD.

NOD random variables include independent random variables and negatively associated random variables as special cases. A number of limit theorems for NOD random variables have been established by many authors. We refer to Volodin (2002) for the Kolmogorov exponential inequality, Asadian et al. (2006) for the Rosenthals type inequality, Kim (2006) for Hájek–Rényi type inequality, Amini et al. (2004), Ko and Kim (2005), Klesov et al. (2005), Wu (2010a), Wu and Zhu (2010), Shen (2011, 2013) and Wu et al. (2013) for almost sure convergence, Kuczmaszewska (2006), Taylor et al. (2002), Wang et al. (2010, 2011) and Sung (2011a) for exponential inequalities, Amini and Bozorgnia (2003), Wu (2010b, 2012), Qiu et al. (2011), Zarei and Jabbari (2011) and Wang et al. (2012, 2013a, 2013b) for complete convergence, Wu and Jiang (2011) for M-estimator, Yang et al. (2012) and Wang et al. (2013c) for consistency for estimator of non parametric regression model based on NOD errors, and so forth.

The following definition of slowly varying function plays an important role throughout this paper.

Definition 1.2. A real-valued function $l(x)$, positive and measurable on $(0, \infty)$, is said to be slowly varying if

$$\lim_{x\to\infty} \frac{l(x\lambda)}{l(x)} = 1 \tag{1.4}$$

for each $\lambda > 0$.

We consider the following assumption in this paper:
(A) There exists a random variable X and a constant $C > 0$ such that for all $x \geq 0$,

$$\sup_{i\geq 1} P\left(|X_i| > x\right) \leq CP\left(|X| > x\right). \tag{1.5}$$

The first result presents the complete convergence for the partial sums of NOD random variables.

Theorem 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables. $\alpha > 1/2$, $\alpha p \geq 1$. Assume that $EX_i = 0$ for each $i \geq 1$ if $\alpha \leq 1$. Let $l(x) > 0$ be a non decreasing slowly varying function satisfying $l(x) \geq C$ for some $C > 0$ if $\alpha p = 1$ and $l(x) \to \infty$ if $\alpha p = 1$ and $\alpha > 1$. If there exists a random variable X such that (1.5) holds and*

$$E|X|^p l\left(|X|^{1/\alpha}\right) < \infty, \tag{1.6}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(|S_n| \geq \varepsilon n^{\alpha}\right) < \infty. \tag{1.7}$$

The following theorem presents the complete moment convergence for NOD random variables.

Theorem 1.2. *Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with zero mean. Suppose that the conditions Theorem 1.1 hold and $p > 1$, then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\left(|S_n| - \varepsilon n^{\alpha}\right)^{+} < \infty. \tag{1.8}$$

Remark 1.1. If we take $l(x) \equiv 1$, $\alpha = 1$ and $p = 2$ in Theorem 1.1, then we can obtain the Hsu–Robbins-type theorem (see Hsu and Robbins, 1947) for NOD random variables.

Remark 1.2. We point out that (1.8) implies (1.7). This can be obtained by the following inequality:

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\left(|S_n| - \varepsilon n^{\alpha}\right)^{+} &= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_0^{\infty} P\left(|S_n| - \varepsilon n^{\alpha} > t\right) dt \\ &\geq \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_0^{\varepsilon n^{\alpha}} P\left(|S_n| - \varepsilon n^{\alpha} > t\right) dt \\ &\geq \varepsilon \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(|S_n| > 2\varepsilon n^{\alpha}\right).\end{aligned}$$

Hence, the complete moment convergence is more general than complete convergence.

Remark 1.3. Sung (2011b) obtained the following complete moment convergence of moving average processes:

$$\sum_{n=1}^{\infty} n^{r-2-1/t} l(n) E\left(\left|\sum_{i=-\infty}^{\infty} a_{ni} X_i\right| - \varepsilon n^{1/t}\right)^{+} < \infty \tag{1.9}$$

for dependent random variables, including NOD random variables. Here, $1 \leq t < 2$ and $r > 1$ (see Theorem 2.1 of Sung, 2011b). If we take $r = \alpha p$, $t = 1/\alpha$ and

$$a_{ni} = \begin{cases} 1, & 1 \leq i \leq n, \\ 0, & \text{otherwise}, \end{cases}$$

in (1.9), then we can obtain (1.8) immediately. But we should point out that $r = \alpha p$ in Theorem 1.2 can take value 1, while is not in Theorem 2.1 of Sung (2011b) and $t = 1/\alpha \in (0, 2)$ in Theorem 1.2, while $t = 1/\alpha \in [1, 2)$ in Theorem 2.1 of Sung (2011b). So we can see that the scope of t and r in Theorem 1.2 are larger than those in Theorem 2.1 of Sung (2011b).

2. Preparations

To prove the main results of this paper, we need the following lemmas.

Lemma 2.1 *(cf. Asadian et al., 2006). Let $p \geq 2$ and $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for every $n \geq 1$. Then there exists a positive constant C depending only on p such that for every $n \geq 1$,*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq C\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}. \tag{2.1}$$

Lemma 2.2 *(cf. Bai and Su, 1985). If $l(x) > 0$ is a slowly varying function, then*

(i) $\lim_{x\to\infty} \frac{l(tx)}{l(x)} = 1$ *for each* $t > 0$; $\lim_{x\to\infty} \frac{l(x+u)}{l(x)} = 1$ *for each* $u \geq 0$

(ii) $\lim_{k\to\infty} \sup_{2^k \leq x < 2^{k+1}} \frac{l(x)}{l(2^k)} = 1$;

(iii) $\lim_{x\to\infty} x^{\delta} l(x) = \infty$, $\lim_{x\to\infty} x^{-\delta} l(x) = 0$ *for each* $\delta > 0$;

(iv) $C_1 2^{kr} l(\varepsilon 2^k) \leq \sum_{j=1}^{k} 2^{jr} l(\varepsilon 2^j) \leq C_2 2^{kr} l(\varepsilon 2^k)$ *for every* $r > 0$, $\varepsilon > 0$, *positive integer* k *and some* $C_1 > 0$, $C_2 > 0$;

(v) $C_3 2^{kr} l(\varepsilon 2^k) \leq \sum_{j=k}^{\infty} 2^{jr} l(\varepsilon 2^j) \leq C_4 2^{kr} l(\varepsilon 2^k)$ *for every* $r < 0$, $\varepsilon > 0$, *positive integer* k *and some* $C_3 > 0$, $C_4 > 0$.

Lemma 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $l(x) > 0$ be a non decreasing slowly varying function. Let s and t be arbitrary positive constants. If (1.5) holds, then*

$$\sup_{i\geq 1} E|X_i|^s l(|X_i|^t) \ll E|X|^s l(|X|^t). \tag{2.2}$$

Proof. Since $l(x) > 0$ is a non decreasing slowly varying function, it follows that

$$0 \leq E|X_i|^s l(|X_i|^t) I(0 < |X_i|^t < 2) \leq 2^{\frac{s}{t}} l(2) := C^*,$$

where C^* is a positive number which does not depend on i. Therefore, we have by Lemma 2.2 that

$$\begin{aligned}
\sup_{i\geq 1} E|X_i|^s l(|X_i|^t) &\leq \sup_{i\geq 1} \sum_{j=1}^{\infty} E|X_i|^s l(|X_i|^t) I\left(2^j \leq |X_i|^t < 2^{j+1}\right) + C^* \\
&\ll \sup_{i\geq 1} \sum_{j=1}^{\infty} 2^{\frac{s}{t}j} l\left(2^j\right) P\left(2^j \leq |X_i|^t < 2^{j+1}\right) \\
&\ll \sup_{i\geq 1} \sum_{j=1}^{\infty} \sum_{k=1}^{j} 2^{\frac{s}{t}k} l\left(2^k\right) P\left(2^j \leq |X_i|^t < 2^{j+1}\right) \\
&= \sup_{i\geq 1} \sum_{k=1}^{\infty} 2^{\frac{s}{t}k} l\left(2^k\right) \sum_{j=k}^{\infty} P\left(2^j \leq |X_i|^t < 2^{j+1}\right) \\
&= \sup_{i\geq 1} \sum_{k=1}^{\infty} 2^{\frac{s}{t}k} l\left(2^k\right) P\left(|X_i|^t \geq 2^k\right)
\end{aligned}$$

and

$$\begin{aligned}
E|X|^s l(|X|^t) &\geq \sum_{j=1}^{\infty} E|X|^s l(|X|^t) I\left(2^j \leq |X|^t < 2^{j+1}\right) \\
&\gg \sum_{j=1}^{\infty} 2^{\frac{s}{t}j} l\left(2^j\right) P\left(2^j \leq |X|^t < 2^{j+1}\right) \\
&\gg \sum_{j=1}^{\infty} \sum_{k=1}^{j} 2^{\frac{s}{t}k} l\left(2^k\right) P\left(2^j \leq |X|^t < 2^{j+1}\right)
\end{aligned}$$

$$= \sum_{k=1}^{\infty} 2^{\frac{s}{t}k} l\left(2^k\right) \sum_{j=k}^{\infty} P\left(2^j \le |X|^t < 2^{j+1}\right)$$

$$= \sum_{k=1}^{\infty} 2^{\frac{s}{t}k} l\left(2^k\right) P\left(|X|^t \ge 2^k\right).$$

The desired result (2.2) follows from the statements above and (1.5) immediately. □

Lemma 2.4. *Let $\{X_n, n \ge 1\}$ be a sequence of random variables satisfying (1.5). Then for any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$E|X_n|^\alpha I\left(|X_n| \le b\right) \le C_1 \left[E|X|^\alpha I\left(|X| \le b\right) + b^\alpha P\left(|X| > b\right)\right], \tag{2.3}$$

$$E|X_n|^\alpha I\left(|X_n| > b\right) \le C_2 E|X|^\alpha I\left(|X| > b\right), \tag{2.4}$$

where C_1 and C_2 are positive constants.

For the proof of Lemma 2.4, one can refer to Wu (2006) or Wang et al. (2014).

Lemma 2.5 *(cf. Bozorgnia et al., 1996). Let random variables $X_1, X_2, \ldots, X_n$ be NOD, $f_1, f_2, \ldots, f_n$ be all non decreasing (or all non increasing) functions, then random variables $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are NOD.*

3. Proofs of the main results

Proof of Theorem 1.1. For fixed $n \ge 1$, denote

$$Y_{ni} = -n^\alpha I(X_i < -n^\alpha) + X_i I(|X_i| \le n^\alpha) + n^\alpha I(X_i > n^\alpha), \; i = 1, 2, \ldots.$$

It is easily seen that for any $\varepsilon > 0$,

$$\begin{aligned}(|S_n| \ge \varepsilon n^\alpha) &= \left(|S_n| \ge \varepsilon n^\alpha, \max_{1\le i\le n}|X_i| > n^\alpha\right) \bigcup \left(|S_n| \ge \varepsilon n^\alpha, \max_{1\le i\le n}|X_i| \le n^\alpha\right) \\ &\subset \left(\max_{1\le i\le n}|X_i| > n^\alpha\right) \bigcup \left(\left|\sum_{i=1}^n (Y_{ni} - EY_{ni})\right| \ge \varepsilon n^\alpha - \left|\sum_{i=1}^n EY_{ni}\right|\right),\end{aligned}$$

which implies that

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(|S_n| \ge \varepsilon n^\alpha\right) \\ &\quad \le \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\max_{1\le i\le n}|X_i| > n^\alpha\right) \\ &\qquad + \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\left|\sum_{i=1}^n (Y_{ni} - EY_{ni})\right| \ge \varepsilon n^\alpha - \left|\sum_{i=1}^n EY_{ni}\right|\right) \\ &\quad := I + J.\end{aligned} \tag{3.1}$$

For I, we have by (1.5), (1.6) and Lemma 2.2 that

$$
\begin{aligned}
I &\leq \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \sum_{i=1}^{n} P(|X_i| > n^{\alpha}) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \sum_{i=1}^{n} P(|X| > n^{\alpha}) = \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P(|X| > n^{\alpha}) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) \sum_{j=n}^{\infty} P\left(j < |X|^{1/\alpha} \leq j+1\right) \\
&= \sum_{j=1}^{\infty} P\left(j < |X|^{1/\alpha} \leq j+1\right) \sum_{n=1}^{j} n^{\alpha p-1} l(n) \\
&\leq \sum_{j=1}^{\infty} P\left(j < |X|^{1/\alpha} \leq j+1\right) \sum_{i=1}^{\lfloor \log_2 j \rfloor +1} \sum_{n=2^{i-1}}^{2^i} n^{\alpha p-1} l(n) \qquad (3.2) \\
&\ll \sum_{j=1}^{\infty} P\left(j < |X|^{1/\alpha} \leq j+1\right) \sum_{i=1}^{\lfloor \log_2 j \rfloor +1} 2^{i\alpha p} l\left(2^i\right) \\
&\ll \sum_{j=1}^{\infty} P\left(j < |X|^{1/\alpha} \leq j+1\right) 2^{(\lfloor \log_2 j \rfloor +1)\alpha p} l\left(2^{\lfloor \log_2 j \rfloor +1}\right) \\
&\ll \sum_{j=1}^{\infty} P\left(j < |X|^{1/\alpha} \leq j+1\right) j^{\alpha p} l(j) \\
&\ll E|X|^p l\left(|X|^{1/\alpha}\right) < \infty.
\end{aligned}
$$

For J, we first show that

$$
Q := n^{-\alpha} \left| \sum_{i=1}^{n} EY_{ni} \right| \to 0, \quad n \to \infty. \qquad (3.3)
$$

Indeed,

$$
Q \leq \sum_{i=1}^{n} P(|X_i| > n^{\alpha}) + n^{-\alpha} \left| \sum_{i=1}^{n} EX_i I(|X_i| \leq n^{\alpha}) \right| := A + B.
$$

For A, if $\alpha p > 1$, then

$$
A \ll nP(|X| > n^{\alpha}) \ll n^{1-\alpha p} l^{-1}(n) \to 0;
$$

if $\alpha p = 1$, then we have by (1.6) that

$$
A \ll nP\left(|X|^p l\left(|X|^{1/\alpha}\right) > Cn\right) \to 0.
$$

For B, we will consider the following four cases:

Case 1. $\alpha > 1, p > 1$

By Lemma 2.3 and (1.6), we have

$$
B \leq n^{1-\alpha} \sup_{i \geq 1} E|X_i| I(|X_i| \leq n^{\alpha}) \leq n^{1-\alpha} \sup_{i \geq 1} E|X_i| \ll n^{1-\alpha} E|X| \to 0, \ n \to \infty.
$$

Case 2. $\alpha > 1, p = 1$

By Lemma 2.3 and (1.6) again, we can obtain

$$\begin{aligned} B &\leq n^{1-\alpha} \sup_{i\geq 1} E|X_i| I(|X_i| \leq n^{\alpha}) \\ &\ll n^{1-\alpha} \sup_{i\geq 1} E|X_i| l\left(|X_i|^{1/\alpha}\right) \sup_{x\leq n} l^{-1}(x) \\ &\ll n^{1-\alpha} E|X| l\left(|X|^{1/\alpha}\right) \sup_{x\leq n} l^{-1}(x) \to 0, \ n \to \infty. \end{aligned}$$

Case 3. $\alpha > 1, p < 1$

Note that $l(x) \to \infty$ as $x \to \infty$ when $\alpha p = 1$ and $\alpha > 1$. It follows from Lemma 2.3 and (1.6) that

$$\begin{aligned} B &\leq n^{1-\alpha} \sup_{i\geq 1} E|X_i| I(|X_i| \leq n^{\alpha}) \\ &\ll n^{1-\alpha} \sup_{i\geq 1} E|X_i|^p l\left(|X_i|^{1/\alpha}\right) n^{\alpha(1-p)} l^{-1}(n) \\ &\ll n^{1-\alpha p} l^{-1}(n) E|X|^p l\left(|X|^{1/\alpha}\right) \to 0, \ n \to \infty. \end{aligned}$$

Case 4. $1/2 < \alpha \leq 1$

Note that $p \geq 1$ when $1/2 < \alpha \leq 1$ and $\alpha p \geq 1$. It follows from $EX_i = 0$, Lemma 2.3 and (1.6) that

$$\begin{aligned} B &\leq n^{1-\alpha} \sup_{i\geq 1} E|X_i| I(|X_i| > n^{\alpha}) \\ &\ll n^{1-\alpha p} l^{-1}(n) \sup_{i\geq 1} E\left(|X_i|^p l\left(|X_i|^{1/\alpha}\right) I(|X_i| > n^{\alpha})\right) \\ &\ll n^{1-\alpha p} l^{-1}(n) E\left(|X|^p l\left(|X|^{1/\alpha}\right) I(|X| > n^{\alpha})\right) \to 0, \ n \to \infty. \end{aligned}$$

From the statements above, we can obtain (3.3) immediately. To prove $J < \infty$, it suffices to show that

$$J' := \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\left|\sum_{i=1}^{n} (Y_{ni} - EY_{ni})\right| \geq \varepsilon n^{\alpha}\right) < \infty. \tag{3.4}$$

It follows from Lemma 2.5 that for any fixed $n \geq 1$, $\{Y_{ni} - EY_{ni}, i \geq 1\}$ are still NOD random variables with mean zero. By Lemma 2.1, Lemma 2.4 and (3.2), we have for $q \geq \max(2, p)$ that

$$\begin{aligned} J' &\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) E\left(\left|\sum_{i=1}^{j} (Y_{ni} - EY_{ni})\right|^q\right) \\ &\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) \left[\sum_{i=1}^{n} E|Y_{ni}|^q + \left(\sum_{i=1}^{n} EY_{ni}^2\right)^{q/2}\right] \\ &\ll \sum_{n=1}^{\infty} n^{\alpha(p-q)-1} l(n) \left[E|X|^q I(|X| \leq n^{\alpha}) + n^{\alpha q} P(|X| > n^{\alpha})\right] \\ &\quad + \sum_{n=1}^{\infty} n^{\alpha(p-q)-2+q/2} l(n) \left[EX^2 I(|X| \leq n^{\alpha}) + n^{2\alpha} P(|X| > n^{\alpha})\right]^{q/2} \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&\ll \sum_{n=1}^{\infty} n^{\alpha(p-q)-1} l(n) E|X|^q I(|X| \le n^\alpha) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha(p-q)-2+q/2} l(n) \left[EX^2 I(|X| \le n^\alpha)\right]^{q/2} \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2+q/2} l(n) \left[P(|X| > n^\alpha)\right]^{q/2} \\
&:= J'_1 + J'_2 + D.
\end{aligned}
$$

If $p \ge 2$, then we take $q > p$. Hence, we have by Lemma 2.2 and (1.6) that

$$
\begin{aligned}
J'_1 &= \sum_{n=1}^{\infty} n^{\alpha(p-q)-1} l(n) E|X|^q I(|X| \le n^\alpha) \\
&\le \sum_{n=1}^{\infty} n^{\alpha(p-q)-1} l(n) \sum_{j=1}^{n} j^{\alpha q} P\left(j-1 < |X|^{1/\alpha} \le j\right) \\
&= \sum_{j=1}^{\infty} j^{\alpha q} P\left(j-1 < |X|^{1/\alpha} \le j\right) \sum_{n=j}^{\infty} n^{\alpha(p-q)-1} l(n) \\
&\le \sum_{j=1}^{\infty} j^{\alpha q} P\left(j-1 < |X|^{1/\alpha} \le j\right) \sum_{i=\lfloor \log_2 j \rfloor}^{\infty} \sum_{n=2^i}^{2^{i+1}} n^{\alpha(p-q)-1} l(n) \qquad (3.6) \\
&\ll \sum_{j=1}^{\infty} j^{\alpha q} P\left(j-1 < |X|^{1/\alpha} \le j\right) \sum_{i=\lfloor \log_2 j \rfloor}^{\infty} 2^{i\alpha(p-q)} l\left(2^i\right) \\
&\ll \sum_{j=1}^{\infty} j^{\alpha q} P\left(j-1 < |X|^{1/\alpha} \le j\right) j^{\alpha(p-q)} l(j) \\
&= \sum_{j=1}^{\infty} j^{\alpha p} l(j) P\left(j-1 < |X|^{1/\alpha} \le j\right) \\
&\ll E|X|^p l\left(|X|^{1/\alpha}\right) < \infty.
\end{aligned}
$$

If $p < 2$, then we take $q = 2$. Similar to the proof of (3.6), we can still obtain $J'_1 < \infty$.

For J'_2 and D, we consider the following three cases:

Case 1. $\alpha p = 1$

Note that $p = 1/\alpha < 2$. Similar to the proof of (3.6) and (3.2), we can obtain $J'_2 < \infty$ and $D < \infty$ by taking $q = 2$, respectively.

Case 2. $\alpha p > 1$ and $p \ge 2$

Taking $q > \max(p, \frac{\alpha p-1}{\alpha-\frac{1}{2}})$, which implies that $\alpha(p-q) - 2 + q/2 < -1$. Hence,

$$
D \le \sum_{n=1}^{\infty} n^{-1-(\alpha p-1)(q/2-1)} l(n) \left(E|X|^p\right)^{q/2} < \infty, \qquad (3.7)
$$

and

$$
J'_2 \ll \sum_{n=1}^{\infty} n^{\alpha(p-q)-2+q/2} l(n) \left(EX^2\right)^{q/2} < \infty.
$$

Case 3. $\alpha p > 1$ and $p < 2$

Take $q = 2$. We can obtain $J'_2 < \infty$ and $D < \infty$ by (3.6) and (3.2), respectively.

Thus, $J'_2 < \infty$ and $D < \infty$ from the statements above. Together with $J'_1 < \infty$ yield that $J' < \infty$. This completes the proof of the theorem. □

Proof of Theorem 1.2. For any $\varepsilon > 0$, we have by Theorem 1.1 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\left(|S_n| - \varepsilon n^{\alpha}\right)^{+} \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_0^{\infty} P\left(|S_n| - \varepsilon n^{\alpha} > t\right) dt \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_0^{n^{\alpha}} P\left(|S_n| - \varepsilon n^{\alpha} > t\right) dt \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^{\alpha}}^{\infty} P\left(|S_n| - \varepsilon n^{\alpha} > t\right) dt \\
&\leq \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(|S_n| > \varepsilon n^{\alpha}\right) + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^{\alpha}}^{\infty} P\left(|S_n| > t\right) dt \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^{\alpha}}^{\infty} P\left(|S_n| > t\right) dt.
\end{aligned} \tag{3.8}$$

Hence, it suffices to show that

$$H := \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^{\alpha}}^{\infty} P\left(|S_n| > t\right) dt < \infty. \tag{3.9}$$

For $t > 0$, denote

$$Z_{ti} = -tI(X_i < -t) + X_i I(|X_i| \leq t) + tI(X_i > t), \quad i = 1, 2, \ldots$$

and

$$U_{ti} = tI(X_i < -t) + X_i I(|X_i| > t) - tI(X_i > t), \quad i = 1, 2, \ldots.$$

Since $X_i = U_{ti} + Z_{ti}$, it follows that

$$\begin{aligned}
H &\leq \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^{\alpha}}^{\infty} P\left(\left|\sum_{i=1}^{n} U_{ti}\right| > t/3\right) dt \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^{\alpha}}^{\infty} P\left(\left|\sum_{i=1}^{n} EZ_{ti}\right| > t/3\right) dt \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^{\alpha}}^{\infty} P\left(\left|\sum_{i=1}^{n} (Z_{ti} - EZ_{ti})\right| > t/3\right) dt \\
&:= H_1 + H_2 + H_3.
\end{aligned} \tag{3.10}$$

Note that $|U_{ti}| \leq 2|X_i|I(|X_i| > t)$. Similar to the proof of (3.2), we have by Markov's inequality, Lemma 2.2 and Lemma 2.4 that

$$
\begin{aligned}
H_1 &\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-1} E\left(\left|\sum_{i=1}^{n} U_{ti}\right|\right) dt \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-1} E\left[|X| I(|X| > t)\right] dt \\
&= \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \sum_{m=n}^{\infty} \int_{m^\alpha}^{(m+1)^\alpha} t^{-1} E\left[|X| I(|X| > t)\right] dt \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \sum_{m=n}^{\infty} m^{-1} E\left[|X| I(|X| > m^\alpha)\right] \\
&= \sum_{m=1}^{\infty} m^{-1} E\left[|X| I(|X| > m^\alpha)\right] \sum_{n=1}^{m} n^{\alpha p-1-\alpha} l(n) \\
&\ll \sum_{m=1}^{\infty} m^{-1} E\left[|X| I(|X| > m^\alpha)\right] m^{\alpha p-\alpha} l(m) \qquad (3.11) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) E\left[|X| I(|X| > n^\alpha)\right] \\
&= \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \sum_{m=n}^{\infty} E\left[|X| I(m < |X|^{1/\alpha} \leq m+1)\right] \\
&= \sum_{m=1}^{\infty} E\left[|X| I(m < |X|^{1/\alpha} \leq m+1)\right] \sum_{n=1}^{m} n^{\alpha p-1-\alpha} l(n) \\
&\ll \sum_{m=1}^{\infty} E\left[|X| I(m < |X|^{1/\alpha} \leq m+1)\right] m^{\alpha p-\alpha} l(m) \\
&\ll E|X|^p l\left(|X|^{1/\alpha}\right) < \infty.
\end{aligned}
$$

According to the proof of (3.11), we have by Markov's inequality and Lemma 2.2 that

$$
\begin{aligned}
H_2 &\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-1} \left|\sum_{i=1}^{n} EZ_{ti}\right| dt \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-1} \sum_{i=1}^{n} E[|X_i| I(|X_i| > t)] dt \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-1} E\left[|X| I(|X| > t)\right] dt < \infty. \qquad (3.12)
\end{aligned}
$$

For any $t > 0$, it is easily seen that $\{Z_{ti}, i \geq 1\}$ is still a sequence of NOD random variables by Lemma 2.5. By Markov's inequality and Lemma 2.1, we have that for any $q \geq 2$,

$$
\begin{aligned}
H_3 &\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-q} E\left(\left|\sum_{i=1}^{n} (Z_{ti} - EZ_{ti})\right|^q\right) dt \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-q} \sum_{i=1}^{n} E\left|Z_{ti}\right|^q dt \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-q} \left(\sum_{i=1}^{n} EZ_{ti}^2\right)^{q/2} dt \\
&:= H_{31} + H_{32}.
\end{aligned} \tag{3.13}
$$

We still consider the following three cases:

Case 1. $\alpha p > 1$ and $p \geq 2$

Take q large enough such that $q > \max(p, \frac{\alpha p-1}{\alpha - \frac{1}{2}})$, which implies that $\alpha p - 2 - \alpha q + q/2 < -1$. We have by Lemma 2.2, Lemma 2.4 and (3.11) that

$$
\begin{aligned}
H_{31} &\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-q} \sum_{i=1}^{n} \left\{E\left[|X_i|^q I\left(|X_i| \leq t\right)\right] + t^q P\left(|X_i| > t\right)\right\} dt \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-q} E\left[|X|^q I\left(|X| \leq t\right)\right] dt \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \int_{n^\alpha}^{\infty} P\left(|X| > t\right) dt \\
&\leq \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-q} E\left[|X|^q I\left(|X| \leq t\right)\right] dt \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-1} E\left[|X| I\left(|X| > t\right)\right] dt \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-q} E\left[|X|^q I\left(|X| \leq t\right)\right] dt.
\end{aligned}
$$

Hence, similar to the proof of (3.6), we can see that

$$
\begin{aligned}
H_{31} &\ll \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \sum_{m=n}^{\infty} \int_{m^\alpha}^{(m+1)^\alpha} t^{-q} E\left[|X|^q I\left(|X| \leq t\right)\right] dt \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \sum_{m=n}^{\infty} m^{\alpha-1-\alpha q} E\left[|X|^q I\left(|X| \leq (m+1)^\alpha\right)\right] \\
&= \sum_{m=1}^{\infty} m^{\alpha-1-\alpha q} E\left[|X|^q I\left(|X| \leq (m+1)^\alpha\right)\right] \sum_{n=1}^{m} n^{\alpha p-1-\alpha} l(n) \\
&\ll \sum_{m=1}^{\infty} m^{\alpha-1-\alpha q} E\left[|X|^q I\left(|X| \leq (m+1)^\alpha\right)\right] m^{\alpha p-\alpha} l(m) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q} l(n) E\left[|X|^q I\left(|X| \leq (n+1)^\alpha\right)\right] \qquad (3.14) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q} l(n) E\left[|X|^q I\left(n^\alpha < |X| \leq (n+1)^\alpha\right)\right]
\end{aligned}
$$

$$+\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q} l(n) E\left[|X|^q I\left(|X| \leq n^\alpha\right)\right]$$

$$\ll \sum_{n=1}^{\infty} n^{-1} E\left[|X|^p l\left(|X|^{1/\alpha}\right) I\left(n^\alpha < |X| \leq (n+1)^\alpha\right)\right] + E|X|^p l\left(|X|^{1/\alpha}\right)$$

$$\ll E|X|^p l\left(|X|^{1/\alpha}\right) < \infty.$$

By Lemma 2.2 and Lemma 2.4 again, we can obtain

$$H_{32} \ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha+q/2} l(n) \int_{n^\alpha}^{\infty} t^{-q}\left[E\left(X^2 I\left(|X| \leq t\right)\right)\right]^{q/2} dt$$

$$+\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha+q/2} l(n) \int_{n^\alpha}^{\infty} \left[P\left(|X| > t\right)\right]^{q/2} dt$$

$$\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha+q/2} l(n) \left(EX^2\right)^{q/2} \int_{n^\alpha}^{\infty} t^{-q} dt$$

$$+\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha+q/2} l(n) \left(E|X|^p\right)^{q/2} \int_{n^\alpha}^{\infty} t^{-pq/2} dt$$

$$\ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q+q/2} l(n) \left(EX^2\right)^{q/2} + \sum_{n=1}^{\infty} n^{-1-(\alpha p-1)(q/2-1)} l(n) \left(E|X|^p\right)^{q/2}$$

$$< \infty.$$

Case 2. $\alpha p > 1$ and $1 < p < 2$

Take $q = 2$. Similar to the proof of (3.13) and (3.14), we can obtain

$$H_3 \ll \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-2} \sum_{i=1}^{n} E\left|Z_{ti}\right|^2 dt < \infty. \tag{3.15}$$

Case 3. $\alpha p = 1$

Note that $1 < p = 1/\alpha < 2$. Take $q = 2$ and similar to the proof of (3.15), we still have $H_3 < \infty$.

From the statements above, we have proved (3.9). This completes the proof of the theorem. □

Acknowledgments

The authors are most grateful to the Editor-in-Chief N. Balakrishnan and anonymous referee for careful reading of the manuscript and valuable suggestions which helped in improving an earlier version of this paper.

Funding

Supported by the National Natural Science Foundation of China (11501004, 11501005, 11526033), the Natural Science Foundation of Anhui Province (1508085J06), the Provincial Natural Science Research Project of Anhui Colleges (KJ2015A018), the Key Projects for Academic Talent of Anhui Province (gxbjZD2016005), the Quality Engineering Project of Anhui Province (2015jyxm045) and the Quality Improvement Project for Undergraduate Education of Anhui University (ZLTS2015035).

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