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Asymptotic probability for the bootstrapped means deviations from the sample mean

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Abstract

In this paper, the asymptotic probability for the bootstrapped means deviations from the sample mean is obtained, without imposing any assumptions on joint distribution of the original sequence of random variables from which the bootstrap sample is withdrawn. A non-restrictive assumption of stochastic domination by a random variable is imposed on marginal distributions of this sequence.

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1. Introduction

The main focus of the present investigation is to obtain asymptotic results for the probability of bootstrapped means deviations from the sample mean.

The work on the consistency of the bootstrap estimators has received a lot of attention in recent years due to a growing demand for the procedure, both theoretically and practically. As it is mentioned in Mikosch (1994), the sample mean is fundamental for parameter estimation in statistics. Therefore, most of the recent literature on bootstrap is devoted to statistics of this type. This literature is mainly concerned with bootstrap consistency, that is, to show that a statistic and its bootstrap version have the same asymptotic distributional behaviour.

However, the limiting behaviour of bootstrap statistics is also of interest since it is by no means clear whether the bootstrap version of a consistent estimator is consistent. From our point of view this explains the usefulness and impact of deviations from the sample means in the "exogenously generated" bootstrap samples on the statistical inference. Furthermore, asymptotic probabilities for the bootstrapped means deviations are quite a useful tool for the study of bootstrap moments. It is important to note that exponential inequalities are of practical use in establishing the strong asymptotic validity of the bootstrapped mean.

We mention the special issue of the journal *Statistical Science* (2003) volume 18, number 2, devoted to the Silver Anniversary of the Bootstrap, where the wide applications of the bootstrap procedure to statistics are discussed.

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables (not necessarily independent or identically distributed) defined on a probability space $(\Omega, \mathscr{F}, P)$. For $\omega \in \Omega$ and $n \geqslant 1$, let $P_n(\omega) = n^{-1}\sum_{i=1}^{n} \delta_{X_i(\omega)}$ denote the empirical measure and let $\{\hat{X}_{n,j}^{(\omega)}, 1 \leqslant j \leqslant m(n)\}$ be i.i.d. random variables with law $P_n(\omega)$, where $\{m(n), n \geqslant 1\}$ is a sequence of positive integers. In other words, the random variables $\{\hat{X}_{n,j}^{(\omega)}, 1 \leqslant j \leqslant m(n)\}$ result by sampling $m(n)$ times with replacement from the n observations $X_1(\omega), \ldots, X_n(\omega)$ such that for each of the $m(n)$ selections, each $X_j(\omega)$ has probability n^{-1} of being chosen. For each $n \geqslant 1, \{\hat{X}_{n,j}^{(\omega)}, 1 \leqslant j \leqslant m(n)\}$ is the so-called Efron (1979) *bootstrap sample* from $X_1, \ldots, X_n$ with bootstrap sample size $m(n)$.

We are using the following standard notations. Let $\overline{X}_n(\omega) = 1/n \sum_{j=1}^{n} X_j(\omega)$ denote the sample mean of $\{X_j(\omega), 1 \leqslant j \leqslant n\}, n \geqslant 1$. In order to distinguish the conditional probability of the bootstrap variables from the unconditional probability P we denote

$$P^*\{\cdot\} = P\{\cdot | X_1(\omega), \ldots, X_n(\omega)\}.$$

The main focus of the paper is to obtain asymptotic results for the following probability:

$$P^*\left\{\frac{1}{n^{1/\alpha}}\left|\sum_{j=1}^{m(n)} \hat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega)\right| \geqslant \varepsilon\right\},$$

as $n \to \infty$ for any $\varepsilon > 0$ and $0 < \alpha < 2$.

For expository purpose, we begin with a brief discussion of results in the literature pertaining to the bootstrap of the mean. We also refer to an overview paper Csörgő and Rosalsky (2003) where a detailed and comprehensive survey of limit laws for the bootstrapped sums is given. In the special case when $\{X_n, n \geqslant 1\}$ is a sequence of independent and identically distributed random variables, strong laws of large numbers (SLLNs) were proved by Athreya (1983), Hu (1991), and Csörgő (1992) for bootstrapped means. Arenal-Gutiérrez et al. (1996) analyzed the results of Athreya (1983) and Csörgő (1992). Then, by taking into account the different growth rates for the resampling size $m(n)$, they gave new and simple proofs of those results. They also provided examples that show that the sizes of resampling required by their results to ensure almost sure (a.s.) convergence are not far from optimal.

Another article which is important for the paper is the work carried out by Mikosch (1994). He established a series of useful exponential inequalities (Theorem 3) that are the important tool in deriving results on the consistency of the bootstrapped mean. Based on these exponential inequalities, he proved an almost sure convergence result for the bootstrapped means (Theorem 1). Next, with the same Mikosch's exponential inequalities, the Baum-Katz/Erdös/Hsu-Robbins/Spitzer type complete convergence result for the bootstrapped means and a moment result for the supremum of normed bootstrapped sums were established by Li et al. (1999) (Theorem 2).

The following result was proved by Mikosch (1994, Proposition 3.3).

Theorem 1. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of independent and identically distributed random variables and* $0<\alpha<2$. *If*

$$E|X_1|^\alpha |\log|X_1||^\alpha < \infty,$$

then for almost every $\omega \in \Omega$:

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} (\hat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega)) \to 0 \ \ a.s.$$

The following result was proved by Li et al. (1999, Theorem 2.1).

Theorem 2. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of pairwise independent identically distributed random variables and* $0<\alpha<2$. *If*

$$E|X_1|^\alpha |\log|X_1||^\alpha < \infty,$$

then for every real number q, *every* $\varepsilon>0$ *and almost every* $\omega \in \Omega$:

$$\sum_{n=1}^{\infty} n^q P^* \left\{ \frac{1}{n^{1/\alpha}} \left| \sum_{j=1}^{n} \hat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega) \right| \geqslant \varepsilon \right\} < \infty.$$

Remark 1. Taking $q=0$ it follows from the Borel–Cantelli lemma and the conclusion of Theorem 2, that for almost every $\omega \in \Omega$

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} (\hat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega)) \to 0 \ \ \text{a.s.}$$

The initial objective of an investigation resulting in the present paper was only to extend the above-mentioned results on the SLLNs for bootstrap of the mean. But it appears that we are able to establish a more general result. Theorem 2 deals with the moment assumption $E|X_1|^\alpha |\log|X_1||^\alpha < \infty$, while our Theorem 4 deals with much more general moment assumption. Moreover, we establish that no independence condition is important and identically distribution assumption can be relaxed to stochastic domination by a random variable which is sufficient.

The main tool is the following result from Mikosch (1994, Lemma 5.1). We mention that this result was proved by Mikosch (1994) under the assumption that $\{X_n, n \geq 1\}$ is a sequence of independent or identically distributed random variables. But a careful analysis of the proof shows that this assumption is unnecessary.

Theorem 3. *Let $\{a_n, n \geq 1\}$ and $\{h_n, n \geq 1\}$ be two sequences of positive real numbers and let $\{X_n, n \geq 1\}$ be a sequence of (not necessarily independent or identically distributed) random variables. Then for $\omega \in \Omega$ and $n \geq 1$ such that $h_n M_n(\omega) < 1$ and all $\varepsilon > 0$ the following inequality holds*:

$$P^*\left\{\left|\sum_{j=1}^{m(n)} (\hat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega))\right| \geq \varepsilon a_n\right\} \leq 2\exp\left\{-\varepsilon \frac{h_n a_n}{m(n)} + \frac{h_n^2 B_n(\omega)}{2(1 - h_n M_n(\omega))}\right\},$$

where

$$M_n(\omega) = \frac{1}{m(n)} \max_{1 \leq j \leq n} |X_j(\omega) - \overline{X}_n(\omega)|$$

and

$$B_n(\omega) = \frac{1}{n\, m(n)} \sum_{j=1}^{n} (X_j(\omega) - \overline{X}_n(\omega))^2.$$

The following notion is well known.

Definition. We will say that a sequence of random variables $\{X_n,\ n \geq 1\}$ is *stochastically dominated* by a random variable X, if there exists a constant $C > 0$ such that

$$P\{|X_n| > t\} \leq CP\{|X| > t\}$$

for all $t \geq 0$ and all $n \geq 1$.

2. A few technical lemmas

In this section we present a few technical results that we will use in proofs of the main results of the paper. The first lemma is the special case of Adler and Rosalsky (1987) with a_n identically 1, hence we omit a proof. Note that there is no independence assumption.

Lemma 1. *Let $\phi(t), t > 0$, be a continuous function that is positive, strictly increasing and satisfying the condition $\phi(t) \to \infty$ as $t \to \infty$. Put $b_n = \phi^{-1}(n), n \geq 1$, where $\phi^{-1}(t)$ is the inverse function of $\phi(t)$. Let, moreover, $\{Y_n, n \geq 1\}$ be a sequence of random variables stochastically dominated by a random variable Y. If*

$$\sum_{n=k}^{\infty} b_n^{-1} = \mathcal{O}(k b_k^{-1}) \quad \text{and} \quad E\phi(|Y|) < \infty,$$

then

$$\frac{1}{b_n}\sum_{j=1}^{n} Y_j \to 0 \ \ a.s.$$

The second lemma in this section deals with convergence of maximums of random variables. Again, no assumption of independence is made.

Lemma 2. *Let $\psi(t), t \geqslant 0$ be an increasing function and let $\{b_n,\ n \geqslant 1\}$ be a sequence of positive numbers such that $b_n = \psi^{-1}(n),\ n \geqslant 1$, where $\psi^{-1}(t)$ is the inverse function of $\psi(t)$. Let, moreover, $\{X_n,\ n \geqslant 1\}$ be a sequence of positive random variables stochastically dominated by a random variable X such that $E\psi(CX) < \infty$ for all $C > 0$. Then*

$$\frac{1}{b_n}\max_{1 \leqslant j \leqslant n} X_j \to 0 \ \ a.s.$$

Proof. As we already mentioned in the proof of Lemma 1, the assumptions

$$\sum_{n=1}^{\infty} P\{X_n > \varepsilon b_n\} < \infty$$

for any $\varepsilon > 0$, and

$$E\psi\left(\frac{X}{\varepsilon}\right) < \infty$$

are equivalent. Then by the Borel–Cantelli lemma $X_n/b_n \to 0$ a.s. For arbitrary $n \geqslant k \geqslant 2$,

$$\begin{aligned}
\frac{1}{b_n}\max_{1 \leqslant j \leqslant n} X_j &\leqslant \frac{1}{b_n}\max_{1 \leqslant j \leqslant k-1} X_j + \frac{1}{b_n}\max_{k \leqslant j \leqslant n} X_j \\
&\leqslant \frac{1}{b_n}\max_{1 \leqslant j \leqslant k-1} X_j + \max_{k \leqslant j \leqslant n} X_j/b_j \quad \text{(since } \{b_n, n \geqslant 1\} \text{ is nondecreasing)} \\
&\leqslant \frac{1}{b_n}\max_{1 \leqslant j \leqslant k-1} X_j + \sup_{j \geqslant k} X_j/b_j \to 0
\end{aligned}$$

as first $n \to \infty$ and then $k \to \infty$. $\square$

Unfortunately, it is not possible to find the inverse function to the function $\phi(t) = t^{1/\beta}/\log^{\gamma} t,\ t > 0,\ \beta > 0$, and $\gamma > 0$ in the closed form. But the following lemma gives a good "approximation" to the inverse function.

Lemma 3. *Let $\phi(t) = t^{1/\beta}\log^{-\gamma/\beta} t$ and $\psi(t) = t^{\beta}\log^{\gamma} t,\ \ t \geqslant e,\ \beta > 0$, and $0 < \gamma < e$. Then for any $\varepsilon > 0$ and for all sufficiently large t*

$$\beta^{-\gamma}(1-\varepsilon)t \leqslant \psi(\phi(t)) \leqslant \beta^{-\gamma} t.$$

Proof. Mention that

$$\psi(\phi(t)) = \frac{t}{\beta^{\gamma}}\left(1 - \gamma\frac{\log\log t}{\log t}\right)^{\gamma}$$

and

$$\frac{\log\log t}{\log t}\downarrow 0$$

for $t \geqslant e^e$, which can be established by the differentiation. □

The main idea of Lemma 3 is that for a positive random variable Y, the assumptions $E\phi^{-1}(Y)<\infty$ and $E\psi(Y)<\infty$ are equivalent.

Lemma 4. *Let $\beta>1$ and $b_n = n^\beta \log^{-2} n,\ n\geqslant 2$, then*

$$\sum_{j=n}^{\infty} b_j^{-1} = \mathcal{O}(nb_n^{-1}).$$

Proof. Really,

$$\begin{aligned}\sum_{j=n}^{\infty}\frac{1}{b_j} &= \sum_{j=n}^{\infty}\frac{\log^2 j}{j^\beta} = \sum_{m=1}^{\infty}\sum_{k=nm}^{n(m+1)-1}\frac{\log^2 k}{k^\beta}\\ &\leqslant \sum_{m=1}^{\infty}\frac{n\log^2(mn)}{(nm)^\beta} \quad \left(\text{since the sequence}\left\{\frac{\log^2 k}{k^\beta}, k\geqslant e^{2/\beta}\right\}\text{ is strictly decreasing}\right)\\ &\leqslant \frac{2n\log^2 n}{n^\beta}\sum_{m=1}^{\infty}\frac{1+\log^2 m/\log^2 2}{m^\beta} = C\frac{n\log^2 n}{n^\beta} \quad (\text{since } \beta>1)\\ &= C\frac{n}{b_n}. \qquad \square\end{aligned}$$

3. Main results

With the preliminaries accounted for, we can formulate and prove the main result of the paper, that is the asymptotic probability of deviations for the bootstrap of the mean. We mention that there is no independence and identical distribution assumption on the original sequence of random variables $\{X_n,\ n\geqslant 1\}$.

Theorem 4. *Let $\psi(t),\ t\geqslant 0$ be an increasing function such that*

$$\sum_{j=n}^{\infty}\frac{1}{(\psi^{-1}(j))^2} = \mathcal{O}\left(\frac{n}{(\psi^{-1}(n))^2}\right), \quad n\geqslant 1, \tag{$*$}$$

where $\psi^{-1}(t)$ is the inverse function of $\psi(t)$. Let, moreover, $\{X_n, n\geqslant 1\}$ be a sequence of random variables stochastically dominated by a random variable X such that $E\psi(C|X|)<\infty$ for all $C>0$ and $\{a_n, n\geqslant 1\}$ be a sequence of positive constants. Then for almost every $\omega\in\Omega$, any $\varepsilon>0$, and any real number r

$$P^*\left\{\left|\sum_{j=1}^{m(n)}(\hat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega))\right|\geqslant \varepsilon a_n\right\} = \mathcal{O}\left(\exp\left\{-r\frac{a_n}{\psi^{-1}(n)} + \frac{m(n)}{n}\mathrm{o}(1)\right\}\right).$$

Proof. Fix any constants r and $\varepsilon > 0$ and let $h_n = rm(n)/\varepsilon\psi^{-1}(n)$. The fact that $h_n M_n \leqslant (r/\varepsilon)\,(1/\psi^{-1}(n)) \max_{1 \leqslant j \leqslant n} |X_j| \to 0$ a.s. follows directly from Lemma 2.

Next, in Lemma 1 consider $Y_j = X_j^2$, $Y = X^2$, and $\phi(t) = \psi(\sqrt{t})$. Then $\phi^{-1}(n) = (\psi^{-1}(n))^2$ and $E\phi(Y) = E\psi(|X|) < \infty$. By Lemma 1

$$h_n^2 B_n = \frac{r^2}{\varepsilon^2} \frac{m(n)}{n} \frac{1}{(\psi^{-1}(n))^2} \sum_{j=1}^{n} X_j^2 = \frac{m(n)}{n} \mathrm{o}(1) \ \text{ a.s.}$$

Hence,

$$\frac{h_n^2 B_n}{2(1 - h_n M_n)} = \frac{m(n)}{n} \ \mathrm{o}(1) \ \text{ a.s.}$$

We also mention that

$$\varepsilon \frac{h_n a_n}{m(n)} = r \frac{a_n}{\psi^{-1}(n)}.$$

Now the result follows directly from Theorem 3. $\square$

Remark 2. The conclusion of Theorem 4 is of course stronger the larger r is taken. The constant r does not play a role in any assumptions and it can be taken arbitrarily large.

Now we can derive different results on asymptotic probability of deviations for the bootstrap of the mean from the sample mean using different moment assumptions.

Corollary 1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables stochastically dominated by a random variable X and $0 < \alpha < 2$. If*

$$E|X|^{\alpha} < \infty,$$

then for almost every $\omega \in \Omega$

$$P^*\left\{ \frac{1}{n^{1/\alpha}} \left| \sum_{j=1}^{n} (\hat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega)) \right| \geqslant \varepsilon \right\} = \mathrm{o}(1),$$

that is,

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} \left(\hat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega) \right) \to 0 \quad \textit{in probability}.$$

Proof. Let $\psi(t) = t^{\alpha}$, then $\psi^{-1}(n) = n^{1/\alpha}$. The relation (*) holds trivially since $2/\alpha > 1$. If we take $a_n = n^{1/\alpha}$ and $m(n) = n$, then according to Theorem 4 for any $\varepsilon > 0$ and any $r > 0$ and for all sufficiently big n

$$P^*\left\{ \frac{1}{n^{1/\alpha}} \left| \sum_{j=1}^{n} \hat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega) \right| \geqslant \varepsilon \right\} \leqslant C \exp\{-r\},$$

that is,

$$\frac{1}{n^{1/\alpha}}\sum_{j=1}^{n}\left(\hat{X}_{n,j}^{(\omega)}-\overline{X}_n(\omega)\right)\to 0 \quad \text{in probability.} \qquad \square$$

Corollary 2. *Let* $\{X_n, n\geqslant 1\}$ *be a sequence of random variables stochastically dominated by a random variable* X *and* $0<\alpha<2$. *If*

$$E|X|^{\alpha}|\log|X||^{\alpha}<\infty,$$

then for every real number r, *every* $\varepsilon>0$, *and almost every* $\omega\in\Omega$

$$P^{*}\left\{\frac{1}{n^{1/\alpha}}\left|\sum_{j=1}^{n}\hat{X}_{n,j}^{(\omega)}-\overline{X}_n(\omega)\right|\geqslant\varepsilon\right\}=\mathcal{O}(n^{-r}).$$

Proof. Let $\psi(t)=t^{\alpha}\log^{\alpha}t, t\geqslant 1$, then according to Lemma 3 the sequence $\psi^{-1}(n)$ is equivalent to $n^{1/\alpha}/\log n,\ n\geqslant 2$. The relation (*) holds by Lemma 4.

For fixed $r,\ \varepsilon>0$, $m(n)=n$, and $a_n=n^{1/\alpha}$, applying Theorem 4 we obtain the result. $\square$

Remark 3. Theorem 2 easily follows from Corollary 2. It is sufficient to take for any constant q from Theorem 2, the constant $r=q+2$ and apply Corollary 2.

Corollary 3. *Let* $\{X_n, n\geqslant 1\}$ *be a sequence of random variables stochastically dominated by a random variable* X *and* $0<\alpha<2$. *If*

$$E|X|^{\delta}<\infty,$$

for some $\alpha<\delta<2$, *then for every* $r,\ \varepsilon>0$ *and almost every* $\omega\in\Omega$

$$P^{*}\left\{\frac{1}{n^{1/\alpha}}\left|\sum_{j=1}^{n}\hat{X}_{n,j}^{(\omega)}-\overline{X}_n(\omega)\right|\geqslant\varepsilon\right\}=\mathcal{O}(\exp\{-rn^{1/\alpha-1/\delta}\}).$$

Proof. Let $\psi(t)=t^{\delta}, t>0$, then $\psi^{-1}(n)=n^{1/\delta}$. The relation (*) holds trivially since $2/\delta>1$. For fixed $r,\ \varepsilon>0$, $m(n)=n$, and $a_n=n^{1/\alpha}$, applying Theorem 4 we obtain the result. $\square$

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