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RATE OF COMPLETE CONVERGENCE FOR ARRAYS OF BANACH SPACE VALUED RANDOM ELEMENTS

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Abstract

We obtain some results on complete convergence for arrays of rowwise independent Banach space valued random elements. This gives a solution to the open problem left by Ahmed et al. [2].

Key words and phrases: Banach space valued random elements, complete convergence, rowwise independent random elements, convergence in probability, Banach space of Rademacher-type p, moving average.

1. Introduction

The concept of complete convergence was first introduced by Hsu and Robbins in [6] as follows: A sequence of random variables $\{U_n,\, n \geq 1\}$ is said to *converge completely* to a constant c if $\sum_{n=1}^{\infty} \mathbb{P}\big\{|U_n - c| > \varepsilon\big\} < \infty$ for all $\varepsilon > 0$. By the Borel–Cantelli Lemma, this implies that $U_n \to c$ almost surely (a. s.), and the converse implication is true if $\{U_n,\, n \geq 1\}$ are independent. Hsu and Robbins proved in [6] that the sequence of arithmetic means of independent identically distributed random variables converges completely to the excepted value if the variance of the summands is finite.

From then on, many authors devote their study to complete convergence (see [5, 7–9]). Some of these articles concern a Banach space setting. A sequence of Banach space valued random elements is said to *converge completely*

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to the zero element of the Banach space if the corresponding sequence of norms converges completely to 0.

In [8], Hu et al. presented a general result establishing complete convergence for the row sums of an array of rowwise independent but not necessarily identically distributed Banach space valued random elements. Their result also specified the corresponding rate of convergence. The result of [8] unifies and extends the results obtained previously, e.g., in [5–8, 10].

In what follows, let $\{V_{nk},\ k \geq 1,\ n \geq 1\}$ be an array of rowwise independent but not necessarily identically distributed random elements taking values in a separable Banach space B, which are stochastically dominated by a random variable X (the technical definitions are given in the next section). Let $\{a_{nk},\ k \geq 1,\ n \geq 1\}$ be an array of constants such that

$$\sup_{k\geq 1} |a_{nk}| = O(n^{-\gamma}) \quad \text{for some} \quad \gamma > 0 \tag{1.1}$$

and, for every n, the series $S_n \equiv \sum_{k=1}^{\infty} a_{nk} V_{nk}$ converges almost surely (a. s.).

The following result is established in [7].

Theorem A. *Suppose that* (1.1) *holds and*

$$\sum_{k=1}^{\infty} |a_{nk}| = O(n^{\alpha}) \quad \textit{for some} \quad \alpha \in [0, \gamma).$$

If $\mathbb{E}|X|^{1+(1+\alpha+\beta)/\gamma} < \infty$ *for some* $\beta \in (-1, \gamma-\alpha-1]$ *and* $S_n \to 0$ *in probability then*

$$\sum_{n=1}^{\infty} n^{\beta} \mathbb{P}\big\{ \|S_n\| > \varepsilon \big\} < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{1.2}$$

We assume in Theorem A that the series S_n converges a. s., since the a. s. convergence is not automatic from the hypotheses. The proof of Theorem A is rather complicated once it uses the Stieltjes integral techniques, summation by parts and so on. The paper [2] not only provides a simpler proof of Theorem A but also establishes a more general result (cf. Theorem B below). The result of Theorem B is more general than the main results of [7], since the rates of convergence for moving averages can be established, which cannot be proved by using Theorem A.

Theorem B. *Suppose that* (1.1) *holds and*

$$\sum_{k=1}^{\infty} |a_{nk}| = O(n^{\alpha}) \quad \textit{for some} \quad \alpha < \gamma.$$

Let β *be such that* $\alpha + \beta \neq -1$. *Fix* $\delta > 0$ *such that* $(\alpha/\gamma) + 1 < \delta \leq 2$. *If* $\mathbb{E}|X|^{\nu} < \infty$, *where* $\nu = \max\big\{1+(1+\alpha+\beta)/\gamma, \delta\big\}$ *and* $S_n \to 0$ *in probability, then* (1.2) *holds.*

Theorem B was generalized in [15] as follows:

Theorem C. *Suppose that* (1.1) *holds and*

$$\sum_{k=1}^{\infty} |a_{nk}|^{\theta} = O(n^{\alpha}) \text{ for some } 0 < \theta \leq 2 \text{ and every } \alpha \text{ such that } \theta + \alpha/\gamma < 2.$$

Let β *be such that* $\alpha + \beta \neq -1$ *and let* $\delta > 0$ *be such that* $(\alpha/\gamma) + 1 < \delta \leq 2$. *If* $\mathbb{E}|X|^{\nu} < \infty$, *where* $\nu = \max\{1+(1+\alpha+\beta)/\gamma, \delta\}$ *and* $S_n \to 0$ *in probability, then* (1.2) *holds.*

If $\beta < -1$ then the conclusion of Theorems A, B, and C are immediate. Hence these theorems are of interest only for $\beta \geq -1$. The case $\beta = -1$ is of special interest. Ahmed et al. conjectured in [2] that if $\beta = -1$ then the assumption $\mathbb{E}|X|^{\nu} < \infty$ with $\nu > 1 + \alpha/\gamma$ can be replaced by the inequality $\mathbb{E}|X|^{1+\alpha/\gamma} \log^{\rho}(|X|) < \infty$ $(\rho > 0)$, where $\log(x) = \max\{1, \ln(x)\}$ and $\ln(x)$ denotes the natural logarithm function. In [14], Sung and Volodin give a positive answer to this problem and prove the following

Theorem D. *Suppose that* (1.1) *holds and*

$$\sum_{k=1}^{\infty} |a_{nk}|^{\theta} = O(n^{\alpha}) \text{ for some } \alpha > 0 \text{ and } \theta > 0 \text{ such that } \theta+\alpha/\gamma < 2. \quad (1.3)$$

If $\mathbb{E}|X|^{\theta+\alpha/\gamma} \log^{\rho}(|X|) < \infty$ *for some* $\rho > 0$ *and* $S_n \to 0$ *in probability then*

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\{\|S_n\| > \varepsilon\} < \infty \text{ for all } \varepsilon > 0.$$

The initial objective of this article was the following simple observation: The normalized partial sums of sequence of independent identically distributed random variables can be considered as a special case of weighted sums S_n in Theorem D. Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed mean zero random variables taking values in the real line $\mathbb{R}$. Given $n \geq 1$, consider an array $\{V_{nk}, k \geq 1, n \geq 1\}$, where $V_{nk} = X_k$ for $1 \leq k \leq n$ and $V_{nk} = 0$ for $k > n$, and the weights $a_{nk} = n^{1/t}$, $1 \leq t = \theta + \alpha/\gamma < 2$, $k \geq 1$. With this assumptions a particular case of Baum and Katz's result (see [3]) states that $\mathbb{E}|V_{11}|^t < \infty$ if and only if $\sum_{n=1}^{\infty} n^{-1}\mathbb{P}(|S_n| \geq \varepsilon) < \infty$ for every $\varepsilon > 0$. Hence the moment assumption $\mathbb{E}|X|^{\theta+\alpha/\gamma} \log^{\rho}|X| < \infty$ for some $\rho > 0$ in Theorem D may be nonoptimal due to the logarithmic term.

In this paper, we prove that Theorem D remains true if we replace the condition $\mathbb{E}|X|^{\theta+\alpha/\gamma} \log^{\rho}|X| < \infty$ by the strictly weaker and, hence, better moment condition $\mathbb{E}|X|^{\theta+\alpha/\gamma} < \infty$.

The article is organized as follows: In Section 2, we recall some definitions pertaining to the current work. In Section 3, we give the main result, Theorem E. As in Theorems A–D, in Theorem E no assumptions are made on the geometry of the underlying Banach space. An application of Theorem E to the case of Rademacher-type Banach spaces is presented in Corollary 1. Finally, in Section 4, we present a result on complete convergence of moving average processes.

2. Preliminaries

Let B be a real separable Banach space with norm $\|\cdot\|$. Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ denote a probability space. A random element X taking values in B is defined as a Borel measurable function from $\{\Omega, \mathcal{F}\}$ into B with the Borel sigma-algebra. The expected value of a B-valued random variable X is defined to be the Bochner integral and denoted by $\mathbb{E}X$.

A Banach space is said to be of *Rademacher-type* p, $1 \le p \le 2$, if there is a constant $C > 0$ such that

$$\mathbb{E}\left\|\sum_{i=1}^{n} X_i\right\|^p \le C \sum_{i=1}^{n} \mathbb{E}\|X_i\|^p$$

for all $n \ge 1$ and each sequence $\{X_n,\, n \ge 1\}$ of independent mean zero random elements taking values in B with finite pth moments.

We know that if B is of Rademacher-type $p > 1$ then, for each r, $1 \le r \le p$, B is of Rademacher-type r. Every separable Hilbert space and finite dimensional Banach space is of Rademacher-type 2.

Let $\{V_{nk},\, k \ge 1,\, n \ge 1\}$ be an array of rowwise independent but not necessarily identically distributed random elements taking values in B. The array of random elements $\{V_{nk},\, k \ge 1,\, n \ge 1\}$ is said to be *stochastically dominated* by a random variable X if there exists a constant D such that

$$\sup_{k\ge1,\, n\ge1} \mathbb{P}\big\{\|V_{nk}\| > x\big\} \le D\mathbb{P}\big\{|X| > x\big\}$$

for all $x > 0$. Let $\{a_{nk},\, k \ge 1,\, n \ge 1\}$ be an array of constants (called *weights*). Consider the sequence of *weighted sums* $S_n \equiv \sum_{k=1}^{\infty} a_{nk}V_{nk}$, $n \ge 1$.

In what follows, the symbol C will be used to denote various positive constants whose exact value is irrelevant.

Next, we present some technical lemmas which will be used to prove our main result.

Lemma 1. *Assume* (1.1) *and* (1.3). *Then for every* $\varepsilon > 0$ *and all* $n \ge 1$

$$\sum_{k=1}^{\infty} \mathbb{P}\big(\|a_{nk}V_{nk}\| > 1\big) \le Cn^{\alpha} \sum_{k=n}^{\infty} k^{\gamma\theta}\mathbb{P}\big(k < |X|^{1/\gamma} \le k+1\big).$$

Proof. Let $\{\phi(j),\, j \geq 1\}$ be an increasing sequence of positive numbers and

$$I_{nj} = \left\{k : \frac{1}{n^\gamma \phi(j+1)} < |a_{nk}| \leq \frac{1}{n^\gamma \phi(j)}\right\}, \quad j \geq 1,\ n \geq 1.$$

Given a finite set A, denote by $\sharp(A)$ the size of A, i.e., the number of elements in A. Then for every $m \geq 1$ we have

$$\sum_{j=1}^{m} \sharp(I_{nj}) \leq n^{\alpha+\gamma\theta} \phi^\theta(m+1).$$

Indeed,

$$\begin{aligned}
\sum_{j=1}^{m} \sharp(I_{nj}) &= \sum_{j=1}^{m} \sum_{k\in I_{nj}} |a_{nk}|^\theta \frac{1}{|a_{nk}|^\theta} \\
&\leq n^{\gamma\theta} \sum_{j=1}^{m} \phi^\theta(j+1) \sum_{k\in I_{nj}} |a_{nk}|^\theta \\
&\leq n^{\alpha+\gamma\theta} \phi^\theta(m+1).
\end{aligned}$$

Consider $\phi(j) = j^\gamma$, $j \geq 1$. We have

$$I_{nj} = \left\{k : \frac{1}{\big(n(j+1)\big)^\gamma} < |a_{nk}| \leq \frac{1}{(nj)^\gamma}\right\} \text{ for } j \geq 1 \text{ and } n \geq 1,$$

$$\sum_{j=1}^{m} \sharp\big(I_{nj}\big) \leq n^{\alpha+\gamma\theta} (m+1)^{\gamma\theta}.$$

Mention that the condition $\sup_{k\geq 1} |a_{nk}| \leq n^{-\gamma}$ implies $\bigcup_{j\geq 1} I_{nj} = \{k \mid a_{nk} \neq 0\}$ for every $n \geq 1$. It follows that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mathbb{P}\big(\|a_{nk}V_{nk}\| > 1\big) \\
&\quad = \sum_{j=1}^{\infty} \sum_{k\in I_{nj}} \mathbb{P}\big(\|a_{nk}V_{nk}\| > 1\big) \\
&\quad \leq \sum_{j=1}^{\infty} \sum_{k\in I_{nj}} \mathbb{P}\big(\|V_{nk}\| > (nj)^\gamma\big) \\
&\quad \leq C \sum_{j=1}^{\infty} \sharp(I_{nj}) \mathbb{P}\big(|X| > (nj)^\gamma\big)
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{j=1}^{\infty}\sharp(I_{nj})\sum_{k=nj}^{\infty}\mathbb{P}\big(k<|X|^{1/\gamma}\le k+1\big)\\
&= C\sum_{k=n}^{\infty}\mathbb{P}\big(k<|X|^{1/\gamma}\le k+1\big)\sum_{j=1}^{[k/n]}\sharp(I_{nj})\\
&\le C\sum_{k=n}^{\infty}\mathbb{P}\big(k<|X|^{1/\gamma}\le k+1\big)n^{\alpha+\gamma\theta}\big([k/n]+1\big)^{\gamma\theta}\\
&\le C2^{\gamma\theta}n^{\alpha}\sum_{k=n}^{\infty}\mathbb{P}\big(k<|X|^{1/\gamma}\le k+1\big)k^{\gamma\theta}. \quad \square
\end{aligned}
$$

The following assertion gives us a useful contraction principle and can be found in [12, Lemma 6.5].

Lemma 2. *Let $\{X_i,\, i\ge 1\}$ be a sequence of symmetric random elements. Let $\{\xi_i,\, i\ge 1\}$ and $\{\zeta_i,\, i\ge 1\}$ be real random variables such that $\xi_i=\phi_i(X_i)$, where $\phi_i\colon B\to\mathbb{R}$ is symmetric (even), and the same is true for ζ_i. If $|\xi_i|\le|\zeta_i|$ almost surely for every i and every $t>0$ then*

$$
\mathbb{P}\left\{\Big\|\sum_i \xi_i X_i\Big\|>t\right\}\le 2\mathbb{P}\left\{\Big\|\sum_i \zeta_i X_i\Big\|>t\right\}.
$$

In particular, this inequality applies to the case in which $\xi_i=I_{\{X_i\in A_i\}}\le 1\equiv\zeta_i$ with the sets A_i symmetric in B (for example, $A_i=\big\{\|x\|\le a_i\big\}$).

Lemma 3 (see below) is a modification of Kuelbs and Zinn's result of [11] concerning the relationship between convergence in probability and mean convergence for sums of independent bounded random variables. We refer to Lemma 2.1 of [8] for the proof.

Lemma 3. *Let the array $\{X_{nk},\, 1\le k\le k_n,\, n\ge 1\}$ of rowwise independent random elements be symmetric and let there exist $\delta>0$ such that $\|X_{nk}\|\le\delta$ a.s. for all $1\le k\le k_n$, $n\ge 1$. If $S_n\xrightarrow{\mathbb{P}}0$ then $\mathbb{E}\|S_n\|\to 0$ as $n\to\infty$.*

The next inequality is a Banach space analog of the classical Marcinkiewicz–Zygmund inequality. We present the inequality in the form of [1], but we should mention that the martingale representation of the difference $\|S_n\|-\mathbb{E}\|S_n\|$ with the useful estimation of the corresponding conditional expectations (from which Lemma 4 follows immediately by the classical Burkhölder inequality) was originally obtained by Yurinsky in [16, 17].

Lemma 4. *Let $\{X_{nk},\, 1\le k\le k_n,\, n\ge 1\}$ be an array of rowwise independent random elements. Then, for every $1\le p\le 2$, there is a positive* *constant* A_p *depending only on* p *such that for all* $n \geq 1$

$$\mathbb{E}\big|\,\|S_n\| - \mathbb{E}\|S_n\|\,\big|^p \leq A_p \sum_{k=1}^{k_n} \mathbb{E}\|X_{nk}\|^p.$$

3. The main results

We now state the main results.

Theorem E. *Assume* (1.1) *and* (1.3). *If* $\mathbb{E}|X|^{\theta+\alpha/\gamma} < \infty$ *and* $S_n \to 0$ *in probability. Then*

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\big\{\|S_n\| > \varepsilon\big\} < \infty \quad \textit{for all} \quad \varepsilon > 0. \tag{3.1}$$

Proof. Note that from (1.3) we have $\theta + \alpha/\gamma < 2$ and all parameters α, γ, and θ are assumed positive. Hence $0 < \theta < 2$. Since $S_n \to 0$ in probability, by the standard symmetric argument, we may assume that $\{V_{nk},\, k \geq 1,\, n \geq 1\}$ are symmetric. Given $\varepsilon > 0$, we have

$$\begin{aligned}\mathbb{P}\big\{\|S_n\| > \varepsilon\big\} \leq \sum_{k=1}^{\infty} \mathbb{P}\big\{\|a_{nk}V_{nk}\| > 1\big\} \\ + \mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}V_{nk}I\big(\|a_{nk}V_{nk}\| \leq 1\big)\right\| > \varepsilon\right\}.\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{\infty} \mathbb{P}\big\{\|a_{nk}V_{nk}\| > 1\big\} \\ &\quad \leq C \sum_{n=1}^{\infty} n^{\alpha-1} \sum_{k=n}^{\infty} k^{\gamma\theta}\mathbb{P}\big(k < |X|^{1/\gamma} \leq k+1\big) \quad \text{(by Lemma 1)} \\ &\quad = C \sum_{k=1}^{\infty} k^{\gamma\theta}\mathbb{P}\big(k < |X|^{1/\gamma} \leq k+1\big) \sum_{n=1}^{k} n^{\alpha-1} \\ &\quad \leq C \sum_{k=1}^{\infty} k^{\alpha+\gamma\theta}\mathbb{P}\big(k < |X|^{1/\gamma} \leq k+1\big) \quad (\text{since } \alpha > 0) \\ &\quad \leq C\mathbb{E}|X|^{\big((\alpha/\gamma)+\theta\big)} < \infty.\end{aligned}$$

Hence, to prove (3.1) it is enough to show that for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}V_{nk}I\big(\|a_{nk}V_{nk}\| \le 1\big)\right\| > \varepsilon\right\} < \infty. \tag{3.2}$$

Put $U_{nk} = V_{nk}I\big(\|a_{nk}V_{nk}\| \le 1\big)$. The sequence $\{U_{nk},\, k \ge 1,\, n \ge 1\}$ is also stochastically dominated by X. Let

$$U_{nk}^{(1)} = U_{nk}I\big(\|U_{nk}\| > n^{\gamma}\big), \quad U_{nk}^{(2)} = U_{nk}I\big(\|U_{nk}\| \le n^{\gamma}\big),$$

i.e., $U_{nk} = U_{nk}^{(1)} + U_{nk}^{(2)}$. Hence, to prove (3.2), it is enough to show that, for $p = 1, 2$ and for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left\{\left\|\sum_{k=1}^{\infty} a_{nk}U_{nk}^{(p)}\right\| > \varepsilon\right\} < \infty. \tag{3.3}$$

Since $S_n \to 0$ in probability, using the contraction principle in Lemma 2, we infer that $\sum_{k=1}^{\infty} a_{nk}U_{nk}^{(p)} \to 0$ in probability, and note that $\big\|a_{nk}U_{nk}^{(p)}\big\| \le 1$ for all $k \ge 1$, $n \ge 1$, and $p = 1, 2$. By Lemma 3, we have

$$\mathbb{E}\left\|\sum_{k=1}^{\infty} a_{nk}U_{nk}^{(p)}\right\| \to 0, \quad p = 1, 2.$$

Hence, in order to prove (3.3), it is enough to show that

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left\{\left|\left\|\sum_{k=1}^{\infty} a_{nk}U_{nk}^{(p)}\right\| - b_n^{(p)}\right| > \varepsilon\right\} < \infty, \quad p = 1, 2,$$

where

$$b_n^{(1)} = 0 \quad \text{for} \quad 0 < \theta < 1,$$

$$b_n^{(1)} = \mathbb{E}\left\|\sum_{k=1}^{\infty} a_{nk}U_{nk}^{(1)}\right\| \quad \text{for} \quad 1 \le \theta < 2,$$

$$b_n^{(2)} = \mathbb{E}\left\|\sum_{k=1}^{\infty} a_{nk}U_{nk}^{(2)}\right\| \quad \text{for} \quad 0 \le \theta < 2.$$

Given $p = 1$, by Markov's inequality and c_r-inequality if $0 < \theta < 1$ (cf. [13, Chapter III, Section 9.3]) and by Lemma 4 if $1 \le \theta < 2$, we have

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left\{\left|\left\|\sum_{k=1}^{\infty} a_{nk}U_{nk}^{(1)}\right\| - b_n^{(1)}\right| > \varepsilon\right\}$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{-1}\mathbb{E}\left|\left\|\sum_{k=1}^{\infty} a_{nk}U_{nk}^{(1)}\right\| - b_n^{(1)}\right|^{\theta} \\
&\le C\sum_{n=1}^{\infty} n^{-1}\sum_{k=1}^{\infty} |a_{nk}|^{\theta}\mathbb{E}\big\|U_{nk}^{(1)}\big\|^{\theta} \\
&\le C\sum_{n=1}^{\infty} n^{-1+\alpha}\mathbb{E}|X|^{\theta} I\big(|X| > n^{\gamma}\big) \\
&\le C\sum_{n=1}^{\infty} n^{-1+\alpha}\sum_{k=n}^{\infty} \mathbb{E}|X|^{\theta} I\big(k^{\gamma} < |X| \le (k+1)^{\gamma}\big) \\
&= C\sum_{k=1}^{\infty} \mathbb{E}|X|^{\theta} I\big(k^{\gamma} < |X| \le (k+1)^{\gamma}\big)\sum_{n=1}^{k} n^{-1+\alpha} \\
&\le \sum_{k=1}^{\infty} k^{\alpha}\mathbb{E}|X|^{\theta} I\big(k^{\gamma} < |X| \le (k+1)^{\gamma}\big) \\
&\le C\mathbb{E}|X|^{\theta+\alpha/\gamma} < \infty.
\end{aligned}
$$

Given $p = 2$, by Markov's inequality and c_r-inequality if $0 < \theta < 1$ and by Lemma 4 if $1 \le \theta < 2$, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left\{\left|\left\|\sum_{n=1}^{\infty} a_{nk}U_{nk}^{(2)}\right\| - \mathbb{E}\left\|\sum_{n=1}^{\infty} a_{nk}U_{nk}^{(2)}\right\|\right| > \varepsilon\right\} \\
&\quad\le C\sum_{n=1}^{\infty} n^{-1}\mathbb{E}\left|\left\|\sum_{n=1}^{\infty} a_{nk}U_{nk}^{(2)}\right\| - \mathbb{E}\left\|\sum_{n=1}^{\infty} a_{nk}U_{nk}^{(2)}\right\|\right|^{2} \\
&\quad\le C\sum_{n=1}^{\infty} n^{-1}\sum_{k=1}^{\infty} |a_{nk}|^{2}\mathbb{E}\left\|U_{nk}^{(2)}\right\|^{2} \\
&\quad\le C\sum_{n=1}^{\infty} n^{-1}\left(\sup_{k\ge 1}|a_{nk}|\right)^{2-\theta}\sum_{k=1}^{\infty} |a_{nk}|^{\theta}\mathbb{E}|X|^{2} I\big(|X| \le n^{\gamma}\big) \\
&\quad\le C\sum_{n=1}^{\infty} n^{-1-\gamma(2-\theta)+\alpha}\sum_{k=1}^{n} \mathbb{E}|X|^{2} I\big((k-1)^{\gamma} < |X| \le k^{\gamma}\big) \\
&\quad\le C\sum_{k=1}^{\infty} \mathbb{E}|X|^{2} I\big((k-1)^{\gamma} < |X| \le k^{\gamma}\big)\sum_{n=k}^{\infty} n^{-1-\gamma(2-\theta)+\alpha} \\
&\quad\le C\sum_{k=1}^{\infty} k^{-\gamma(2-\theta)+\alpha}\mathbb{E}|X|^{2} I\big((k-1)^{\gamma} < |X| \le k^{\gamma}\big) \\
&\quad\le C\mathbb{E}|X|^{\theta+\alpha/\gamma} < \infty. \quad \square
\end{aligned}
$$

If we assume that the Banach space under consideration is of Rademacher-type p, $1 \le p \le 2$, then the following result can be formulated and proved.

Corollary 1. *Let B is of Rademacher-type p, $1 < p \le 2$. Assume that (1.1) holds, $\mathbb{E}V_{nk} = 0$ for all $n \ge 1$ and $k \ge 1$, and*

$$\sum_{k=1}^{\infty} |a_{nk}|^{\theta} = O\big(n^{\alpha}\big) \text{ for some } \alpha > 0 \text{ and } \theta > 0 \text{ such that } \theta + \alpha/\gamma < p.$$

If $\mathbb{E}|X|^{\theta+\alpha/\gamma} < \infty$ then (3.1) holds.

Proof. We need to prove that

$$S_n \equiv \sum_{k=1}^{\infty} a_{nk} V_{nk} \to 0$$

in probability because all other assumptions of Theorem E are obviously satisfied.

For $0 < \theta < 1$ we have

$$\begin{aligned}
&\mathbb{P}\big\{\|S_n\| > \varepsilon\big\} \\
&\quad \le \mathbb{P}\left\{\left\|\sum_{k=1}^{n} a_{nk} V_{nk} I\big(\|V_{nk}\| > n^{\gamma}\big)\right\| > \varepsilon/2\right\} \\
&\qquad + \mathbb{P}\left\{\left\|\sum_{k=1}^{n} a_{nk} V_{nk} I\big(\|V_{nk}\| \le n^{\gamma}\big)\right\| > \varepsilon/2\right\} \\
&\quad = I_1 + I_2.
\end{aligned}$$

For I_1, using Markov's inequality and the c_r-inequality, we infer

$$\begin{aligned}
I_1 &\le C \sum_{k=1}^{\infty} \big|a_{nk}\big|^{\theta} \mathbb{E}\|V_{nk}\|^{\theta} \le C n^{\alpha} \mathbb{E}|X| I\big(|X| > n^{\gamma}\big) \\
&\le C\mathbb{E}|X|^{\theta+\alpha/\gamma} I\big(|X| > \gamma\big) \to 0.
\end{aligned}$$

For I_2, since $0 < \theta < 1$ and $\mathbb{E}V_{nk} = 0$, we have

$$\begin{aligned}
&\left\|\mathbb{E}\sum_{k=1}^{n} a_{nk} V_{nk} I\Big(|V_{nk}| \le n^{\gamma}\Big)\right\| \\
&\qquad \le \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}\|V_{nk}\| I\big(\|V_{nk}\| > n^{\gamma}\big)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{k=1}^{\infty} |a_{nk}| \mathbb{E}|X| I\big(|X| > n^{\gamma}\big) \\
&\le C n^{\alpha-\gamma(1-\theta)} \mathbb{E}|X| I\big(|X| > n^{\gamma}\big) \\
&\le \begin{cases} C\mathbb{E}|X| I\big(|X| > n^{\gamma}\big) & \text{if } \ \alpha - \gamma(1-\theta) \le 0, \\ C\mathbb{E}|X|^{\theta+\alpha/\gamma} I\big(|X| > n^{\gamma}\big) & \text{if } \ \alpha - \gamma(1-\theta) > 0, \end{cases}
\end{aligned}
$$

where the tail expectations on the right-hand side of the inequality vanish in both cases as $n \to \infty$. For n large enough, from (2.1) and the dominated convergence theorem we obtain

$$
\begin{aligned}
I_2 &\le \mathbb{P}\left\{ \left\| \sum_{k=1}^{n} a_{nk} V_{nk} I\big(|V_{nk}| \le n^{\gamma}\big) - \mathbb{E} \sum_{k=1}^{n} a_{nk} V_{nk} I\big(|V_{nk}| \le n^{\gamma}\big) \right\| > \varepsilon/4 \right\} \\
&\le \sum_{n=1}^{\infty} |a_{nk}|^{p} \mathbb{E}\|V_{nk}\|^{p} I\big(\|V_{nk}\| \le n^{\gamma}\big) \le n^{\alpha-\gamma(p-\theta)} \mathbb{E}|X|^{p} I\big(|X| \le n^{\gamma}\big) \\
&\le C n^{\alpha+\gamma\theta} \int_0^1 x^{p-1} \mathbb{P}\big\{|X| > x n^{\gamma}\big\} dx \to 0.
\end{aligned}
$$

If $1 \le \theta < p$ then

$$
\begin{aligned}
&\mathbb{P}\big\{\|S_n\| > \varepsilon\big\} \\
&\quad \le \mathbb{P}\left\{ \left\| \sum_{k=1}^{n} a_{nk} V_{nk} I\big(\|V_{nk}\| > n^{\gamma}\big) - \mathbb{E} \sum_{k=1}^{n} a_{nk} V_{nk} I\big(\|V_{nk}\| > n^{\gamma}\big) \right\| > \varepsilon/2 \right\} \\
&\qquad + \mathbb{P}\left\{ \left\| \sum_{k=1}^{n} a_{nk} V_{nk} I\big(\|V_{nk}\| \le n^{\gamma}\big) - \mathbb{E} \sum_{k=1}^{n} a_{nk} V_{nk} I\big(\|V_{nk}\| \le n^{\gamma}\big) \right\| > \varepsilon/2 \right\} \\
&\quad = I_3 + I_4.
\end{aligned}
$$

Using Markov's inequality and the definition of Rademacher-type p, for I_3 we have

$$
\begin{aligned}
I_3 &\le C \sum_{k=1}^{\infty} |a_{nk}|^{\theta} \mathbb{E}\|V_{nk}\|^{\theta} I\big(\|V_{nk}\| > n^{\gamma}\big) \\
&\le C n^{\alpha} \mathbb{E}|X|^{\theta} I\big(|X| > n^{\gamma}\big) \\
&\le C\mathbb{E}|X|^{\theta+\alpha/\gamma} I\big(|X| > n^{\gamma}\big) \to 0.
\end{aligned}
$$

The proof of $I_4 \to 0$ is similar to the proof of $I_2 \to 0$. $\square$

4. Complete convergence of moving average processes

For the application of Theorem E, we obtain one result about the rate of complete convergence of moving average processes.

Corollary 2. *Assume that* $\{Y_i, -\infty < i < \infty\}$ *is a doubly infinite sequence of independent* B*-valued random elements and is stochastically dominated by a random variable* X. *Let* $\{a_i, -\infty < i < \infty\}$ *be an absolutely summable sequence of real numbers and let* $V_i = \sum_{k=-\infty}^{\infty} a_{i+k} Y_k$, $i \geq 1$. *If* $n^{-1/\nu} \sum_{i=1}^{n} V_i \to 0$ *in probability and* $\mathbb{E}|X|^{\nu} < \infty$, *where* $1 < \nu < 2$, *then*

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left\{ \left\| \sum_{i=1}^{n} V_i \right\| > \varepsilon n^{1/\nu} \right\} < \infty \quad \text{for all } \varepsilon > 0.$$

Proof. Since all series converge absolutely, it is only a matter of renumeration that Theorem E can be applied in this situation. Put

$$a_{nk} = n^{1/\nu} \sum_{i=1}^{n} a_{k+i} \quad \text{and} \quad V_{nk} = Y_k \quad \text{for all} \quad n \geq 1, \; k \geq 1.$$

Then $n^{1/\nu} \sum_{i=1}^{n} V_i = \sum_{k=-\infty}^{\infty} a_{nk} V_{nk}$.

Letting $a = \sum_{i=-\infty}^{\infty} a_i$ and $b = \sum_{i=-\infty}^{\infty} |a_i|$, we have $\sup |a_{nk}| \leq b n^{-1/\nu}$ and, by Lemma 1 of [4], $\sum_{k=-\infty}^{\infty} |a_{nk}| < C|a| n^{1-1/\nu}$. The result follows by Theorem E with $\gamma = 1/\nu$, $\theta = 1$, and $\alpha = 1 - 1/\nu$. □

The proof of the next result is similar to that of Corollary 1 with the only difference that we should refer to Corollary 2 instead of Theorem E. Because of that the proof is omitted.

Corollary 3. *Assume that* $\{Y_i, -\infty < i < \infty\}$ *is a doubly infinite sequence of independent means zero random elements taking values in a separable real Banach space* B *of Rademacher-type* p, $1 < p \leq 2$, *and is stochastically dominated by a random variable* X. *Let* $\{a_i, -\infty < i < \infty\}$ *be an absolutely summable sequence of real numbers and let* $V_i = \sum_{k=-\infty}^{\infty} a_{i+k} Y_k$, $i \geq 1$. *If* $\mathbb{E}|X|^{\nu} < \infty$, *where* $1 \leq \nu < p$, *then*

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left\{ \left\| \sum_{i=1}^{m} V_i \right\| > \varepsilon n^{1/\nu} \right\} < \infty \quad \text{for all } \varepsilon > 0.$$

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