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On Complete Convergence in Mean of Normed Sums of Independent Random Elements in Banach Spaces

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Abstract: For a sequence of random elements $\{T_n, n \geq 1\}$ in a real separable Banach space $\mathscr{X}$, we study the notion of T_n *converging completely to 0 in mean of order* p where p is a positive constant. This notion is stronger than (i) T_n converging completely to 0 and (ii) T_n converging to 0 in mean of order p. When $\mathscr{X}$ is of Rademacher type p $(1 \leq p \leq 2)$, for a sequence of independent mean 0 random elements $\{V_n, n \geq 1\}$ in $\mathscr{X}$ and a sequence of constants $b_n \to \infty$, conditions are provided under which the normed sum $\sum_{j=1}^{n} V_j/b_n$ converges completely to 0 in mean of order p. Moreover, these conditions for $\sum_{j=1}^{n} V_j/b_n$ converging completely to 0 in mean of order p are shown to provide an exact characterization of Rademacher type p Banach spaces. Illustrative examples are provided.

Keywords: Complete convergence; Complete convergence in mean; Convergence in mean; Normed sums of independent random elements; Rademacher type p Banach space; Real separable Banach space.

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1. INTRODUCTION

Let $\{T_n, n \geq 1\}$ be a sequence of random elements in a real separable Banach space $\mathscr{X}$ with norm $\|\cdot\|$. We recall that T_n is said to *converge completely to 0* (denoted $T_n \xrightarrow{c} 0$) if

$$\sum_{n=1}^{\infty} P\{\|T_n\| > \varepsilon\} < \infty \quad \text{for all } \varepsilon > 0$$

and that for $p > 0$, T_n is said to *converge to 0 in mean of order p* (denoted $T_n \xrightarrow{\mathscr{L}_p} 0$) if

$$E\|T_n\|^p \to 0.$$

In general, the modes of convergence $T_n \xrightarrow{c} 0$ and $T_n \xrightarrow{\mathscr{L}_p} 0$ are not comparable. Of course, by the Borel-Cantelli lemma, $T_n \xrightarrow{c} 0$ ensures that $T_n \to 0$ almost surely (a.s.).

In this note, we study the following (very strong) convergence notion. For $p > 0$, T_n is said to *converge completely to 0 in mean of order p* if $\sum_{n=1}^{\infty} E\|T_n\|^p < \infty$. This will be denoted by $T_n \xrightarrow{c,\mathscr{L}_p} 0$. This mode of convergence was apparently first investigated by Chow [2] in the (real-valued) random variable case; our results and his do not entail each other. Clearly, this mode of convergence implies that $T_n \xrightarrow{\mathscr{L}_p} 0$ and (by the Markov inequality) $T_n \xrightarrow{c} 0$. Two examples will be provided in Section 4 showing that $T_n \xrightarrow{\mathscr{L}_p} 0$ and $T_n \xrightarrow{c} 0$ do not imply that $T_n \xrightarrow{c,\mathscr{L}_p} 0$.

The main results of this paper are Theorems 1 and 2. Theorem 1 provides conditions under which a normed sum of independent random elements in a real separable *Rademacher type p* ($1 \leq p \leq 2$) Banach space converges completely to 0 in mean of order p. (Technical definitions such as Rademacher type p will be reviewed in Section 2.) An example will be given in Section 4 showing that Theorem 1 is sharp. In Theorem 2, it is shown that the implication in Theorem 1 provides an exact characterization of Rademacher type p Banach spaces.

2. PRELIMINARIES

Some definitions and preliminary results will be presented prior to establishing the main results.

The *expected value* or *mean* of a random element V, denoted EV, is defined to be the *Pettis integral*, provided it exists. That is, V has expected value $EV \in \mathscr{X}$ if $f(EV) = E(f(V))$ for every $f \in \mathscr{X}^*$ where $\mathscr{X}^*$ denotes the (*dual*) space of all continuous linear functionals on $\mathscr{X}$. If $E\|V\| < \infty$, then (see, e.g., Taylor [10], p. 40) V has an expected value. But, the expected value can exist when $E\|V\| = \infty$. For an example, see Taylor [10], p. 41.

Let $\{Y_n, n \geq 1\}$ be a symmetric *Bernoulli sequence*; that is, $\{Y_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with $P\{Y_1 = 1\} = P\{Y_1 = -1\} = 1/2$. Let $\mathscr{X}^\infty = \mathscr{X} \times \mathscr{X} \times \mathscr{X} \times \cdots$ and define

$$\mathscr{C}(\mathscr{X}) = \left\{(v_1, v_2, \dots) \in \mathscr{X}^\infty : \sum_{n=1}^{\infty} Y_n v_n \text{ converges in probability}\right\}.$$

Let $1 \leq p \leq 2$. Then $\mathscr{X}$ is said to be of *Rademacher type* p if there exists a constant $0 < C < \infty$ such that

$$E\left\|\sum_{n=1}^{\infty} Y_n v_n\right\|^p \leq C \sum_{n=1}^{\infty} \|v_n\|^p \quad \text{for all } (v_1, v_2, \dots) \in \mathscr{C}(\mathscr{X}).$$

Hoffmann-Jørgensen and Pisier [5] proved for $1 \leq p \leq 2$ that a real separable Banach space is of Rademacher type p if and only if there exists a constant $0 < C < \infty$ such that

$$E\left\|\sum_{i=1}^{n} V_i\right\|^p \leq C \sum_{i=1}^{n} E\|V_i\|^p \tag{1}$$

for every finite collection $\{V_1, \dots, V_n\}$ of independent mean 0 random elements.

If a real separable Banach space is of Rademacher type p for some $1 < p \leq 2$, then it is of Rademacher type q for all $1 \leq q < p$. Every real separable Banach space is of Rademacher type (at least) 1 while the $\mathscr{L}_p$-spaces and ℓ_p-spaces are of Rademacher type $2 \wedge p$ for $p \geq 1$. Every real separable Hilbert space and real separable finite-dimensional Banach space is of Rademacher type 2. In particular, the real line R is of Rademacher type 2.

Lemma 1 (Adler and Rosalsky [1]). *Let $\{b_n, n \geq 1\}$ be a sequence of positive constants with $b_n^p/n \uparrow$ for some $p > 0$. Then*

$$\sum_{j=n}^{\infty} \frac{1}{b_j^p} = \mathscr{O}\left(\frac{n}{b_n^p}\right)$$

if and only if

$$\liminf_{n\to\infty} \frac{b_{kn}^p}{b_n^p} > k \quad \text{for some integer } k \geq 2. \tag{2}$$

Lemma 2. *Let $\{V_n, n \geq 1\}$ be a sequence of independent mean 0 random elements in a real separable Banach space. Then for all $p \geq 1$,*

$$E\left\|\sum_{j=1}^{n} V_j\right\|^p \quad \text{is nondecreasing.} \tag{3}$$

Proof. Let $\mathcal{F}_n = \sigma(V_1, \dots, V_n)$, $n \geq 1$. Now it is well known but seems to have been first observed by Scalora [8] that $\{\| \sum_{j=1}^n V_j \|, \mathcal{F}_n, n \geq 1\}$ is a real submartingale and hence so is (see, e.g., Chow and Teicher [3], p. 244) $\{\| \sum_{j=1}^n V_j \|^p, \mathcal{F}_n, n \geq 1\}$ via convexity and monotonicity of the function $g(x) = x^p$, $0 \leq x < \infty$. The conclusion (3) follows immediately. □

The following result of Hoffmann-Jørgensen and Pisier [5] is a random element analogue of a classical result of Kolmogorov.

Proposition 1 (Hoffmann-Jørgensen and Pisier [5]). *Let* $1 \leq p \leq 2$ *and let* $\mathcal{X}$ *be a real separable Banach space. Then the following statements are equivalent.*

(i) $\mathcal{X}$ *is of Rademacher type* p.
(ii) *For every sequence* $\{V_n, n \geq 1\}$ *of independent mean* 0 *random elements in* $\mathcal{X}$, *the condition*

$$\sum_{n=1}^{\infty} \frac{E\|V_n\|^p}{n^p} < \infty \tag{4}$$

implies that the strong law of large numbers (*SLLN*)

$$\frac{\sum_{j=1}^n V_j}{n} \to 0 \quad a.s.$$

obtains.

Proposition 2 (Kuelbs and Zinn [7]). *Let* $\{V_n, n \geq 1\}$ *be a sequence of independent random elements in a real separable Banach space* $\mathcal{X}$ *and suppose that* (4) *holds for some* $1 \leq p \leq 2$. *Then*

$$\frac{\sum_{j=1}^n V_j}{n} \to 0 \ a.s. \quad \text{if and only if} \quad \frac{\sum_{j=1}^n V_j}{n} \xrightarrow{P} 0.$$

Proposition 3 (Etemadi [4]). *Let* $\{V_n, n \geq 1\}$ *be a sequence of independent random elements in a real separable Banach space. Then*

$$\frac{\sum_{j=1}^n V_j}{n} \to 0 \quad a.s.$$

if and only if

$$\frac{\sum_{j=1}^n V_j}{n} \xrightarrow{P} 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n} P\left\{ \left\| \sum_{j=n+1}^{2n} V_j \right\| > n\varepsilon \right\} < \infty \quad \text{for all } \varepsilon > 0.$$

The next result, also due to Etemadi [4], is a Banach space version of a famous result of Spitzer [9]. Note that the summands are assumed to be i.i.d.

Proposition 4 (Etemadi [4]). *Let $\{V_n, n \geq 1\}$ be a sequence of i.i.d. random elements in a real separable Banach space. Then*

$$\frac{\sum_{j=1}^{n} V_j}{n} \to 0 \quad a.s.$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left\{ \left\| \sum_{j=1}^{n} V_j \right\| > n\varepsilon \right\} < \infty \quad \text{for all } \varepsilon > 0. \tag{5}$$

Finally, a remark about notation is in order. The symbol C denotes throughout a generic constant ($0 < C < \infty$), which is not necessarily the same one in each appearance.

3. THE MAIN RESULTS

With the preliminaries accounted for, Theorem 1 may now be established. The sequence $\{b_n, n \geq 1\}$ is not assumed to be monotone increasing.

Theorem 1. *Let $\{V_n, n \geq 1\}$ be a sequence of independent mean 0 random elements in a real separable Rademacher type p $(1 \leq p \leq 2)$ Banach space $\mathscr{X}$ and let $\{b_n, n \geq 1\}$ be a sequence of positive constants with $\sum_{n=1}^{\infty} b_n^{-p} < \infty$. If*

$$\sum_{n=1}^{\infty} \varphi(n) E\|V_n\|^p < \infty, \tag{6}$$

where $\varphi(n) = \sum_{j=n}^{\infty} b_j^{-p}$, $n \geq 1$, then

$$\frac{\sum_{j=1}^{n} V_j}{b_n} \xrightarrow{c, \mathscr{L}_p} 0. \tag{7}$$

Proof. Set $T_n = \sum_{j=1}^{n} V_j / b_n$, $n \geq 1$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} E\|T_n\|^p &\leq C \sum_{n=1}^{\infty} \frac{\sum_{j=1}^{n} E\|V_j\|^p}{b_n^p} \quad \text{(by (1))} \\
&= C \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \frac{E\|V_j\|^p}{b_n^p} \\
&= C \sum_{j=1}^{\infty} E\|V_j\|^p \varphi(j) \\
&< \infty \quad \text{(by (6))}.
\end{aligned}$$

□

Remark 1. (i) Theorem 1 is new even if $\mathscr{X}$ is the real line.

(ii) A perusal of the argument in Theorem 1 reveals that if $p = 1$, then the independence hypothesis and the hypothesis that the $\{V_n, n \geq 1\}$ have mean 0 are not needed for the theorem to hold.

While the proof of Theorem 1 was not difficult, we will show in Theorem 2 that the implication ((6) $\Longrightarrow$ (7)) indeed completely characterizes Rademacher type p Banach spaces.

Theorem 2. *Let $1 \leq p \leq 2$ and let $\mathscr{X}$ be a real separable Banach space. Then the following statements are equivalent.*

(i) *$\mathscr{X}$ is of Rademacher type p.*

(ii) *For every sequence $\{V_n, n \geq 1\}$ of independent mean 0 random elements in $\mathscr{X}$ and every sequence $\{b_n, n \geq 1\}$ of positive constants with $\sum_{n=1}^{\infty} b_n^{-p} < \infty$, the condition* (6) *(where $\varphi(n) = \sum_{j=n}^{\infty} b_j^{-p}$, $n \geq 1$) implies* (7).

(iii) *For every sequence $\{V_n, n \geq 1\}$ of independent mean 0 random elements in $\mathscr{X}$ and every sequence $\{b_n, n \geq 1\}$ of positive constants satisfying*

$$\sum_{j=n}^{\infty} \frac{1}{b_j^p} = \mathscr{O}\left(\frac{n}{b_n^p}\right), \tag{8}$$

the condition

$$\sum_{n=1}^{\infty} \frac{nE\|V_n\|^p}{b_n^p} < \infty$$

implies (7).

(iv) *For every sequence $\{V_n, n \geq 1\}$ of independent mean 0 random elements in $\mathscr{X}$, the condition*

$$\sum_{n=1}^{\infty} \frac{E\|V_n\|^p}{n^p} < \infty \tag{9}$$

implies

$$\frac{\sum_{j=1}^{n} V_j}{n^{\frac{p+1}{p}}} \xrightarrow{c, \mathscr{L}_p} 0. \tag{10}$$

Proof. The implication ((i) $\Longrightarrow$ (ii)) is precisely Theorem 1 whereas the implications ((ii) $\Longrightarrow$ (iii)) and ((iii) $\Longrightarrow$ (iv)) are immediate. It remains to verify the implication ((iv) $\Longrightarrow$ (i)). Assume that (iv) holds. Let $\{V_n, n \geq 1\}$ be a sequence of independent mean 0 random elements in $\mathscr{X}$

such that $\sum_{n=1}^{\infty} E\|V_n\|^p/n^p < \infty$. In view of Proposition 1, it suffices to verify that

$$\frac{\sum_{j=1}^{n} V_j}{n} \to 0 \quad \text{a.s.} \tag{11}$$

Now (10) holds by $\sum_{n=1}^{\infty} E\|V_n\|^p/n^p < \infty$ and (iv) and so

$$\sum_{n=1}^{\infty} \frac{1}{n^{p+1}} E\left\| \sum_{j=1}^{n} V_j \right\|^p = \sum_{n=1}^{\infty} E\left\| \frac{\sum_{j=1}^{n} V_j}{n^{\frac{p+1}{p}}} \right\|^p < \infty. \tag{12}$$

Note that

$$\sum_{m=n}^{\infty} \frac{1}{m^{p+1}} \sim \frac{1}{pn^p}. \tag{13}$$

Thus

$$\begin{aligned} E\left\| \frac{\sum_{j=1}^{n} V_j}{n} \right\|^p &= \frac{1}{n^p} E\left\| \sum_{j=1}^{n} V_j \right\|^p \\ &\le C \sum_{m=n}^{\infty} \frac{1}{m^{p+1}} E\left\| \sum_{j=1}^{n} V_j \right\|^p \quad \text{(by (13))} \\ &\le C \sum_{m=n}^{\infty} \frac{1}{m^{p+1}} E\left\| \sum_{j=1}^{m} V_j \right\|^p \quad \text{(by Lemma 2)} \\ &\to 0 \quad \text{(by (12))}. \end{aligned}$$

Then by the Markov inequality $\sum_{j=1}^{n} V_j/n \xrightarrow{P} 0$, and so (11) holds by Proposition 2, thereby completing the proof of the implication ((iv) $\Longrightarrow$ (i)). □

Remark 2. (i) If $b_n \uparrow$, there is a trade-off between the Rademacher type and the condition (8); the larger is p, the stronger is the condition on the Banach space $\mathscr{X}$ whereas the condition (8) is weaker for larger p. To see that (8) becomes weaker as p increases, note that if $1 \le p_1 < p_2 \le 2$, then

$$b_n^{p_2} \sum_{j=n}^{\infty} \frac{1}{b_j^{p_2}} = b_n^{p_1} \sum_{j=n}^{\infty} \frac{b_n^{p_2-p_1}}{b_j^{p_2-p_1} b_j^{p_1}} \le b_n^{p_1} \sum_{j=n}^{\infty} \frac{1}{b_j^{p_1}} \quad (\text{since } b_j \uparrow)$$

and so if (8) holds with $p = p_1$, then (8) also holds with $p = p_2$.

(ii) If $b_n^p/n \uparrow$, it follows from Lemma 1 that (8) is equivalent to the structurally simpler condition (2).

(iii) For $b_n = n^r$, $n \geq 1$ of Marcinkiewicz-Zygmund growth where $r > 1/p$, the condition (8) automatically holds. This is easy to verify directly, but it also follows from Remark 2(ii) with $k = 2$.

The following corollary follows from Proposition 1 and the implication ((i) $\Longrightarrow$ (iv)) of Theorem 2.

Corollary 1. *Let* $\{V_n, n \geq 1\}$ *be a sequence of independent mean* 0 *random elements in a real separable Rademacher type* p $(1 \leq p \leq 2)$ *Banach space and suppose that* $\sum_{n=1}^{\infty} E\|V_n\|^p/n^p < \infty$. *Then*

$$\frac{\sum_{j=1}^{n} V_j}{n} \to 0 \quad a.s. \quad and \quad \frac{\sum_{j=1}^{n} V_j}{n^{\frac{p+1}{p}}} \xrightarrow{c,\mathscr{L}_p} 0. \tag{14}$$

Note that the second half of (14) is the assertion that

$$\sum_{n=1}^{\infty} \frac{1}{n} E\left\| \frac{\sum_{j=1}^{n} V_j}{n} \right\|^p < \infty$$

and this implies (5) by the Markov inequality.

The last theorem to be presented shows that the limit law (10) ensures that the SLLN $\sum_{j=1}^{n} V_j/n \to 0$ a.s. holds. We emphasize that we are not assuming that the Banach space is of Rademacher type p for some $1 < p \leq 2$.

Theorem 3. *Let* $\{V_n, n \geq 1\}$ *be a sequence of independent random elements in a real separable Banach space. If* (10) *holds for some* $1 \leq p \leq 2$, *then the SLLN*

$$\frac{\sum_{j=1}^{n} V_j}{n} \to 0 \quad a.s. \tag{15}$$

obtains.

Proof. Note that (10) is tantamount to

$$\sum_{n=1}^{\infty} \frac{1}{n^{p+1}} E\left\| \sum_{j=1}^{n} V_j \right\|^p = \sum_{n=1}^{\infty} \frac{1}{n} E\left\| \frac{\sum_{j=1}^{n} V_j}{n} \right\|^p < \infty. \tag{16}$$

Proceeding as in the proof of Theorem 2, we get from (16) that

$$\frac{\sum_{j=1}^{n} V_j}{n} \xrightarrow{P} 0.$$

Moreover, it follows from (16) and the Markov inequality that for arbitrary $\varepsilon > 0$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n} P\left\{ \left\| \sum_{j=n+1}^{2n} V_j \right\| > n\varepsilon \right\} \\
&\leq 2\left(\frac{4}{\varepsilon}\right)^p \sum_{n=1}^{\infty} \frac{1}{2n} E\left\| \frac{\sum_{j=1}^{2n} V_j}{2n} \right\|^p + \left(\frac{2}{\varepsilon}\right)^p \sum_{n=1}^{\infty} \frac{1}{n} E\left\| \frac{\sum_{j=1}^{n} V_j}{n} \right\|^p \\
&< \infty.
\end{aligned}$$

The conclusion (15) then follows from Proposition 3. □

Remark 3. It follows from the *proof* of Theorem 3 that its hypotheses also entail

$$\frac{\sum_{j=1}^{n} V_j}{n} \xrightarrow{\mathscr{L}_p} 0.$$

4. SOME INTERESTING EXAMPLES

We close by presenting four illustrative examples. We recall that ℓ_p (where $p \geq 1$) is the real separable Banach space of absolute pth power summable real sequences $v = \{v_i, i \geq 1\}$ with norm $\|v\| = (\sum_{i=1}^{\infty} |v_i|^p)^{1/p}$. It is well known that ℓ_p is of Rademacher type $2 \wedge p$ and that if $1 \leq p_1 < p_2 \leq 2$, then ℓ_{p_1} is not of Rademacher type p_2. The element of ℓ_p having 1 in its nth position and 0 elsewhere will be denoted by $v^{(n)}$, $n \geq 1$. Define a sequence $\{V_n, n \geq 1\}$ of independent mean 0 random elements in ℓ_p by requiring the $\{V_n, n \geq 1\}$ to be independent with

$$P\{V_n = v^{(n)}\} = P\{V_n = -v^{(n)}\} = \frac{1}{2}, \quad n \geq 1. \tag{17}$$

This sequence of random elements will be used in Examples 1 and 3.

By Theorem 2, if a real separable Banach space is not of Rademacher type p where $1 < p \leq 2$, then there exists a sequence of independent mean 0 random elements for which (9) holds but (10) fails. Example 1 exhibits such a sequence of random elements in the Banach space ℓ_1. This example will also demonstrate that, in general, $T_n \xrightarrow{c} 0$ and $T_n \xrightarrow{\mathscr{L}_p} 0$ do not ensure that $T_n \xrightarrow{c,\mathscr{L}_p} 0$.

Example 1. Let $1 < p \leq 2$ and consider the Banach space ℓ_1 (which is not of Rademacher type p) and the sequence $\{V_n, n \geq 1\}$ of independent mean 0 random elements in ℓ_1 as in (17). Then (9) holds but

$$\sum_{n=1}^{\infty} E\left\| \frac{\sum_{j=1}^{n} V_j}{n^{\frac{p+1}{p}}} \right\|^p = \sum_{n=1}^{\infty} \frac{n^p}{n^{p+1}} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \tag{18}$$

and so (10) fails. Moreover, since for all $\varepsilon > 0$ and all large n

$$P\left\{\frac{\|\sum_{j=1}^n V_j\|}{n^{\frac{p+1}{p}}} > \varepsilon\right\} = 0,$$

it follows that $\sum_{j=1}^n V_j/n^{\frac{p+1}{p}} \xrightarrow{c} 0$. Now by the computation in (18) we have

$$E\left\|\frac{\sum_{j=1}^n V_j}{n^{\frac{p+1}{p}}}\right\|^p = \frac{1}{n} \to 0$$

and so $\sum_{j=1}^n V_j/n^{\frac{p+1}{p}} \xrightarrow{\mathscr{L}_p} 0$. Consequently,

$$T_n \xrightarrow{c} 0 \quad \text{and} \quad T_n \xrightarrow{\mathscr{L}_p} 0 \not\Longrightarrow T_n \xrightarrow{c,\mathscr{L}_p} 0.$$

In the previous example, the Banach space under consideration was *not* of Rademacher type r for any $1 < r \leq 2$ and we showed for the sequence of random elements $\{T_n, n \geq 1\}$ under consideration that for all $1 < p \leq 2$,

$$T_n \xrightarrow{c} 0, \quad T_n \xrightarrow{\mathscr{L}_p} 0, \quad T_n \overset{c,\mathscr{L}_p}{\not\longrightarrow} 0. \tag{19}$$

(It is clear that (19) also holds for all $0 < p \leq 1$.) In the next example, we exhibit a sequence of random elements $\{T_n, n \geq 1\}$ in a Banach space which *is* of Rademacher type r for all $1 < r \leq 2$ and such that (19) holds for all $0 < p \leq 2$.

Example 2. Let $\mathscr{X} = R$ which is of Rademacher type r for all $1 \leq r \leq 2$. Let $\{V_n, n \geq 1\}$ be a symmetric Bernoulli sequence, and set $T_n = \sum_{j=1}^n V_j/n$, $n \geq 1$. Since $EV_1 = 0$ and $EV_1^2 = 1$, by the celebrated theorem of Hsu and Robbins [6] $T_n \xrightarrow{c} 0$. We also have

$$ET_n^2 = \frac{\sum_{j=1}^n EV_j^2}{n^2} = \frac{1}{n} \to 0$$

and so $T_n \xrightarrow{\mathscr{L}_p} 0$ for all $0 < p \leq 2$. Now by the Lévy central limit theorem $\sqrt{n}T_n \xrightarrow{d} N(0, 1)$ and since $E(\sqrt{n}T_n)^2 = 1$, $n \geq 1$, it follows from the moment convergence theorem (see, e.g., Chow and Teicher [3], p. 277) that for all $0 < p \leq 2$,

$$E|\sqrt{n}T_n|^p \to E|Z|^p < \infty$$

where Z denotes a random variable with the $N(0, 1)$ distribution. Thus for all $0 < p \leq 2$

$$E|T_n|^p \sim \frac{C}{n^{p/2}} \geq \frac{C}{n}$$

and so

$$\sum_{n=1}^{\infty} E|T_n|^p = \infty.$$

Thus for all $0 < p \leq 2$,

$$T_n \overset{c,\mathscr{L}_p}{\not\longrightarrow} 0.$$

The third example shows that Theorem 1 is sharp in that it can fail if (6) is weakened to

$$E\|V_n\|^p \sum_{j=n}^{\infty} b_j^{-p} \to 0. \tag{20}$$

Example 3. Let $1 \leq p \leq 2$ and consider the Rademacher type p Banach space ℓ_p and the sequence $\{V_n, n \geq 1\}$ of independent mean 0 random elements in ℓ_p as in (17). Let $b_n = n^{2/p}$, $n \geq 1$. Then $\sum_{n=1}^{\infty} b_n^{-p} < \infty$. Now $E\|V_n\|^p \sum_{j=n}^{\infty} b_j^{-p} \sim n^{-1}$ whence (6) fails but (20) holds. Finally, note that

$$\sum_{n=1}^{\infty} E\left\|\frac{\sum_{j=1}^{n} V_j}{b_n}\right\|^p = \sum_{n=1}^{\infty} \frac{(n^{1/p})^p}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

and so (7) fails.

The last example shows that the converse of Theorem 3 is not valid even for a sequence $\{V_n, n \geq 1\}$ of independent mean 0 random elements in a Rademacher type 2 Banach space where $E\|V_n\|^2 < \infty$ for all $n \geq 1$.

Example 4. Let $\mathscr{X} = R$, which is of Rademacher type r for all $1 \leq r \leq 2$. Set

$$a_1 = 1, \quad a_n = n^4 \sum_{i=1}^{n-1} a_i, \quad n \geq 2.$$

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables and $\{A_n, n \geq 1\}$ be a sequence of independent events where

$$P\{X_n = a_n\} = P\{X_n = -a_n\} = \frac{1}{2}, \quad n \geq 1, \quad P\{A_n\} = \frac{1}{n^2}, \quad n \geq 1,$$

and the sequences $\{X_n, n \geq 1\}$ and $\{I_{A_n}, n \geq 1\}$ are independent. Set $V_n = X_n I_{A_n}$, $n \geq 1$. Then $\{V_n, n \geq 1\}$ is a sequence of independent mean 0 $\mathscr{L}_2$ random variables. Note that V_n is equivalent to 0 in the sense of Khintchine since $\sum_{n=1}^{\infty} P\{A_n\} < \infty$ and so (15) holds. We now verify that (10) fails for all $1 \leq p \leq 2$. Note that for all $n \geq 2$,

$$\sum_{j=1}^{n} V_j = \sum_{j=1}^{n-1} X_j I_{A_j} + X_n I_{A_n}$$

where

$$-\frac{a_n}{n^4} \leq \sum_{j=1}^{n-1} X_j I_{A_j} \leq \frac{a_n}{n^4} \quad \text{a.s.}$$

Hence for all $n \geq 1$ and $1 \leq p \leq 2$,

$$\left| \sum_{j=1}^{n} V_j \right| \geq \frac{a_n}{2} I_{A_n} \quad \text{a.s.}$$

Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^{p+1}} E\left| \sum_{j=1}^{n} V_j \right|^p &\geq \sum_{n=2}^{\infty} \frac{a_n^p}{n^{p+1} 2^p} P\{A_n\} \\
&= \sum_{n=2}^{\infty} \frac{n^{4p} \left(\sum_{i=1}^{n-1} a_i \right)^p}{2^p n^{p+3}} \\
&= \sum_{n=2}^{\infty} \frac{n^{3p-3}}{2^p} \left(\sum_{i=1}^{n-1} a_i \right)^p \\
&\geq \sum_{n=2}^{\infty} \frac{1}{2^p} \left(\sum_{i=1}^{n-1} a_i \right)^p \\
&= \infty
\end{aligned}$$

and so (10) fails.

Remark 4. In view of Theorem 3, Example 4 demonstrates that the implication ((i) $\Longrightarrow$ (iv)) in Theorem 2 is a bona fide improvement of the implication ((i) $\Longrightarrow$ (ii)) in Proposition 1.

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REFERENCES

1. Adler, A., and A. Rosalsky. 1987. Some general strong laws for weighted sums of stochastically dominated random variables. *Stochastic Anal. Appl.* 5:1–16.
2. Chow, Y.S. 1988. On the rate of moment convergence of sample sums and extremes. *Bull. Inst. Math. Acad. Sinica* 16:177–201.
3. Chow, Y.S., and H. Teicher. 1997. *Probability Theory: Independence, Interchangeability, Martingales,* 3rd ed. New York: Springer-Verlag.
4. Etemadi, N. 1985. Tail probabilities for sums of independent Banach space valued random variables. *Sankhyā Ser. A* 47:209–214.
5. Hoffmann-Jørgensen, J., and G. Pisier. 1976. The law of large numbers and the central limit theorem in Banach spaces. *Ann. Probab.* 4:587–599.
6. Hsu, P.L., and H. Robbins. 1947. Complete convergence and the law of large numbers. *Proc. Nat. Acad. Sci. U.S.A.* 33:25–31.
7. Kuelbs, J., and J. Zinn. 1979. Some stability results for vector valued random variables. *Ann. Probab.* 7:75–84.
8. Scalora, F.S. 1961. Abstract martingale convergence theorems. *Pacific J. Math.* 11:347–374.
9. Spitzer, F. 1956. A combinatorial lemma and its application to probability theory. *Trans. Amer. Math. Soc.* 82:323–339.
10. Taylor, R.L. 1978. *Stochastic Convergence of Weighted Sums of Random Elements in Linear Spaces*. Lecture Notes in Mathematics No. 672. Berlin: Springer-Verlag.