Stochastic Analysis and Applications, 26: 595–602, 2008

ISSN 0736-2994 print/1532-9356 online
DOI: 10.1080/07362990802007186

On Complete Convergence for Arrays of Random Elements

Soo Hak Sung,[1] Henar Urmeneta,[2] and Andrei I. Volodin[3]
[1]Department of Applied Mathematics, Pai Chai University, Taejon, South Korea
[2]Departamento de Estadística e Investigación Operativa, de la Universidad Pública de Navarra, Pamplona, Spain
[3]Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada

Abstract: A complete convergence theorem for arrays of rowwise independent random variables was obtained by Kruglov, Volodin, and Hu (*Statistics and Probability Letters* 2006, 76:1631–1640). In this article, we extend the result to a Banach space without any additional conditions. No assumptions are made concerning the geometry of the underlying Banach space.

Keywords: Array of random elements; Complete convergence; Rowwise independence.

AMS Subject Classification: 60B12; 60F15.

1. INTRODUCTION

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [1] as follows. A sequence

Received March 20, 2007; Accepted September 14, 2007
The authors would like to thank the referee for helpful comments that helped to improve the presentation.
Address correspondence to Andrei I. Volodin, Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2, Canada; E-mail: andrei@math.uregina.ca

$\{U_n, n \geq 1\}$ of random variables converges completely to the constant θ if

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \epsilon) < \infty \quad \text{for all } \epsilon > 0.$$

In view of the Borel–Cantelli lemma, this implies that $U_n \to \theta$ almost surely. The converse is true if $\{U_n, n \geq 1\}$ are independent random variables. Hsu and Robbins [1] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [2] proved the converse.

The result of Hsu–Robbins–Erdös has been generalized and extended in several directions. Some of these generalizations are in a Banach space setting. A sequence of Banach space valued random elements is said to converge completely to the 0 element of the Banach space if the corresponding sequence of norms converges completely to 0.

Rowwise independence means that the random elements within each row are independent but that no independence is assumed between rows.

Recently, Kruglov et al. [3] proved the following complete convergence theorem for arrays of rowwise independent random variables $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$, where $\{k_n, n \geq 1\}$ is a sequence of positive integers.

Theorem 1. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random variables and $\{a_n, n \geq 1\}$ a sequence of positive constants. Suppose that the following conditions hold*:

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|X_{ni}| > \epsilon) < \infty$ *for all* $\epsilon > 0$.
(ii) *there exist* $J > 0, \delta > 0,$ *and* $p \geq 1$ *such that*

$$\sum_{n=1}^{\infty} a_n \left(E \left| \sum_{i=1}^{k_n} (X_{ni} I(|X_{ni}| \leq \delta) - EX_{ni} I(|X_{ni}| \leq \delta)) \right|^p \right)^J < \infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P\left(\max_{1 \leq m \leq k_n} \left| \sum_{i=1}^{m} (X_{ni} - EX_{ni} I(|X_{ni}| \leq \delta)) \right| > \epsilon \right) < \infty \quad \textit{for all } \epsilon > 0.$$

In Theorem 1, J is not necessary an integer. In this article, we extend Theorem 1 to a separable Banach space without any additional conditions. No assumptions are made concerning the geometry of the underlying Banach space.

2. MAIN RESULTS

To prove our main results, we will need the following lemma. It was proved in Hu et al. [4] as a version of the famous Hoffmann–Jørgensen inequality for independent, but not necessarily symmetric, random variables.

Lemma 1. *If $X_1, \ldots, X_n$ are independent random variables, then for every integer $j \geq 1$ and $t > 0$*

$$P(|S_n| > 6^j t) \leq C_j P\Big(\max_{1\leq i\leq n} |X_i| > \frac{t}{4^{j-1}}\Big) + D_j \max_{1\leq i\leq n}\left[P\left(|S_i| > \frac{t}{4^j}\right)\right]^{2^j},$$

where C_j and D_j are positive constants depending only on j, and $S_i = \sum_{l=1}^{i} X_l$ for $1 \leq i \leq n$.

Note that Lemma 1 is still valid for independent, but not necessarily symmetric, random elements.

Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent, but not necessarily identically distributed, random elements taking values in a separable Banach space B. In general the case $k_n = \infty$ is not being precluded. In this case we are assuming that the series of random elements in Theorems 2(ii) and 3(ii) converges almost surely.

Theorem 2. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements and $\{a_n, n \geq 1\}$ a sequence of positive constants. Suppose that the following conditions hold*:

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon) < \infty$ *for all* $\epsilon > 0$.
(ii) *there exist* $J > 0$ *and* $\delta > 0$ *such that*

$$\sum_{n=1}^{\infty} a_n \left(E\left\| \sum_{i=1}^{k_n} (X_{ni} I(\|X_{ni}\| \leq \delta) - EX_{ni} I(\|X_{ni}\| \leq \delta))\right\|\right)^J < \infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P\left(\max_{1\leq m\leq k_n} \left\| \sum_{i=1}^{m} (X_{ni} - EX_{ni} I(\|X_{ni}\| \leq \delta))\right\| > \epsilon\right) < \infty \quad \text{for all } \epsilon > 0.$$

Proof. Let $\epsilon > 0$ be given. Without loss of generality, we may assume $0 < \epsilon \leq \delta$. We proceed with two cases.

Case 1: $J \geq 1$.

Take a positive integer j such that $2^j \geq J$. Define $X'_{ni} = X_{ni} I(\|X_{ni}\| \leq \epsilon/(4^{j-1} \cdot 16 \cdot 6^j))$ and $X''_{ni} = X_{ni} I(\epsilon/(4^{j-1} \cdot 16 \cdot 6^j) < \|X_{ni}\| \leq \delta)$ for $1 \leq$

$i \le k_n$ and $n \ge 1$. The set of all natural numbers is partitioned into two subsets

$$A' = \left\{ n: \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon/(4^{j-1} \cdot 16 \cdot 6^j)) \le 1 \right\}$$

and

$$A'' = \left\{ n: \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon/(4^{j-1} \cdot 16 \cdot 6^j)) > 1 \right\}.$$

Applying (i) we obtain

$$\sum_{n \in A''} a_n \le \sum_{n \in A''} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon/(4^{j-1} \cdot 16 \cdot 6^j)) < \infty.$$

By the inequality of Etemadi [5], we have that

$$P\left(\max_{1 \le m \le k_n} \left\| \sum_{i=1}^{m} (X_{ni} - EX_{ni} I(\|X_{ni}\| \le \delta)) \right\| > \epsilon \right)$$
$$\le 4 \max_{1 \le m \le k_n} P\left(\left\| \sum_{i=1}^{m} (X_{ni} - EX_{ni} I(\|X_{ni}\| \le \delta)) \right\| > \frac{\epsilon}{4} \right).$$

Hence it is enough to show that

$$\sum_{n \in A'} a_n \max_{1 \le m \le k_n} P\left(\left\| \sum_{i=1}^{m} (X_{ni} - EX_{ni} I(\|X_{ni}\| \le \delta)) \right\| > \frac{\epsilon}{4} \right) < \infty.$$

Observe that

$$\sum_{n \in A'} a_n \max_{1 \le m \le k_n} P\left(\left\| \sum_{i=1}^{m} (X_{ni} - EX_{ni} I(\|X_{ni}\| \le \delta)) \right\| > \frac{\epsilon}{4} \right)$$
$$\le \sum_{n \in A'} a_n \max_{1 \le m \le k_n} P\left(\left\| \sum_{i=1}^{m} (X'_{ni} - EX'_{ni}) \right\| > \frac{\epsilon}{8} \right)$$
$$+ \sum_{n \in A'} a_n \max_{1 \le m \le k_n} P\left(\left\| \sum_{i=1}^{m} (X''_{ni} - EX''_{ni}) \right\| > \frac{\epsilon}{8} \right)$$
$$+ \sum_{n \in A'} a_n \max_{1 \le m \le k_n} \sum_{i=1}^{m} P(\|X_{ni}\| > \delta)$$
$$=: I_1 + I_2 + I_3.$$

For I_1, we note that

$$\max_{1 \le i \le m} \|X'_{ni} - EX'_{ni}\| \le \epsilon/(4^{j-1} \cdot 8 \cdot 6^j).$$

By Lemma 1, we obtain

$$\begin{aligned}
&\max_{1\le m\le k_n} P\left(\left\|\sum_{i=1}^{m}(X'_{ni}-EX'_{ni})\right\| > \frac{\epsilon}{8}\right)\\
&\quad\le C_j \max_{1\le m\le k_n} P\left(\max_{1\le i\le m}\|X'_{ni}-EX'_{ni}\| > \frac{\epsilon}{4^{j-1}\cdot 8\cdot 6^j}\right)\\
&\qquad + D_j \max_{1\le m\le k_n}\max_{1\le i\le m}\left[P\left(\left\|\sum_{l=1}^{i}(X'_{nl}-EX'_{nl})\right\| > \frac{\epsilon}{4^j\cdot 8\cdot 6^j}\right)\right]^{2^j}\\
&\quad\le D_j \max_{1\le i\le k_n}\left[P\left(\left\|\sum_{l=1}^{i}(X'_{nl}-EX'_{nl})\right\| > \frac{\epsilon}{4^j\cdot 8\cdot 6^j}\right)\right]^{J}\\
&\quad\le D_j\left(\frac{4^j\cdot 8\cdot 6^j}{\epsilon}\right)^J \max_{1\le i\le k_n}\left(E\left\|\sum_{l=1}^{i}(X'_{nl}-EX'_{nl})\right\|\right)^J\\
&\quad= D_j\left(\frac{4^j\cdot 8\cdot 6^j}{\epsilon}\right)^J \left(E\left\|\sum_{i=1}^{k_n}(X'_{ni}-EX'_{ni})\right\|\right)^J.
\end{aligned}$$

The last equality follows by Lemma 3 in Etemadi [6]. By the triangle inequality, we have

$$\begin{aligned}
&E\left\|\sum_{i=1}^{k_n}(X'_{ni}-EX'_{ni})\right\|\\
&\quad= E\left\|\sum_{i=1}^{k_n}(X_{ni}I(\|X_{ni}\|\le\delta)-EX_{ni}I(\|X_{ni}\|\le\delta)) - \sum_{i=1}^{k_n}(X''_{ni}-EX''_{ni})\right\|\\
&\quad\le E\left\|\sum_{i=1}^{k_n}(X_{ni}I(\|X_{ni}\|\le\delta)-EX_{ni}I(\|X_{ni}\|\le\delta))\right\| + E\left\|\sum_{i=1}^{k_n}(X''_{ni}-EX''_{ni})\right\|\\
&\quad\le E\left\|\sum_{i=1}^{k_n}(X_{ni}I(\|X_{ni}\|\le\delta)-EX_{ni}I(\|X_{ni}\|\le\delta))\right\| + 2\sum_{i=1}^{k_n}E\|X''_{ni}\|\\
&\quad\le E\left\|\sum_{i=1}^{k_n}(X_{ni}I(\|X_{ni}\|\le\delta)-EX_{ni}I(\|X_{ni}\|\le\delta))\right\|\\
&\qquad + 2\delta\sum_{i=1}^{k_n}P(\|X_{ni}\| > \epsilon/(4^{j-1}\cdot 16\cdot 6^j)).
\end{aligned}$$

It follows that

$$\begin{aligned}
I_1 \le D_j\left(\frac{4^j\cdot 8\cdot 6^j}{\epsilon}\right)^J \sum_{n\in A'} a_n\Bigg(E&\left\|\sum_{i=1}^{k_n}(X_{ni}I(\|X_{ni}\|\le\delta)-EX_{ni}I(\|X_{ni}\|\le\delta))\right\|\\
&+ 2\delta\sum_{i=1}^{k_n}P(\|X_{ni}\| > \epsilon/(4^{j-1}\cdot 16\cdot 6^j))\Bigg)^J
\end{aligned}$$

$$\le D_j\left(\frac{4^j\cdot 8\cdot 6^j}{\epsilon}\right)^J 2^{J-1}$$

$$\times\sum_{n\in A'} a_n\left[\left(E\left\|\sum_{i=1}^{k_n}(X_{ni}I(\|X_{ni}\|\le\delta)-EX_{ni}I(\|X_{ni}\|\le\delta))\right\|\right)^J\right.$$

$$\left.+\left(2\delta\sum_{i=1}^{k_n}P(\|X_{ni}\|>\epsilon/(4^{j-1}\cdot 16\cdot 6^j))\right)^J\right]$$

$$\le D_j\left(\frac{4^j\cdot 8\cdot 6^j}{\epsilon}\right)^J 2^{J-1}$$

$$\times\sum_{n\in A'} a_n\left[\left(E\left\|\sum_{i=1}^{k_n}(X_{ni}I(\|X_{ni}\|\le\delta)-EX_{ni}I(\|X_{ni}\|\le\delta))\right\|\right)^J\right.$$

$$\left.+(2\delta)^J\sum_{i=1}^{k_n}P(\|X_{ni}\|>\epsilon/(4^{j-1}\cdot 16\cdot 6^j))\right],$$

since $n\in A'$ and $J\ge 1$. Thus $I_1<\infty$ by (i) and (ii).

For I_2, we have by Markov's inequality that

$$I_2\le\frac{8}{\epsilon}\sum_{n\in A'}a_n\max_{1\le m\le k_n}E\left\|\sum_{i=1}^{m}(X''_{ni}-EX''_{ni})\right\|$$

$$\le\frac{16}{\epsilon}\sum_{n\in A'}a_n\sum_{i=1}^{k_n}E\|X''_{ni}\|$$

$$\le\frac{16\delta}{\epsilon}\sum_{n\in A'}a_n\sum_{i=1}^{k_n}P(\|X_{ni}\|>\epsilon/(4^{j-1}\cdot 16\cdot 6^j)).$$

Thus $I_2<\infty$ by (i). Obviously $I_3<\infty$ by (i).

Case 2: $0<J<1$.

The set of all natural numbers is partitioned into two subsets

$$B'=\left\{n:\ E\left\|\sum_{i=1}^{k_n}(X_{ni}I(\|X_{ni}\|\le\delta)-EX_{ni}I(\|X_{ni}\|\le\delta))\right\|\le 1\right\}$$

and

$$B''=\left\{n:\ E\left\|\sum_{i=1}^{k_n}(X_{ni}I(\|X_{ni}\|\le\delta)-EX_{ni}I(\|X_{ni}\|\le\delta))\right\|>1\right\}.$$

Applying (ii) we obtain

$$\sum_{n\in B''}a_n\le\sum_{n\in B''}a_n\left(E\left\|\sum_{i=1}^{k_n}(X_{ni}I(\|X_{ni}\|\le\delta)-EX_{ni}I(\|X_{ni}\|\le\delta))\right\|\right)^J<\infty.$$

Hence it is enough to show that

$$\sum_{n\in B'} a_n P\left(\max_{1\le m\le k_n}\left\|\sum_{i=1}^{m}(X_{ni}-EX_{ni}I(\|X_{ni}\|\le\delta))\right\|>\epsilon\right)<\infty.$$

For $1\le i\le k_n$ and $n\ge 1$, define

$$Y_{ni}=\begin{cases}X_{ni} & \text{if } n\in B',\\ 0 & \text{if } n\in B''.\end{cases}$$

Then $\{Y_{ni}, 1\le i\le k_n, n\ge 1\}$ is an array of rowwise independent random elements. Further, we have by (i) and (ii) that

$$\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|Y_{ni}\|>\epsilon)<\infty \quad \text{for all } \epsilon>0$$

and

$$\sum_{n=1}^{\infty} a_n E\left\|\sum_{i=1}^{k_n}(Y_{ni}I(\|Y_{ni}\|\le\delta)-EY_{ni}I(\|Y_{ni}\|\le\delta))\right\|<\infty.$$

Applying Case 1 (with $J=1$) to the array $\{Y_{ni}, 1\le i\le k_n, n\ge 1\}$, we obtain

$$\sum_{n=1}^{\infty} a_n P\left(\max_{1\le m\le k_n}\left\|\sum_{i=1}^{m}(Y_{ni}-EY_{ni}I(\|Y_{ni}\|\le\delta))\right\|>\epsilon\right)<\infty \quad \text{for all } \epsilon>0.$$

But

$$\begin{aligned}&\sum_{n=1}^{\infty} a_n P\left(\max_{1\le m\le k_n}\left\|\sum_{i=1}^{m}(Y_{ni}-EY_{ni}I(\|Y_{ni}\|\le\delta))\right\|>\epsilon\right)\\ &\quad=\sum_{n\in B'} a_n P\left(\max_{1\le m\le k_n}\left\|\sum_{i=1}^{m}(X_{ni}-EX_{ni}I(\|X_{ni}\|\le\delta))\right\|>\epsilon\right).\end{aligned}$$

Hence

$$\sum_{n\in B'} a_n P\left(\max_{1\le m\le k_n}\left\|\sum_{i=1}^{m}(X_{ni}-EX_{ni}I(\|X_{ni}\|\le\delta))\right\|>\epsilon\right)<\infty. \qquad \square$$

By using Theorem 2, we can obtain the following theorem which extends Theorem 1 to a Banach space without any additional conditions.

Theorem 3. *Let* $\{X_{ni}, 1\le i\le k_n, n\ge 1\}$ *be an array of rowwise independent random elements and* $\{a_n, n\ge 1\}$ *a sequence of positive* *constants. Suppose that the following conditions hold*:

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon) < \infty$ *for all* $\epsilon > 0$.
(ii) *there exist* $J > 0$, $\delta > 0$, *and* $p \geq 1$ *such that*

$$\sum_{n=1}^{\infty} a_n \left(E \left\| \sum_{i=1}^{k_n} (X_{ni} I(\|X_{ni}\| \leq \delta) - EX_{ni} I(\|X_{ni}\| \leq \delta)) \right\|^p \right)^J < \infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P\left(\max_{1 \leq m \leq k_n} \left\| \sum_{i=1}^{m} (X_{ni} - EX_{ni} I(\|X_{ni}\| \leq \delta)) \right\| > \epsilon \right) < \infty \quad \textit{for all } \epsilon > 0.$$

Proof. By (ii), we have that

$$\sum_{n=1}^{\infty} a_n \left(E \left\| \sum_{i=1}^{k_n} (X_{ni} I(\|X_{ni}\| \leq \delta) - EX_{ni} I(\|X_{ni}\| \leq \delta)) \right\| \right)^{pJ}$$
$$\leq \sum_{n=1}^{\infty} a_n \left(E \left\| \sum_{i=1}^{k_n} (X_{ni} I(\|X_{ni}\| \leq \delta) - EX_{ni} I(\|X_{ni}\| \leq \delta)) \right\|^p \right)^J < \infty.$$

The result follows from Theorem 2 with J replaced by pJ. □

REFERENCES

1. Hsu, P.L., and Robbins, H. 1947. Complete convergence and the law of large numbers. *Proceedings of the National Academy of Sciences, USA* 33:25–31.
2. Erdös, P. 1949. On a theorem of Hsu and Robbins. *Annals of Mathematics and Statistics* 20:286–291.
3. Kruglov, V.M., Volodin, A., and Hu, T.-C. 2006. On complete convergence for arrays. *Statistics and Probability Letters* 76:1631–1640.
4. Hu, T.-C., Ordóñez Cabrera, M., Sung, S.H., and Volodin, A. 2003. Complete convergence for arrays of rowwise independent random variables. *Communications of the Korean Mathematical Society* 18:375–383.
5. Etemadi, N. 1985. On some classical results in probability theory. *Sankhyā Series A* 47:215–221.
6. Etemadi, N. 1987. On sums of independent random vectors. *Communications of Statistics and Theory Methods* 16:241–252.