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An application of φ-subgaussian technique to Fourier analysis

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ABSTRACT

We present an application to random Fourier series for φ-subgaussian random variables. In particular, we generalize previously known notions of dependence for random variables by introducing the concept of F-manageable random variables, and consider Fourier series of F-manageable φ-subgaussian random variables. Gaussian series are then a particular case of the series considered in the paper. Conditions for the uniform convergence of such series in probability are established and a rate of convergence is investigated. Moreover, the results are new even for the case of independent random variables.

1. Introduction

As noted by Giuliano Antonini et al. [10], one of the most interesting applications of series of subgaussian random variables is to be found in Fourier analysis. In the present paper, we consider Fourier series with dependent terms. Although there is a well-established literature on uniform convergence with probability one of random series with independent terms including Fourier series, several results of which we will review momentarily, there does not seem to be any consideration made to series of dependent terms of the kind considered herein.

The first papers that investigated convergence of random series with independent terms are from 1930s and include the fundamental papers by Paley and Zygmund [36,37] and Paley, Wiener, and Zygmund [35] who considered series of the form $\sum_{n=1}^{\infty} \xi_n a_n e^{itn}$, where ξ_n is a sequence of independent random variables. In particular, they studied the cases when (i) the ξ_n are Gaussian and (ii) the $\xi_n = \pm 1$ each with probability 1/2.

Hunt [13] significantly generalized these results by considering series of the form $\sum_{n=1}^{\infty} \xi_n a_n e^{it\lambda_n}$, where ξ_n is a sequence of independent random variables with $E\xi_n = 0$ and $E|\xi_n|^p < \infty$ for some $1 < p \le 2$, and he obtained conditions for uniform convergence of such series. As corollaries, he found conditions for the uniform convergence of integrals of the form $\int_{-\infty}^{\infty} e^{i\lambda t} d\xi(\lambda)$, where $\xi(\lambda)$ is a stochastic process with independent increments. This allowed Hunt to obtain conditions for the continuity of Gaussian stationary processes. In fact, these were the best results known for such processes at the time.

Different properties of random Fourier and Taylor series were investigated by Kahane [19,17,18], Billard [1,2], Jain and Marcus [15,16], and Marcus and Pisier [33]. Series with more general terms and representations of different random fields were investigated by Kozačenko [20,21], Gladkaja and Kozačenko [11], and Kozačenko and Jadrenko [22].

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By the end of the 1960s, investigations into conditions for the convergence of random series with independent terms taking values in a Banach space had begun. One of the first papers in this area was by Walsh [38], but perhaps the most important early paper was by Itô and Nisio [14] whose topological methods of proof were considered the most important. This lead Buldygin [3] to a general theory of convergence with probability one of series of independent random elements taking values in topological spaces.

Recently there has been a resurgence in interest in properties of random series due to their widespread use in the theory of random processes. In particular, there has been a desire to establish conditions for convergence in various norms and rates of such convergence. We mention the papers by Giuliano Antonini et al. [8,9], Kozachenko et al. [30], Kozachenko and Rozora [27,28], and Kozachenko and Pogoriliak [25], where models of random processes for a wide class of random variables (not only Gaussian) are considered. Also constructed are approximations to these processes accurate in the uniform metric to a given reliability. Moreover, Kozachenko and Slivka [29], Dovgay and Kozachenko [7], and Kozachenko and Veresh [31] apply properties of random series to investigate solutions to boundary value problems in mathematical physics by Fourier methods, and Buldygin and Runovska [5,6] investigate series whose terms are a Markov sequence. Other authors, including Kozachenko et al. [24], Kozachenko and Polosmak [26], Kurbanmuradov and Sabelfeld [32], and Kozachenko et al. [23], have investigated properties of random processes that are decompositions by wavelet bases. Note that in these cases the terms of the series are usually dependent, and so it is important to study series with dependent terms belonging to a wider class of random variables than just Gaussian. It is such a class that we consider in this paper, namely the φ-subgaussian random variables (see below for the relevant definitions).

Giuliano Antonini et al. [10] study the almost sure convergence of weighted sums of φ-subgaussian m-acceptable random variables and introduce a new notion for the dependence structure for a sequence of random variables, namely acceptable random variables. The main results are then applied to the specific cases of negatively dependent and m-dependent subgaussian random variables. Finally we would like to stress that one of the most important features of the present paper is that it investigates Fourier series, including φ-subgaussian series, of which the usual Gaussian series is a special case. Motivated by the need to consider series of dependent random variables (which are important for applications), we introduce a new dependence structure for a sequence of random variables. Hence, the results obtained herein for dependent random variables are new even in the case of independent random variables.

2. Definitions and technical lemmas

In this section we present a number of definitions and technical results that will be used in the proofs of our main results. To begin, recall that a continuous even convex function $\varphi(x)$, $x \in \mathbf{R}$, is called an *N-function*, if

(a) $\varphi(0) = 0$ and $\varphi(x)$ monotone increasing for $x > 0$, and
(b) $\lim_{x\to 0} \frac{\varphi(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$.

The following condition is important to ensure that the class of φ-subgaussian random variables (cf. definition below) is nonempty. We say that an N-function $\varphi(x)$ satisfies *condition Q* if $\lim_{x\to 0} \frac{\varphi(x)}{x^2} = C > 0$, where $C \in (0, \infty]$ is a non-negative extended real number (i.e., $C = \infty$ is allowed).

Hence, from this point forward, $\varphi(x)$ will always represent an N-function satisfying condition Q.

The function $\varphi^*(x)$, $x \in \mathbf{R}$, defined by $\varphi^*(x) = \sup_{y\in\mathbf{R}} (xy - \varphi(y))$ is called the *Young–Fenchel transform* of $\varphi(x)$. It is well known that $\varphi^*(x)$ is also an N-function, and if $\varphi(x) = |x|^p/p$, $p > 1$ for sufficiently large x, then $\varphi^*(x) = |x|^q/q$ for sufficiently large x, where $1/p + 1/q = 1$.

A random variable X is said to be *φ-subgaussian* if there exists a constant $a > 0$ such that, for every $\lambda \in \mathbf{R}$, we have $E \exp\{\lambda X\} \le \exp\{\varphi(a\lambda)\}$. The *$\varphi$-subgaussian standard* $\tau_\varphi(X)$ is defined as

$$\tau_\varphi(X) = \inf\{a > 0 : E \exp\{\lambda X\} \le \exp\{\varphi(a\lambda)\}, \lambda \in \mathbf{R}\}.$$

We refer the reader to the monograph by Buldygin and Kozachenko [4] and the paper by Giuliano Antonini et al. [8,9] where this notion is discussed in detail and important examples are provided. Note that if $\varphi(x) = x^2/2$, then φ-subgaussianity is simply the subgaussianity in the *classical* sense; cf., for example, [12].

Let $\{X_n, n \ge 1\}$ be a sequence of φ-subgaussian random variables, $\alpha_n = \tau_\varphi(X_n)$, $n \ge 1$, and $\{a_n, n \ge 1\}$ be a sequence of constants. We say that the sequence of random variables $\{X_n, n \ge 1\}$ is *F-manageable*, if for all $1 \le r < n$, there exists a positive function $F(r, n, a_i, \alpha_i, r \le i \le n)$ such that

1. $F(r, n, a_i, \alpha_i, r \le i \le n) \le F(r, n, c_i, \alpha_i, r \le i \le n)$ if $|a_i| \le |c_i|$, $r \le i \le n$,
2. $F(r, n, a_i, \alpha_i, r \le i \le n) \le F(1, n, a_i, \alpha_i, 1 \le i \le n)$,
3. $F(r, n, a_i, \alpha_i, r \le i \le n) \le F(r, n + 1, a_i, \alpha_i, 1 \le i \le n + 1)$,
4. $\tau_\varphi\left(\sum_{i=r}^{n} a_j X_j\right) \le F(r, n, a_i, \alpha_i, r \le i \le n)$, and
5. for any increasing sequence $\{n_n, k \ge 1\}$ of counting numbers, there exists $1 \le Q < 2$ such that for any $r < m$,

$$\sum_{k=r}^{m} F(n_k + 1, nk + 1, a_i, \alpha_i, n_k + 1 \le i \le n_{k+1}) \le QF(n_r - 1, nm + 1, a_i, \alpha_i, n_r + 1 \le i \le n_{m+1}).$$

For simplicity, we will let $F_r^n(a, \alpha) = F(r, n, a_i, \alpha_i, r \le i \le n)$.

Example of F-manageable sequences of random variables.

The following concepts were introduced by Giuliano Antonini et al. [10]. We say that a finite family of random variables $X_1, X_2, \ldots, X_n$ is *acceptable* if for any $\lambda \in \mathbf{R}$,

$$E\exp\left\{\lambda\sum_{i=1}^{n}X_i\right\} \le \prod_{i=1}^{n}E\exp\{\lambda X_i\}.$$

A sequence of random variables $\{X_n, n \ge 1\}$ is *acceptable* if every finite subfamily is acceptable.

Let $m > 1$ be a fixed integer. A sequence of random variables $\{X_n, n \ge 1\}$ is said to be *m-acceptable* if for any $n \ge 2$ and any $i_1, \ldots, i_n$ such that $|i_k - i_j| \ge m$ for all $1 \le k \ne j \le n$, the family $X_{i_1}, \ldots, X_{i_n}$ is acceptable.

Note that both a sequence of negatively dependent random variables, as well as a sequence of m-dependent random variables, are examples of sequences of m-acceptable random variables. The reader is referred to Giuliano Antonini et al. [10] for further discussion as well as for some other interesting examples.

Let $\{a_n, n \ge 1\}$ and $\{\alpha_n, n \ge 1\}$ be sequences of real numbers, and assume that $\alpha_n \ge 0$ for all $n \ge 1$. For $p > 0$, $1 \le r < n \le \infty$ (i.e., n is not necessarily finite), let

$$A^{(p)}(r,n) = \sum_{i=r}^{n}(\alpha_i|a_i|)^p.$$

In Lemma 4 of Giuliano Antonini et al. [10], it was proved that if $\alpha_n = \tau_\varphi(X_n)$ where $\{X_n, n \ge 1\}$ is a sequence of m-acceptable φ-subgaussian random variables and the function $\varphi(|x|^{1/p})$ is convex for some $p \in [1, 2]$, then

$$\tau_\varphi\left(\sum_{i=r}^{n}a_iX_i\right) \le (2m)^{1-1/p}\left(A^{(p)}(n,r)\right)^{1/p},$$

where $\{a_n, n \ge 1\}$ is a sequence of constants. From this we conclude that a sequence of m-acceptable φ-subgaussian random variables provides an example of the F-manageable sequence with

$$F_r^n(a,\alpha) = (2m)^{1-1/p}\left(\sum_{i=r}^{n}(\alpha_i|a_i|)^p\right)^{1/p}.$$

Remark. Actually, in Lemma 4 of Giuliano Antonini et al. [10] even more accurate estimation of the subgaussian norm of $\tau_\varphi(\sum_{i=r}^{n} a_iX_i)$ is given. It is connected with blocks of size m of the sequence $\{X_n, n \ge 1\}$ and, for our purposes, is equivalent to $A^{(p)}$. We decided not to provide details here because of the cumbersome notations involved, and instead we refer the interested reader to the above mentioned paper.

Let $\{X_n, n \ge 1\}$ and $\{Y_n, n \ge 1\}$ be two sequences of F-manageable random variables such that $\tau_\varphi(X_n) = \tau_\varphi(Y_n) = \alpha_n, n \ge 1$ and let $\{a_n, n \ge 1\}$ be a sequence of constants. For $0 \le t \le 2\pi$ and integers $k \ge 1$, consider a random trigonometric polynomial of the form $Z_k(t) = a_k[X_k\cos(kt) + Y_k\sin(kt)]$. The main result of the paper concerns the series $S(t)$ which is defined as

$$S(t) = \sum_{k=1}^{\infty}Z_k(t) = \sum_{k=1}^{\infty}a_k[X_k\cos(kt) + Y_k\sin(kt)].$$

Finally, set

$$S_n(t) = \sum_{k=1}^{n}Z_k(t), \quad \text{and} \quad S_r^n(t,c) = \sum_{k=r}^{n}c_kZ_k(t),$$

where $\{c_k, k \ge 1\}$ is a sequence of constants.

Remark. Note that the representation $Z_k(t) = a_kW_k\cos(kt + D_k)$ is sometimes used in the literature. Our scheme is somewhat more general because if $X_k = W_k\cos(D_k)$ and $Y_k = W_k\sin(D_k)$, then $Z_k(t) = a_k[W_k\cos(D_k)\cos(kt) + W_k\sin(D_k)\sin(kt)]$.

Lemma 1. *We have* $\tau_\varphi\left(S_r^n(t,c)\right) \le 2F_r^n(a\cdot c,\alpha)$ *where* $a \cdot c$ *is understood as the sequence of coordinatewise products* $\{a_kc_k, k \ge 1\}$.

Proof. To begin, observe that

$$\tau_\varphi\left(S_r^n(t,c)\right) = \tau_\varphi\left(\sum_{k=r}^{n} c_k a_k X_k \cos(kt) + c_k a_k Y_k \sin(kt)\right) \le \tau_\varphi\left(\sum_{k=r}^{n} c_k a_k X_k \cos(kt)\right) + \tau_\varphi\left(\sum_{k=r}^{n} c_k a_k Y_k \sin(kt)\right)$$

because $\tau_\varphi(\cdot)$ is a norm. However, since $\{X_n, n \ge 1\}$ and $\{Y_n, n \ge 1\}$ are sequences of F-manageable random variables such that $\tau_\varphi(X_n) = \tau_\varphi(Y_n) = \alpha_n$, the previous expression is bounded above by

$$\begin{aligned} &\le F(r, n, c_k a_k \cos(kt), \alpha_k, r \le k \le n) + F(r, n, c_k a_k \sin(kt), \alpha_k, r \le k \le n) \\ &\le F(r, n, c_k a_k, \alpha_k, r \le k \le n) + F(r, n, c_k a_k, \alpha_k, r \le k \le n) \\ &= 2F_r^n(a \cdot c, \alpha), \end{aligned}$$

using properties of the function F. □

For what follows, consider the complete separable Banach space $C[0, 2\pi]$ of all continuous functions on $[0, 2\pi]$ with the norm given by $\|T\| = \sup_{0\le t\le 2\pi} |T(t)|$ for $T \in C[0, 2\pi]$. The following lemma, which is needed for the proof of our main results, provides a useful integral estimation of the exponent of the trigonometric polynomial's norm.

Lemma 2. *If $T_n(t)$, $n \ge 1$, is a trigonometric polynomial of the form*

$$T_n(t) = \sum_{k=0}^{n} a_k \cos(kt) + b_k \sin(kt),$$

where $\{a_k, k \ge 1\}$ and $\{b_k, k \ge 1\}$ are sequences of constants, then for all $\lambda > 0$, $0 < \theta < 1$, it follows that

$$\exp\{\lambda\|T_n\|\} \le \frac{n}{2\theta}\int_0^{2\pi} \exp\left\{\frac{\lambda}{1-\theta}|T_n(u)|\right\} du.$$

Proof. It follows from the Bernstein inequality (see, for example, [39, p. 11]) that $\|T_n'\| \le n\|T_n\|$. For convenience, let $\|T_n\| = T_n(t_0) > 0$ so that by the mean value theorem there exists some $\tilde{t} \in [t, t_0]$ such that the following statement holds

$$T_n(t_0) - T_n(t) \le |T_n'(\tilde{t})|\,|t_0 - t| \le T_n(t_0) \cdot n \cdot |t_0 - t|.$$

If A is a subset of $[0, 2\pi]$ such that $|t_0 - t| \le \frac{\theta}{n}$, then for $t \in A$,

$$T_n(t_0) - T_n(t) \le \theta T_n(t_0) \quad \text{and} \quad T_n(t) \ge (1-\theta)T_n(t_0).$$

That is, for $t \in A$, we have

$$\|T_n\| \le \frac{T_n(t)}{1-\theta},$$

and so for any $\lambda > 0$ we conclude

$$\frac{2\theta}{n}\exp\{\lambda\|T_n\|\} = \int_A \exp\{\lambda\|T_n\|\}\,du \le \int_A \exp\left\{\frac{\lambda}{1-\theta}|T_n(u)|\right\} du \le \int_0^{2\pi} \exp\left\{\frac{\lambda}{1-\theta}|T_n(u)|\right\} du$$

which establishes the lemma. □

Finally, we need a lemma that is a slight modification of Theorem 2.2 of Móricz et al. [34]. The difference is that we state the result for Banach space valued random elements, whereas Móricz et al. [34] only proved it in the case of real valued random variables. But this is minor since the proof is identical modulo the obvious change from absolute values to norms.

Lemma 3. *Let $\{X_n, n \ge 1\}$ be a sequence of random elements taking values in a real separable Banach space with norm $\|\cdot\|$. For any $1 \le i \le j$, let*

$$S(i,j) = \sum_{k=i}^{j} X_k \quad \textit{and} \quad M(i,j) = \max_{i\le k\le j} \|S(i,k)\|.$$

Moreover, for any $1 \le i \le j$, *let* $g(i,j) \ge 0$ *be a function such that* $g(i,j) \le g(i,j+1)$ *and* $g(i,j) + g(j+1,k) \le Qg(i,k)$ *where* $1 \le Q < 2$. *Suppose there exist constants* $K \ge 1$ *and* $t_0 > 0$ *such that*

$$P\{\|S(i,j)\| \ge t\} \le K \exp\{f(t)/g(i,j)\}$$

for all $0 < t < t_0$ *and* $1 \le i \le j$, *where* $f(t) > 0$ *also satisfies*

$$\inf_{0<t<t_0} f(Ct)/f(t) = \xi(C) > 0 \quad \text{with} \quad \lim_{C\to 1-} \xi(C) = 1$$

for every $C \in (0,1)$. *Then there exist constants* $A \ge 1$ *and* $B \ge 1$, *depending only on* Q *and* ξ, *such that*

$$P\{M(i,j) \ge t\} \le AK \exp\left\{\frac{f(t)}{Bg(i,j)}\right\}$$

for all $0 < t < t_0$.

3. Main results

In this section we state and prove the primary results of the paper. We begin with the following proposition which is crucial for the proof of the main result, and provides us with a useful estimate of the exponent of the random trigonometric polynomial's norm.

Proposition 1. *Let* $\{c_k, k \ge 1\}$ *be an unbounded strictly increasing sequence of constants. If* $\lambda > 0$, $0 < \theta < 1$, *then*

$$E\exp\{\lambda\|S_n - S_{r-1}\|\} \le \frac{2\pi}{\theta}\exp\left\{\varphi\left(\frac{2\lambda}{1-\theta}\sum_{k=r}^{n} d_k F_r^k(a\cdot c,\alpha)\right) + \frac{4\lambda}{1-\theta}\sum_{k=r}^{n} d_k F_r^k(a\cdot c,\alpha)\frac{\log k}{\varphi^{-1}(\log k)}\right\}$$

where

$$d_k = \frac{1}{c_k} - \frac{1}{c_{k+1}}, \quad \text{for } r \le k < n, \quad \text{and} \quad d_n = \frac{1}{c_n}.$$

Proof. It is easy to see that

$$S_n(t) - S_{r-1}(t) = \sum_{k=r}^{n-1} S_n^k(t,c)\left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) + S_r^n(t,c)\frac{1}{c_n} = \sum_{k=r}^{n} S_r^k(t,c) d_k,$$

and so

$$\|S_n - S_{r-1}\| \le \sum_{k=r}^{n} \|S_r^k(c,\cdot)\| d_k.$$

Let $p_k > 1$, $r \le k \le n$, be such that $\sum_{k=r}^{n} 1/p_k \le 1$. It then follows from the generalized Hölder and the above inequalities that

$$E\exp\{\lambda\|S_n - S_{r-1}\|\} \le \prod_{k=r}^{n}(E\exp\{\lambda p_k d_k \|S_r^k(c,\cdot)\|\})^{1/p_k},$$

and from Lemmas 2 and 1, we deduce that

$$\begin{aligned} E\exp\{\lambda p_k d_k\|S_r^k(c)\|\} &\le \frac{k}{2\theta}\int_0^{2\pi} E\exp\left\{\frac{\lambda p_k d_k}{1-\theta}|S_r^k(t,c)|\right\} dt \\ &\le \frac{k}{2\theta}\int_0^{2\pi} 2\exp\left\{\varphi\left(\frac{\lambda p_k d_k}{1-\theta}\tau_\varphi(S_r^k(t,c))\right)\right\} dt \\ &\le \frac{2\pi k}{\theta}\exp\left\{\varphi\left(\frac{\lambda p_k d_k}{1-\theta}2F_r^k(a\cdot c,\alpha)\right)\right\}. \end{aligned}$$

If we then set $u_k = d_k F_r^k(a\cdot c,\alpha)$, $r \le k \le n$, then it follows from the inequalities above that

$$\begin{aligned} E\exp\{\lambda\|S_n - S_{r-1}\|\} &\le \prod_{k=r}^{n}\left(\frac{2\pi}{\theta}\right)^{1/p_k} k^{1/p_k}\exp\left\{\frac{1}{p_k}\varphi\left(\frac{2\lambda p_k u_k}{1-\theta}\right)\right\} \\ &\le \frac{2\pi}{\theta}\exp\left\{\sum_{k=r}^{n}\frac{1}{p_k}\left(\log k + \varphi\left(\frac{2\lambda p_k u_k}{1-\theta}\right)\right)\right\} dt. \end{aligned} \tag{1}$$

Let

$$p_k = \frac{1-\theta}{2\lambda u_k}\,\varphi^{-1}\left(\varphi\left(\frac{2\lambda\sum\limits_{k=r}^{n} u_k}{1-\theta}\right)+\log k\right)$$

and note that

$$\sum_{k=r}^{n}\frac{1}{p_k} \le \sum_{k=r}^{n}\frac{2\lambda u_k}{(1-\theta)\varphi^{-1}\left(\varphi\left(\frac{2\lambda\sum_{k=r}^{n}u_k}{1-\theta}\right)\right)} = 1,$$

from which we conclude that

$$\begin{aligned}\sum_{k=r}^{n}\frac{1}{p_k}\left(\log k+\varphi\left(\frac{2\lambda p_k u_k}{1-\theta}\right)\right) &= \sum_{k=r}^{n}\frac{1}{p_k}\log k+\sum_{k=r}^{n}\frac{1}{p_k}\left(\varphi\left(\frac{2\lambda\sum\limits_{k=r}^{n}u_k}{1-\theta}\right)+\log k\right)\\ &\le 2\sum_{k=r}^{n}\frac{1}{p_k}\log k+\varphi\left(\frac{2\lambda}{1-\theta}\sum_{k=r}^{n}u_k\right)\\ &\le \frac{4\lambda}{1-\theta}\sum_{k=r}^{n}u_k\frac{\log k}{\varphi^{-1}(\log k)}+\varphi\left(\frac{2\lambda}{1-\theta}\sum_{k=r}^{n}u_k\right).\end{aligned}$$

Substituting the above inequality into (1), we obtain the result. □

The first corollary provides a useful exponential estimate.

Corollary 1. *Let $\{c_k, k \ge 1\}$ be an unbounded strictly increasing sequence of constants, and for $0 < \theta < 1$, let*

$$R_r^n = \frac{4}{1-\theta}\sum_{k=r}^{n} d_k F_r^k(a\cdot c,\alpha)\frac{\log k}{\varphi^{-1}(\log k)}.$$

If r is sufficiently large and $x > R_r^n$, then

$$P\{\|S_n - S_{r-1}\| > x\} \le \frac{2\pi}{\theta}\exp\left\{-\varphi^*\left(2\frac{x-R_r^n}{R_r^n}\right)\right\}.$$

Proof. Set

$$D_r^n = \frac{2}{1-\theta}\sum_{k=r}^{n} d_k F_r^k(a\cdot c,\alpha)$$

and note that $\varphi(t) \ge t$ for sufficiently large t by property (b) of N-functions. This implies $D_r^n \le R_r^n/2$ for sufficiently large r. It now follows from Chebyshev's inequality and Proposition 1 that for any $\lambda > 0$,

$$\begin{aligned}P\{\|S_n - S_{r-1}\| > x\} &\le \exp\{-\lambda x\}E\exp\{\lambda\|S_n - S_{r-1}\|\}\\ &\le \frac{2\pi}{\theta}\exp\{\varphi(\lambda D_r^n)+\lambda(R_r^n - x)\}\\ &= \frac{2\pi}{\theta}\exp\left\{-\left(\lambda D_r^n\frac{x-R_r^n}{D_r^n}-\varphi(\lambda D_r^n)\right)\right\},\end{aligned}$$

and so for $x > R_r^n$,

$$\sup_{\lambda D_r^n > 0}\left(\lambda D_r^n\frac{x-R_r^n}{D_r^n}-\varphi(\lambda D_r^n)\right) = \varphi^*\left(\frac{x-R_r^n}{D_r^n}\right) \ge \varphi^*\left(2\frac{x-R_r^n}{R_r^n}\right)$$

which completes the proof. □

The second corollary now establishes our first convergence in probability result.

Corollary 2. *Let $\{c_k, k \geq 1\}$ be an unbounded strictly increasing sequence of constants. If*

$$\sum_{k=r}^{n} d_k F_r^k(a \cdot c, \alpha) \frac{\log k}{\varphi^{-1}(\log k)} \to 0$$

as $r, n \to \infty$, then there exists a stochastic process $S(t)$, $t \in [0, 2\pi]$, having a sample continuous stochastic modification, such that $\|S(t) - S_n(t)\| \to 0$ in probability as $n \to \infty$; that is, $S_n(t) \to S(t)$ as $n \to \infty$ uniformly in probability on $[0, 2\pi]$.

Proof. Note that in the notation of Corollary 1, the assumption of Corollary 2 implies that $R_r^n \to 0$ as $r, n \to \infty$. Let $\varepsilon > 0$. If r and n are sufficiently large, then $R_r^n < \varepsilon$ and so by Corollary 1, we see that

$$P\{\|S_n - S_{r-1}\| > \varepsilon\} \to 0$$

as $n, r \to \infty$. Consider the space $L_0(C[0, 2\pi])$ of all random elements with values in $C[0, 2\pi]$ with the topology of convergence in probability. For example, we can consider the metric

$$\rho(S, T) = E \frac{\|S - T\|}{1 + \|S - T\|},$$

for any $S, T \in L_0(C[0, 2\pi])$. We now conclude that $\{S_n, n \geq 1\}$ is a Cauchy sequence in the complete space $L_0(C[0, 2\pi])$. Hence, there exists a stochastic process $S(t)$, $t \in [0, 2\pi]$ such that $S_n(t) \to S(t)$ as $n \to \infty$ uniformly in probability on $[0, 2\pi]$. Since, for each fixed ω, the functions $t \mapsto S_n(t, \omega)$ are continuous, we conclude that $S(t)$ is sample continuous as required. □

Remark 1. Note that we did not use properties 3–5 of the function F in the proof of Proposition 1 and Corollaries 1 and 2.

Proposition 2. *If there exists an unbounded strictly increasing sequence of positive numbers $\{c_k, k \geq 1\}$ such that*

$$\sum_{k=2}^{\infty} F_1^k(a \cdot c, \alpha) \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) \frac{\log k}{\varphi^{-1}(\log k)} < \infty,$$

then there exists a stochastic process $S(t)$, $t \in [0, 2\pi]$, having a sample continuous stochastic modification, such that $\|S(t) - S_n(t)\| \to 0$ in probability as $n \to \infty$. In this case, if $x > R_r$ and $0 < \theta < 1$, then

$$P\{\|S - S_{r-1}\| > x\} \leq \frac{2\pi}{\theta} \exp\left\{-\varphi^*\left(2\frac{x - R_r}{R_r}\right)\right\},$$

where

$$R_r = \frac{4}{1-\theta} \sum_{k=r}^{\infty} F_1^k(a \cdot c, \alpha) \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) \frac{\log k}{\varphi^{-1}(\log k)}.$$

Proof. Note that $R_r^n \leq R_r$ by property (2) and so using properties of the function F, we find

$$\begin{aligned}
\sum_{k=r}^{n} d_k F_r^k(a \cdot c, \alpha) \frac{\log k}{\varphi^{-1}(\log k)} &= \sum_{k=r}^{n-1} \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) F_r^k(a \cdot c, \alpha) \frac{\log k}{\varphi^{-1}(\log k)} + \frac{1}{c_n} F_r^n(a \cdot c, \alpha) \frac{\log n}{\varphi^{-1}(\log n)} \\
&\leq \sum_{k=r}^{n-1} \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) F_1^k(a \cdot c, \alpha) \frac{\log k}{\varphi^{-1}(\log k)} + \frac{1}{c_n} F_1^n(a \cdot c, \alpha) \frac{\log n}{\varphi^{-1}(\log n)} \\
&\leq \sum_{k=r}^{n-1} \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) F_1^k(a \cdot c, \alpha) \frac{\log k}{\varphi^{-1}(\log k)} \\
&\quad + F_1^n(a \cdot c, \alpha) \frac{\log n}{\varphi^{-1}(\log n)} \sum_{k=n}^{\infty} \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) \\
&\leq \sum_{k=r}^{n-1} \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) F_1^k(a \cdot c, \alpha) \frac{\log k}{\varphi^{-1}(\log k)} \\
&\quad + \sum_{k=n}^{\infty} F_1^k(a \cdot c, \alpha) \frac{\log n}{\varphi^{-1}(\log n)} \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right).
\end{aligned}$$

By Corollary 2 there exists $S(t)$ such that $\|S - S_n\| \to 0$ in probability as $n \to \infty$. The second statement of the theorem (exponential inequality) now follows from Corollary 1 when $n \to \infty$. □

We are now in a position to state our main result on almost sure convergence.

Theorem 1. *Let $\varphi(\cdot)$ be an N-function such that $\varphi^*(|x|^{1/p})$ is convex. If there exists an unbounded strictly increasing sequence of positive numbers $\{c_k, k \geq 1\}$ such that*

$$\sum_{k=2}^{\infty} F_1^k(a \cdot c, \alpha)\left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right)\frac{\log k}{\varphi^{-1}(\log k)} < \infty,$$

then there exists a stochastic process $S(t)$, $t \in [0, 2\pi]$, having a sample continuous stochastic modification, such that $\|S(t) - S_n(t)\| \to 0$ almost surely as $n \to \infty$; that is, $S_n(t) \to S(t)$ as $n \to \infty$ uniformly a.s. on $[0, 2\pi]$.

Proof. From the previous proposition, we know that there exists $S(t)$ such that $\|S - S_n\| \to 0$ in probability as $n \to \infty$.

We will now show that $\|S(t) - S_n(t)\| \to 0$ almost surely as $n \to \infty$. First we note that there exists a subsequence $\{n_k, k \geq 1\}$ such that $\|S - S_{n_k}\| \to 0$ almost surely.

Set $M_k = \max_{n_k \leq j < n_{k+1}} \|S_{n_{k+1}} - S_{j-1}\|$ and fix any $\varepsilon > 0$. Note that for sufficiently large $r < n$ we have $\varepsilon > R_r^n$. Moreover, let r, n be so large that

$$\frac{\varepsilon}{R_r^n} - 1 > \frac{\varepsilon}{2R_r^n}.$$

This can be achieved because $R_r^n \to 0$ as $r, n \to \infty$. Let $n_k \leq j < n_{k+1}$.

The following inequality was established in the proof of Corollary 1, namely

$$\begin{aligned} P\{\|S_{n_{k+1}} - S_{j-1}\| > \varepsilon\} &\leq \frac{2\pi}{\theta} \exp\left\{-\varphi^*\left(\frac{2\varepsilon}{R_j^{n_{k+1}}} - 1\right)\right\} \\ &\leq \frac{2\pi}{\theta} \exp\left\{-\varphi^*\left(\frac{\varepsilon}{R_j^{n_{k+1}}}\right)\right\} \\ &\leq \frac{2\pi}{\theta} \exp\left\{-\frac{C\varepsilon^p}{(R_j^{n_{k+1}})^p}\right\}. \end{aligned}$$

Note that for the last inequality, since the function $\varphi^*(|x|^{1/p})$ is convex, there exists a positive constant C such that $\varphi^*(x) \geq Cx^p$ holds for $|x| > 1$.

Set $g(j, n) = (R_j^n)^p$, and observe that

$$g(j, n) + g(n+1, k) = (R_j^n)^p + (R_{n+1}^k)^p \leq (R_j^n + R_{n+1}^k)^p \leq (R_j^k)^p = g(j, k)$$

for any $j < n < k$.

As a consequence of these definitions, we have

$$P\{\|S_{n_{k+1}} - S_{j-1}\| > \varepsilon\} \leq \frac{2\pi}{\theta} \exp\left\{-\left(\frac{C\epsilon^p}{g(j, n_{k+1})}\right)\right\}.$$

By Lemma 3 with $f(t) = t^p$ we know that

$$P\{M_k > \varepsilon\} \leq C_1 \exp\left\{-\frac{C_2\epsilon^p}{g(n_k+1, n_{k+1})}\right\} \leq \frac{C_1}{C_2\epsilon^p} g(n_k+1, n_{k+1})$$

where we have used the relation $e^{-x} \leq 1/x$ which holds for all $x > 0$. Hence

$$\begin{aligned} \sum_{k=1}^{\infty} P\{|M_k| > \epsilon\} &\leq \frac{C_1}{C_2\epsilon^q} \sum_{k=1}^{\infty} g(n_k+1, n_{k+1}) = C \sum_{k=1}^{\infty} (R_{n_k+1}^{n_{k+1}})^p \\ &\leq C\left(\sum_{k=1}^{\infty} R_{n_k+1}^{n_{k+1}}\right)^p \\ &\leq C(R_1)^p < \infty. \end{aligned}$$

By the Borel–Cantelli lemma, $M_k \to 0$ as $k \to \infty$ almost surely. Hence, for every j with $n_k \leq j < n_{k+1}$, we have

$$\|S_j - S\| \leq \|S_j - S_{n_{k+1}}\| + \|S_{n_{k+1}} - S\| \leq M_k + \|S_{n_{k+1}} - S\|,$$

so that, almost surely

$$\lim_{n \to \infty} \|S_n - S\| = 0$$

and the proof is complete. □

Remark 2. A careful analysis of the proofs of Theorem 1 and Corollary 1 shows that for any $x > R_r$ and $0 < \theta < 1$ a more precise inequality holds, namely

$$P\{\|S - S_{r-1}\| > x\} \le \frac{2\pi}{\theta} \exp\left\{-\varphi^*\left(2\frac{x - R_r}{D_r}\right)\right\},$$

where R_r is as above and

$$D_r = \frac{2}{1-\theta} \sum_{k=r}^{\infty} F_1^k(a \cdot c, \alpha) \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right).$$

One of the assumptions of Theorem 1 seems to be difficult to check. The following three lemmas discuss how this condition may be simplified.

Lemma 4. *If there exists an unbounded strictly increasing positive sequence* $\{c_k, k \ge 1\}$ *such that*

$$\sum_{k=1}^{\infty} \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) \frac{\log k}{\varphi^{-1}(\log k)} < \infty$$

and

$$\sup_{k \ge 1} F_1^k(a \cdot c, \alpha) < \infty,$$

then

$$\sum_{k=2}^{\infty} F_1^k(a \cdot c, \alpha) \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) \frac{\log k}{\varphi^{-1}(\log k)} < \infty.$$

Proof. Since

$$\sum_{k=2}^{\infty} F_1^k(a \cdot c, \alpha) \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) \frac{\log k}{\varphi^{-1}(\log k)} \le \sup_{k \ge 1} F_1^k(a \cdot c, \alpha) \sum_{k=2}^{\infty} \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) \frac{\log k}{\varphi^{-1}(\log k)} < \infty$$

the lemma follows immediately. □

The following result whose proof is obvious provides an example of a sequence $\{c_n, n \ge 1\}$ in Lemma 4 for a special choice of the function φ.

Lemma 5. *If* $\varphi(t) = t^p/p$ *for large values of* t *and* $c_k = (\log k)^{1+\varepsilon-\frac{1}{p}}$ *for some* $\varepsilon > 0$, *then*

$$\sum_{k=1}^{\infty} \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) \frac{\log k}{\varphi^{-1}(\log k)} < \infty.$$

Finally, our last lemma discusses the special choice of the function F that we use for m-acceptable φ-subgaussian random variables.

Lemma 6. *Let*

$$F_r^n(a, \alpha) = C\left(\sum_{i=r}^{n} (\alpha_i |a_i|)^p\right)^{1/p} \quad \text{and} \quad c_k = \left(\sum_{j=k}^{\infty} \alpha_j^p |a_j|^p\right)^{-1}.$$

If

$$\sum_{j=1}^{\infty} (\alpha_j |a_j|)^p < \infty$$

and

$$\sum_{k=1}^{\infty} \left(\sum_{i=k+1}^{\infty} \alpha_i^p |a_j|^p\right)^{1/p\ -1} \alpha_{k+1}^p |a|_{k+1}^p \frac{\log k}{\varphi^{-1}(\log k)} < \infty,$$

then

$$\sum_{k=2}^{\infty} F_1^k(a \cdot c, \alpha) \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) \frac{\log k}{\varphi^{-1}(\log k)} < \infty.$$

Proof. We have

$$\sum_{j=1}^{k} \alpha_j^p |a_j|^p c_j^p = \sum_{j=1}^{k} \frac{\alpha_j^p |a_j|^p}{\left(\sum\limits_{i=j}^{\infty} \alpha_i^p |a_i|^p\right)^p} = \sum_{i=1}^{k} \int_{z_{j+1}}^{z_j} \frac{1}{\left(\sum\limits_{i=j}^{\infty} (\alpha_i^p |a_i|)^p\right)^p}\, dx,$$

where $z_j = \sum_{i=j}^{\infty} \alpha_i^p |a_j|^p$. Therefore

$$\sum_{j=1}^{k} \alpha_j^p |a_j|^p c_j^p \le \sum_{j=1}^{k} \int_{z_{j+1}}^{z_j} \frac{1}{x^p}\, dp = \frac{1}{p-1} (z_{k+1}^{1-p} - z_1^{1-p}) \le \frac{1}{p-1} z_{k+1}^{1-p}$$

and so

$$\sum_{k=2}^{\infty} F_1^k(a \cdot c, \alpha) \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) \frac{\log k}{\varphi^{-1}(\log k)} = \sum_{k=2}^{\infty} \left(\sum_{j=1}^{k} (\alpha_j |a_j| c_j)^p\right)^{1/p} \left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right) \frac{\log k}{\varphi^{-1}(\log k)}$$
$$\le \sum_{k=2}^{\infty} \frac{1}{(p-1)^{1/p}} \left(\sum_{i=k+1}^{\infty} (\alpha_j |a_j| c_j)^p\right)^{1/p-1} \alpha_{k+1}^p |a_{k+1}|^p \frac{\log k}{\varphi^{-1}(\log k)} < \infty$$

as required. □

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