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On the asymptotic behavior of the sequence and series of running maxima from a real random sequence

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ABSTRACT

For a sequence $\{X_n, n \geq 1\}$ of random variables, set $Y_n = \max_{1\leq k\leq n} X_k - a_n$, where $\{a_n, n \geq 1\}$ is a sequence of constants to be specified. We obtain the limiting behavior of the sequences of positive and negative parts of $\{Y_n, n \geq 1\}$ when the tail distribution of $\{X_n, n \geq 1\}$ satisfies suitable "exponential-type" conditions. Next, we consider the rate convergence of the positive part to zero (results similar to complete convergence).

1. Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables (not necessarily independent or identically distributed). We are interested in giving suitable conditions on the tail distribution of X_n assuring the existence of a sequence $\{a_n,\ n \geq 1\}$ such that, putting

$$Y_n = \max_{1\leq k\leq n} X_k - a_n,$$

the sequence $\{Y_n,\ n \geq 1\}$ converges to 0 as $n \to \infty$. The main result of the paper is Theorem 1. To prove it, we split the sequence $\{Y_n,\ n \geq 1\}$ into two parts:

$$Y_n^+ = \max(Y_n, 0) \quad \text{and} \quad Y_n^- = \max(-Y_n, 0), \quad n \geq 1,$$

and we derive the convergence to 0 for the sequences $\{Y_n^+,\ n \geq 1\}$ and $\{Y_n^-, n \geq 1\}$ separately. Our results (Propositions 1 and 2) show clearly that

(i) the behavior of $\{Y_n^+,\ n \geq 1\}$ (resp. $\{Y_n^-,\ n \geq 1\}$) is governed by the right (resp. left) tail distribution of X_n;
(ii) the sequence $\{a_n,\ n \geq 1\}$ is intimately related to the function ψ appearing in the bounds for the (left and right) tail distribution of X_n (see the assumptions of Lemma 1 and Proposition 2).

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The origin of this kind of result traces back to Pickands (1967), which treats the case of Gaussian random variables. Later in Giuliano Antonini (1995) the classical subgaussian case was studied.

The aim of the present investigation is to generalize the results of Giuliano Antonini (1995) to the case of φ-subgaussian random variables. Note that φ-subgaussian case is relevant for applications, see the monograph Buldygin and Kozachenko (2000) and the paper Giuliano Antonini et al. (2003) where the notion of φ-subgaussianity is discussed in detail and important examples are provided.

Anyway, it turns out that, with some more effort, something more can be done: more precisely, the rate of convergence of the sequence $\{Y_n^+,\ n \geq 1\}$ can be examined; this is done in Section 3 of the paper.

2. Preliminary results

In this section we present definitions and a few technical results that we will use in the proof of the main result of the paper.

A continuous even convex function $\varphi(x),\ x \in \mathbf{R}$, is called an *N-function*, if

(a) $\varphi(0) = 0$ and $\varphi(x)$ is monotone increasing for $x > 0$.
(b) $\lim_{x\to 0} \frac{\varphi(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$.

In the following the notation $\varphi(x)$ always stands for an N-function. It is obvious that the function $\varphi(x) = |x|^r/r,\ r > 1$ is an example of an N-function.

The function $\psi(x),\ x \in \mathbf{R}$, defined by $\psi(x) = \sup_{y\in\mathbf{R}} (xy - \varphi(y))$ is called the *Young–Fenchel transform* of $\varphi(x)$. It is well known that $\psi(x)$ is an N-function, too, and if $\varphi(x) = |x|^r/r,\ r > 1$ for all x, then $\psi(x) = |x|^q/q$ for all x, where $\frac{1}{r} + \frac{1}{q} = 1$.

Any N-function $\varphi(x)$ can be expressed in the form

$$\varphi(x) = \int_0^{|x|} p_\varphi(t)\, dt$$

for a suitable density $p_\varphi(t)$. This function $p_\varphi(t)$ is nondecreasing and admits a generalized inverse

$$q_\varphi(t) = \sup\{u \geq 0 : p_\varphi(u) \leq t\}.$$

It can be proved that

$$\psi(x) = \int_0^{|x|} q_\varphi(t)\, dt.$$

As a consequence we have that $\psi(x)$ is differentiable and

$$(\psi)' = q_\varphi.$$

A random variable X is said to be *φ-subgaussian* if there exists a constant $a > 0$ such that, for every $t \in \mathbf{R}$, we have $E\exp\{tX\} \leq \exp\{\varphi(at)\}$. The *$\varphi$-subgaussian standard* $\tau_\varphi(X)$ is defined as

$$\tau_\varphi(X) = \inf\{a > 0 : E\exp\{tX\} \leq \exp\{\varphi(at)\},\ t \in \mathbf{R}\}.$$

The definition of a φ-subgaussian random variable is presented in terms of expectations, but it is essentially a condition on the tail of the distribution. Namely, the following result holds (Buldygin and Kozachenko, 2000, Chapter 2, Lemma 4.3).

Lemma 1. *If φ is an N-function and a random variable X is φ-subgaussian, then for every $x > 0$ we have*

$$P\{X > x\} \leq \exp\left\{-\psi\left(\frac{x}{\tau_\varphi(X)}\right)\right\}.$$

We refer to the monograph Buldygin and Kozachenko (2000) and the paper Giuliano Antonini et al. (2003) where the notion of φ-subgaussianity is discussed in detail and important examples are provided. In the case $\varphi(x) = x^2/2$ the notion of φ-subgaussianity reduces to the classical one of *subgaussianity* (cf. for example in Hoffmann-Jørgensen, 1994, Section 4.29).

We present a brief discussion of results in the literature concerning to a sequence of classical subgaussian random variables.

Consider a sequence $\{X_n,\ n \geq 1\}$ of random variables, and set

$$Y_n = \max_{1\leq k\leq n} X_k - \sqrt{2\log n}.$$

It is well known that if the random variables X_n are independent and each of them has the standard normal distribution, then $\lim_{n\to\infty} Y_n = 0$, see, for instance, Pickands (1967, p. 198–199).

Remind that we set $Y_n^+ = \max(Y_n, 0)$. In Giuliano Antonini (1995) the following proposition is proved.

Proposition A (Giuliano Antonini, 1995, Lemma 1). *Assume that for each $n \geq 1$ and for each $x > 0$ we have*

$$P\{X_n > x\} \leq \exp\{-x^2/2\}.$$

Then $\lim_{n\to\infty} Y_n^+ = 0$ *a.s.*

Note that the only assumption on the random variables $X_n,\ n \geq 1$, imposed in Proposition A is satisfied by classical subgaussian random variables. As we already mentioned, a (classical) subgaussian random variable is φ-subgaussian with $\varphi(x) = x^2/2$. It is easy to see that in this case the Young–Fenchel transformation of φ is $\psi(x) = x^2/2$, so that $\sqrt{2\log n} = \psi^{-1}(\log n)$, where $\psi^{-1}(x)$ is the inverse function to $\psi(x)$. It is natural to think about an extension of Proposition A to the general φ-subgaussian setting, and this is done in the next section.

Next, in Giuliano Antonini (1995) the behavior of $\{Y_n^-, n \geq 1\}$ is studied also, but an inequality on the left tail distribution of X_n is required, that cannot be obtained by the subgaussianity assumption (see Remark 2). In contrast to Proposition A, the independence assumption is also needed here.

Proposition B (Giuliano Antonini, 1995, Lemma 2). *Let $\{X_n,\ n \geq 1\}$ be a sequence of independent random variables and there exists $C > 0$ such that for all $n \geq 1$ and all $x > 0$, we have*

$$P\{X_n < x\} \leq \exp\{-Ce^{-x^2/2}\}.$$

Then $\lim_{n\to\infty} Y_n^- = 0$ *a.s.*

3. On the running maxima of some φ-subgaussian random sequences

Let φ be an N-function and let p_φ be its density. Consider an arbitrary sequence $\{X_n,\ n \geq 1\}$ of random variables (not necessarily independent or identically distributed) and set

$$Y_n = \max_{1\leq k\leq n} X_k - \psi^{-1}(\log n)$$

$$Z_n = X_n - \psi^{-1}(\log n).$$

We shall use throughout the notations Y_n^+ and Y_n^- previously introduced.

The following proposition generalizes Proposition A to the case of φ-subgaussian random variables.

Proposition 1. *Suppose that there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \leq \varepsilon_0$, the density $p_\varphi(x)$ of the N-function $\varphi(x)$ satisfies the conditions*

$$\int_0^\infty q_\varphi(x)\exp\{-\varepsilon q_\varphi(x)\}\,dx < \infty \tag{1}$$

and

$$\sup_{n\geq 1} \tau_\varphi(X_n) = c \leq 1. \tag{2}$$

Then $\lim_{n\to\infty} Y_n^+ = 0$ *a.s.*

Remark 1. In the case of $\varphi(x) = x^2/2$ (that is classical subgaussianity), assumption (1) is clearly in force. Noticing this, by Lemma 1 and assumption (2)

$$P\{X_n > x\} \leq \exp\left\{-\frac{x^2}{2}\right\},$$

we that Proposition 1 generalizes Proposition A of Giuliano Antonini (1995).

Both proofs of Proposition A (presented in Giuliano Antonini (1995) and Proposition 1 (presented below) are based on the following fact.

Lemma 2. *For any $\varepsilon > 0$*

$$\{Y_n^+ > \varepsilon \text{ i.o. }\} = \{Z_n^+ > \varepsilon \text{ i.o. }\},$$

where i.o., as usual, stands for "infinitely often".

We have two different proofs of Lemma 2. The most elegant is based on the following lemma.

Lemma 3. *Let $\{x_n,\ n \geq 1\}$ and $\{a_n,\ n \geq 1\}$ be two sets of real numbers with a_n increasing to infinity. Then the following two statements are equivalent.*

(a) *The set $A = \{n : x_n > a_n\}$ is infinite.*
(b) *The set $B = \{n : \max_{1\leq k\leq n} x_k > a_n\}$ is infinite.*

Proof. The implication (a) $\Rightarrow$ (b) is obvious because $A \subset B$.

For the reverse implication assume that the set B is infinite. For every $n \in B$ put

$$\kappa_n = \min\{k : k \le n, x_k > a_n\}.$$

For any $n \in B$ we have that $x_{\kappa_n} > a_n \ge a_{\kappa_n}$ (since the sequence $\{a_n,\ n \ge 1\}$ is nondecreasing), hence $\kappa_n \in A$.

Moreover, for any two numbers $m, n \in B$ with $m < n$, we have that $\kappa_m \le \kappa_n$. In fact, either $\kappa_n > m \ge \kappa_m$, or $\kappa_n \le m$. Hence it is enough to prove that the nondecreasing sequence $\{\kappa_n,\ n \ge 1\} \subset A$ is infinite. Assume the contrary. If the sequence is finite, then there exists k such that $\kappa_n = k$ for every sufficiently large $n \in B$. This implies $x_k = x_{\kappa_n} > a_n$ for every sufficiently large $n \in B$. This is impossible because a_n is increasing to infinity. $\square$

Now we are ready to prove the first proposition.

Proof of Proposition 1. First of all note that the statement $\lim_{n\to\infty} Y_n^+ = 0$ a.s. is equivalent to $P\{Y_n^+ > \varepsilon \text{ i.o.}\} = 0$ for any $\varepsilon > 0$. According to Lemma 2 $\{Y_n^+ > \varepsilon \text{ i.o.}\} = \{Z_n^+ > \varepsilon \text{ i.o.}\}$. We shall prove, using the (first) Borel–Cantelli lemma, that the last event has probability zero. For this we need to show that

$$\sum_{n=1}^{\infty} P\{Z_n^+ > \varepsilon\} < \infty.$$

By the assumption of φ-subgaussianity and the properties of the function ψ we have

$$\begin{aligned} P\{Z_n^+ > \varepsilon\} &= P\{Z_n > \varepsilon\} = P\{X_n > \psi^{-1}(\log n) + \varepsilon\} \\ &\le \exp\left\{-\psi\left(\frac{\psi^{-1}(\log n) + \varepsilon}{\tau_\varphi(X_n)}\right)\right\} \\ &\le \exp\left\{-\psi\left(\psi^{-1}(\log n) + \varepsilon\right)\right\} \quad \text{since } c = \sup_{n\ge1} \tau_\varphi(X_n) \le 1. \end{aligned}$$

Since the series

$$\sum_{n=1}^{\infty} \exp\left\{-\psi\left(\psi^{-1}(\log n) + \varepsilon\right)\right\}$$

has the same behavior as the integral

$$\int_1^{\infty} \exp\left\{-\psi\left(\psi^{-1}(\log x) + \varepsilon\right)\right\}\, dx,$$

we can study the convergence of this one.

By means of the change of variable $t = \psi^{-1}(\log x) + \varepsilon$, the previous integral becomes

$$\int_\varepsilon^{\infty} \psi'(t-\varepsilon)\exp\{-\psi(t) + \psi(t-\varepsilon)\}\, dt = \int_\varepsilon^{\infty} \psi'(t-\varepsilon)\exp\{-\psi'(\xi)\varepsilon\}\, dt$$

by the Lagrange theorem with $t - \varepsilon < \xi < t$.

We know that ψ is an N-function with the density q_φ, that is, $\psi' = q_\varphi$, which is nondecreasing. Hence we have that the last integral equals

$$\begin{aligned} \int_\varepsilon^{\infty} q_\varphi(t-\varepsilon)\exp\{-q_\varphi(\xi)\varepsilon\}\, dt &\le \int_\varepsilon^{\infty} q_\varphi(t-\varepsilon)\exp\{-q_\varphi(t-\varepsilon)\varepsilon\}\, dt \\ &= \int_0^{\infty} q_\varphi(x)\exp\{-\varepsilon q_\varphi(x)\}\, dx < \infty \end{aligned}$$

by the assumption. $\square$

As we already mentioned, the limiting behavior of the sequence $\{Y_n^-,\ n \ge 1\}$ cannot be described in terms of subgaussianity. Roughly speaking, an opposite type of the inequality is required (see Remark 2). Anyway, the following is the generalization of Proposition B.

Proposition 2. *Assume that the $\{X_n,\ n \ge 1\}$ is a sequence of independent random variables and there exists a number $C > 0$ such that, for every $n \ge 1$ and all $x > 0$, we have*

$$P\{X_n < x\} \le \exp\left\{-Ce^{-\psi(x)}\right\},$$

where $\psi(x)$ is a positive differentiable function with $q(x) = \psi'(x)$ nondecreasing for $x > 0$. Suppose that there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \le \varepsilon_0$

$$\int_0^{+\infty} \exp\left\{\psi(x) - Ce^{\varepsilon q(x-\varepsilon)}\right\} q(x)dx < \infty.$$

Then $\lim_{n\to\infty} Y_n^- = 0$ a.s.

Remark 2. It is easy to see that the assumption imposed on the distribution function of X_n in Proposition 2 is implied by an "exponential type" condition imposed on the tail of X_n, which is, in some sense is converse of the one implied by φ-subgaussianity.

According to Lemma 1, if X is φ-subgaussian, then for every $x > 0$ we have

$$P\{X > x\} \leq \exp\left\{-\psi\left(\frac{x}{\tau_\varphi(X)}\right)\right\},$$

where ψ is the Young–Fenchel transformation of φ. If we consider in some sense "opposite" to the previous one inequality

$$P\{X > x\} \geq C\exp\{-\psi(x)\},$$

then, recalling that $t \leq \exp(t-1)$ we have

$$P\{X < x\} \leq \exp\{-P\{X \geq x\}\} \leq \exp\left\{-Ce^{-\psi(x)}\right\},$$

which we assume in Proposition 2.

Proof of Proposition 2. Once more by applying the Borel–Cantelli lemma, we shall prove that $P\{Y_n^- > \varepsilon \text{ i.o.}\} = 0$. In fact,

$$\begin{aligned} P\{Y_n^- > \varepsilon\} &= P\{Y_n < -\varepsilon\} = P\{\max_{1\leq k\leq n} X_k < \psi^{-1}(\log n) - \varepsilon\} \\ &\leq \exp\left\{-Cn\exp\left\{-\psi(\psi^{-1}(\log n) - \varepsilon)\right\}\right\}, \end{aligned}$$

and the series

$$\sum_{n=1}^{\infty} \exp\left\{-Cn\exp\left\{-\psi\left(\psi^{-1}(\log n) - \varepsilon\right)\right\}\right\}$$

has the same behavior as the integral

$$\int_1^{+\infty} \exp\left\{-Cx\exp\left\{-\psi\left(\psi^{-1}(\log x) - \varepsilon\right)\right\}\right\} dx.$$

By means of the change of variable $u = \psi^{-1}(\log x)$, the above integral can be transformed into

$$\int_0^{+\infty} \exp\left\{\psi(u) - Ce^{\psi(u)}e^{-\psi(u-\varepsilon)}\right\}\psi'(u)du = \int_0^{+\infty} \exp\left\{\psi(u) - Ce^{\psi(u)-\psi(u-\varepsilon)}\right\} q(u)du.$$

By Lagrange Theorem, for every u there exists a number t, with $u - \varepsilon \leq t \leq u$, such that

$$\psi(u) - \psi(u-\varepsilon) = \varepsilon\psi'(t) = \varepsilon q(t) \geq \varepsilon q(u-\varepsilon),$$

since $q(t)$ is nondecreasing.

Inserting this relation in the second member of the last integral we estimate it as

$$\int_0^{+\infty} \exp\left\{\psi(u) - Ce^{\psi(u)-\psi(u-\varepsilon)}\right\} q(u)du \leq \int_0^{+\infty} \exp\left\{\psi(u) - Ce^{\varepsilon q(u-\varepsilon)}\right\} q(u)du < \infty. \quad \square$$

The following lemma may be useful if we would like to check that a particular function satisfies the assumption from Proposition 2.

Lemma 4. *In the notations of Proposition 2, if for some $C' < C$ and all $\varepsilon > 0$ we have that*

$$\int_0^{+\infty} \exp\left\{\psi(x) - C'e^{\varepsilon q(x)}\right\} dx < \infty$$

and

$$\liminf_{x\to\infty} \frac{q(x-\varepsilon)}{q(x)} = \ell > 0,$$

then

$$\int_0^{+\infty} \exp\left\{\psi(x) - Ce^{\varepsilon q(x-\varepsilon)}\right\} q(x)dx < \infty$$

holds for all $\varepsilon > 0$.

Proof. By the second assumption can simplify the statement of the lemma, that is, we need to prove only that

$$\int_0^{+\infty} \exp\left\{\psi(x) - Ce^{\varepsilon q(x)}\right\} q(x)dx < \infty \quad \text{for all } \varepsilon > 0.$$

Let $\varepsilon > 0$, and let $D = C - C'$. Since

$$q(x) \le \exp\{Dq(x)\},$$

the integral

$$\int_0^{+\infty} \exp\left\{\psi(x) - Ce^{\varepsilon q(x)}\right\} q(x)dx \le \int_0^{+\infty} \exp\left\{\psi(x) - Ce^{\varepsilon q(x)} + Dq(x)\right\} dx.$$

Using again the estimation $q(x) \le \exp\{\varepsilon q(x)\}$, we see that the above integral converges if

$$\int_0^{+\infty} \exp\left\{\psi(x) - Ce^{\varepsilon q(x)} + De^{\varepsilon q(x)}\right\} dx = \int_0^{+\infty} \exp\left\{\psi(x) - C'e^{\varepsilon q(x)}\right\} dx < \infty. \quad \square$$

Remark 3. It is not difficult to see that the assumption from Lemma 3 holds if φ is a regularly varying function with exponent $\alpha \ge 1$. In particular it holds for $\varphi(x) = x^2/2$, the case considered in Giuliano Antonini (1995, Lemma 2).

The following result combines Propositions 1 and 2 together.

Theorem 1. *Assume that the $\{X_n,\ n \ge 1\}$ is a sequence of independent random variables and $\varepsilon_0 > 0$. Let $\varphi(x)$ be an N-function with the density $p_\varphi(x)$.*

Next, assume that for all $\varepsilon \le \varepsilon_0$, some $C > 0$, and some $A \ge 1$

$$\int_0^{\infty} q_\varphi(x) \exp\{-\varepsilon q_\varphi(x)\}\, dx < \infty$$

and

$$\int_0^{+\infty} \exp\left\{\psi(x) - Ce^{\varepsilon q_\varphi(x-\varepsilon)}\right\} q_\varphi(x)dx < \infty.$$

Let moreover for every $n \ge 1$ and $x > 0$

$$C \exp\{-\psi(x)\} \le P\{X_n > x\} \le \exp\{-\psi(Ax)\}.$$

Then $\lim_{n\to\infty} Y_n = 0$ a.s.

Proof. It is just a combination of Propositions 1 and 2. Note that the statement $P\{X_n > x\} \le \exp\left\{-\psi\left(\frac{x}{c}\right)\right\}$ is equivalent to $\sup_{n\ge 1} \tau_\varphi(X_n) = c$ due to Lemma 1 if we take $A = 1/c$.

For the left hand side of the assumption on tail probabilities we used the well known inequality $s \ge 1 - e^{-s}$. $\square$

4. Convergence of some series of maxima of random sequences

In this section we deal with the convergence of some series connected with the maxima. In particular, we prove that under the assumption of φ- subgaussianity of a sequence $\{X_n,\ n \ge 1\}$ of random variables and some assumptions on their norms, then

$$\sum_{k=1}^{\infty} k^{-\alpha} P\{Y_k^+ > \varepsilon\} < \infty$$

for a suitable constant α. We also show that the result is sharp.

The behavior of the series

$$\sum_{k=1}^{\infty} k^{-\alpha} P\{Y_k^- > \varepsilon\}$$

is still an *open problem.*

The main idea of the proof of Proposition 1 from the previous section was that instead of the complicated sequence $\{Y_n,\ n \ge 1\}$, we considered the much simpler sequence $\{Z_n,\ n \ge 1\}$. This was possible to achieve because

$$\{Y_n^+ > \varepsilon \text{ i.o.}\} = \{Z_n^+ > \varepsilon \text{ i.o.}\}.$$

Our initial attempt to prove the results in this section was also to consider the sequence $\{Z_n,\ n \ge 1\}$. Unfortunately, it appears that the series of probabilities we are interested in this section for these two sequences can behave very differently. This fact may be shown by the following simple example provided to us by Professor Victor Kruglov (Moscow State University).

Example. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a discrete sample $\Omega = \{\omega_k,\ k \geq 1\}$, sigma-algebra $\mathcal{F}$ of all subsets of Ω and probability $P\{\omega_k\} = C/k^2$, $k \geq 1$, where $C = 1/\left(\sum_{k=1}^{\infty} 1/k^2\right) = 6/\pi^2$.

Consider a sequence of one-point sets $A_n = \{\omega_n\},\ n \geq 1$ and let $B_n = \cup_{k=n}^{\infty} A_k$. Then

$$\{A_k \text{ i.o. }\} = \cap_{m=1}^{\infty} \cup_{k=m}^{\infty} A_k = \cap_{m=1}^{\infty} \cup_{k=m}^{\infty} B_k = \{B_k \text{ i.o. }\},$$

while

$$\sum_{k=1}^{\infty} k^{-\alpha} P\{A_k\} = C \sum_{k=1}^{\infty} k^{-2-\alpha}$$

converges for $\alpha > -1$ and

$$\sum_{k=1}^{\infty} k^{-\alpha} P\{B_k\} = C \sum_{k=1}^{\infty} k^{-\alpha} \sum_{m=k}^{\infty} k^{-2-\alpha} \equiv \sum_{k=1}^{\infty} k^{-1-\alpha}$$

converges for $\alpha > 0$ only.

As in the previous section, we split the sequence $\{Y_n,\ n \geq 1\}$ into two parts $Y_n^+ = \max(Y_n, 0)$ and $Y_n^- = \max(-Y_n, 0),\ n \geq 1$, and derive the results for the sequences $\{Y_n^+,\ n \geq 1\}$ and $\{Y_n^-,\ n \geq 1\}$ separately. The main goal of this section is to deduce the probability series convergence when the tail distribution of $X_n,\ n \geq 1$ satisfies suitable "exponential-type" conditions.

The next proposition is very unusual and was not what we expected at the beginning of the project. It appears that we can derive a rate of convergence for $P\{Y_n^+ > 0\}$, while usually only results for $P\{Y_n^+ > \varepsilon\}$, where $\varepsilon > 0$ can be obtained. Certainly, because $P\{Y_n^+ > \varepsilon\} \leq P\{Y_n^+ > 0\}$ for any $\varepsilon > 0$ we can obtain from the assumptions of the proposition that $\sum_{k=1}^{+\infty} k^{-\alpha} P\{Y_n^+ > \varepsilon\} < +\infty$.

Proposition 3. *Let $\{X_n,\ n \geq 1\}$ be a sequence of φ-subgaussian random variables such that*

$$\sup_{n \geq 1} \tau_\varphi(X_n) = c$$

and let $\alpha > 2 - \frac{1}{c}$. Then

$$\sum_{k=1}^{+\infty} k^{-\alpha} P\{Y_n^+ > 0\} < +\infty.$$

Remark 4. The statement of Proposition 3 is trivial for $\alpha > 1$. Because of that, only the case $c < 1$ is interesting.

Proof. It is easy to see that

$$\begin{aligned}
P\{Y_k^+ > 0\} &= P\{\max_{1 \leq m \leq k} X_m > \psi^{-1}(\log k)\} \\
&\leq \sum_{m=1}^{k} P\{X_m > \psi^{-1}(\log k)\} \\
&\leq \sum_{m=1}^{k} \exp\left\{-\psi\left(\frac{\psi^{-1}(\log k)}{\tau_\varphi(X_m)}\right)\right\} \\
&\leq k \exp\left\{-\psi\left(\frac{\psi^{-1}(\log k)}{c}\right)\right\} \text{ by Lemma 1, properties of } \psi \text{, and assumptions of the proposition} \\
&\leq k e^{-\log k/c} = k^{1-\frac{1}{c}} \quad \text{since } \psi(\theta x) \geq \psi(x), \theta \geq 1, \text{ which is true for any } N\text{-function.}
\end{aligned}$$

Hence we can write

$$\sum_{k=1}^{\infty} k^{-\alpha} P\{Y_n^+ > 0\} \leq \sum_{k=1}^{+\infty} k^{-\alpha+1-\frac{1}{c}} < \infty. \quad \square$$

The following result shows that, in some sense, Proposition 3 is sharp. But before we need to prove the following lemma.

Lemma 5. *Let $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ be a strictly increasing function. Then for any $t > 0$ and $\varepsilon > 0$:*

$$t \exp\left\{-\psi[\psi^{-1}(\log t) + \varepsilon]\right\} \leq 1.$$

Proof. Let $u = \psi^{-1}(\log t) + \varepsilon$, then $t = \exp\{\psi(u - \varepsilon)\}$ and

$$t \exp\left\{-\psi[\psi^{-1}(\log t) + \varepsilon]\right\} = \exp\{\psi(u - \varepsilon) - \psi(u)\} \le 1. \quad \square$$

Proposition 4. *Let $(X_n)_{n\ge1}$ be a sequence of independent random variables with the property that there exist a strictly increasing differentiable function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ and a positive real number t_0 such that, for every $n \ge 1$ and for every $t > t_0$ we have*

$$P\{X_n > t\} \ge \exp\{-\psi(t)\}.$$

Assume that there exists $\varepsilon_0 > 0$ such that

$$\limsup_{x\to+\infty} \frac{\max\limits_{x\le\xi\le x+\varepsilon_0} \psi'(\xi)}{\psi(x)} = l < \infty. \tag{3}$$

Then, for every real number $\alpha < 1$ and for every $0 < \varepsilon < (1-\alpha)/l$ we have

$$\sum_{k=1}^{\infty} k^{-\alpha} P(Y_k^+ > \varepsilon) = \infty.$$

Here we adopt the convention $1/0 = \infty$, that is, if $l = 0$, the result holds for every $\varepsilon > 0$.

Remark 5. A good example of a function ψ that satisfies the assumptions of Proposition 4 is $\psi(x) = x^p/p$, where $p \ge 1$.

Remark 6. Note that assumption (3) implies that for all $\varepsilon < \varepsilon_0$

$$\limsup_{x\to+\infty} \frac{\max\limits_{x\le\xi\le x+\varepsilon} \psi'(\xi)}{\psi(x)} = l < \infty.$$

Proof of Proposition 4. We have

$$\begin{aligned}
P\{Y_k^+ > \varepsilon\} &= P\left\{\max_{1\le m\le k} X_m > \psi^{-1}(\log n) + \varepsilon\right\} \\
&= 1 - \prod_{m=1}^{k}\left(1 - P\left\{X_m > \psi^{-1}(\log k) + \varepsilon\right\}\right) \\
&\ge 1 - \left(1 - \exp\left\{-\psi\left[\psi^{-1}(\log k) + \varepsilon\right]\right\}\right)^k \\
&\ge 1 - \exp\left\{-k\exp\left\{-\psi\left[\psi^{-1}(\log k) + \varepsilon\right]\right\}\right\} \\
&\quad \text{since, for } t > 0, \text{ we have } 1 - a \le e^{-a} \text{ and we take } a = \exp\left\{-\psi\left[\psi^{-1}(\log k) + \varepsilon\right]\right\} \\
&\ge \exp\left\{-k\exp\left\{-\psi\left[\psi^{-1}(\log k) + \varepsilon\right]\right\}\right\} \cdot k\exp\left\{-\psi\left[\psi^{-1}(\log k) + \varepsilon\right]\right\} \\
&\quad \text{since } 1 - e^{-a} = \int_0^a e^{-x}dx > e^{-a}\cdot a \text{ and we take } a = k\exp\left\{-\psi\left[\psi^{-1}(\log k) + \varepsilon\right]\right\} \\
&\ge e^{-1}k\exp\left\{-\psi\left[\psi^{-1}(\log k) + \varepsilon\right]\right\},
\end{aligned}$$

by Lemma 4 applied to the expression under first exponent.

From this we obtain that

$$\sum_{k=1}^{\infty} k^{-\alpha} P\{Y_k^+ > \varepsilon\} \ge e^{-1}\sum_{k=1}^{\infty} k^{1-\alpha}\exp\{-\psi[\psi - 1(\log k) + \varepsilon]\}.$$

By the integral test, the divergence of the last integral diverges together with

$$\int_1^\infty t^{1-\alpha}\exp\left\{-\psi\left[\psi^{-1}(\log t) + \varepsilon\right]\right\}dt.$$

With the change of variable

$$x = \psi^{-1}(\log t), \qquad t = \exp\{\psi(x)\}$$

the last integral becomes

$$\int_\varepsilon^\infty \exp\{(2-\alpha)\psi(x) - \psi(x+\varepsilon)\}\psi'(x)dx = \int_\varepsilon^\infty \exp\left\{(1-\alpha)\psi(x) - \varepsilon\psi'(\xi)\right\}\psi'(x)dx$$

by the Lagrange theorem with $x \le \xi \le x + \varepsilon$.

By the assumption (3), for any $\delta > 0$, the last integral has the same behavior as

$$\int_{\varepsilon}^{\infty} \exp\{(1-\alpha)\psi(x) - \varepsilon(1+\delta)l\psi(x)\}\,\psi'(x)dx = \int_{\psi^{-1}(\varepsilon)}^{\infty} \exp\{(1-\alpha-\varepsilon(1+\delta)l)\,u\}\,du,$$

where $u = \psi(x)$. Since $\varepsilon < (1-\alpha)/l$, we conclude that $\delta = \frac{1-\alpha}{l\epsilon} - 1 > 0$. For this choice of δ we have $1-\alpha-\varepsilon(1+\delta)l = 0$. In this case the integral written above diverges. $\square$

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