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ON THE COMPLETE CONVERGENCE FOR ARRAYS OF ROWWISE EXTENDED NEGATIVELY DEPENDENT RANDOM VARIABLES

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Abstract. A general result for the complete convergence of arrays of rowwise extended negatively dependent random variables is derived. As its applications eight corollaries for complete convergence of weighted sums for arrays of rowwise extended negatively dependent random variables are given, which extend the corresponding known results for independent case.

1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins ([5]) as follows. A sequence $\{U_n, n \geq 1\}$ of random variables *converges completely* to the constant θ if

$$\sum_{n=1}^{\infty} P\{|U_n - \theta| > \epsilon\} < \infty \text{ for all } \epsilon > 0.$$

Moreover, they proved that the sequence of arithmetic means of independent identically distribution (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. This result has been generalized and extended in several directions, see Gut ([3], [4]), Hu *et al.* ([7], [8]), Chen *et al.* ([2]), Sung ([14], [15], [17]), Zarei and Jabbari ([20]), Baek *et al.* ([1]). In particular, Sung ([14]) obtained the following two Theorems A and B.

Theorem A. *Let $\{X_n, n \geq 1\}$ be a sequence of independent zero-mean random variables which are stochastically dominated by a random variable X, i.e.,*

$$P(|X_n| > x) \leq CP(|X| > x) \text{ for all } x \geq 0 \text{ and } n \geq 1,$$

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where C is a positive constant. Assume that $E|X|^\gamma < \infty$, where $\gamma = p(t+\beta+1) \geq 1$ and $p > 0$. Let $\{b_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers satisfying

$$\sup_{n,i} |b_{ni}| < \infty, \sum_{i=1}^{\infty} |b_{ni}|^q = O(n^\beta) \text{ for some } q < \gamma. \tag{1.1}$$

Assume that $\sum_{i=1}^\infty b_{ni}X_{ni}$ is finite a.s. for any $n \geq 1$.

(i) *If $1 \leq \gamma < 2$, then*

$$\sum_{n=1}^{\infty} n^t P\left(n^{-1/p}\left|\sum_{i=1}^{\infty} b_{ni}X_i\right| > \varepsilon\right) < \infty \text{ for all } \varepsilon > 0. \tag{1.2}$$

(ii) *If $\gamma \geq 2$, and*

$$\sum_{i=1}^{\infty} b_{ni}^2 = O(n^\alpha) \text{ for some } \alpha < 2/p, \tag{1.3}$$

then (1.2) *holds.*

Theorem B. *Let $\{X_n, n \geq 1\}$ be a sequence of independent zero-mean random variables which are stochastically dominated by a random variable X satisfying*

$$E|X|^\gamma \log(1+|X|) < \infty,$$

where $\gamma = p(t+\beta+1) \geq 1$ and $p > 0$. Let $\{b_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers satisfying

$$\sup_{n,i} |b_{ni}| < \infty, \sum_{i=1}^{\infty} |b_{ni}|^{p(t+\beta+1)} = O(n^\beta). \tag{1.4}$$

Assume that $\sum_{i=1}^\infty b_{ni}X_{ni}$ is finite a.s. for any $n \geq 1$.

(i) *If $1 \leq \gamma < 2$, then* (1.2) *holds.*

(ii) *If $\gamma \geq 2$, and $\{b_{ni}, i \geq 1, n \geq 1\}$ satisfies* (1.3), *then* (1.2) *holds.*

Baek et al. ([1]) announced the following complete convergence result.

Theorem C. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise pairwise zero-mean ND random variables which are stochastically dominated by a random variable X, i.e.,*

$$P(|X_{ni}| > x) \leq CP(|X| > x) \text{ for all } x \geq 0 \text{ and all } i \geq 1 \text{ and } n \geq 1,$$

where C is a positive constant. Assume that $t \geq -1$ and $p > 0$ and that $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of real numbers satisfying

$$\sup_{i \geq 1} |a_{ni}| = O(n^{-\mu}) \text{ for some } \mu > 0 \tag{1.5}$$

and

$$\sum_{i=1}^{\infty} |a_{ni}| = O(n^\tau) \text{ for some } \tau \in [0, \mu). \tag{1.6}$$

(i) *If* $\tau + t + 1 > 0$ *and there exists some* $\delta > 0$ *such that* $(\tau/\mu) + 1 < \delta \leq 2, \gamma = \max\{1 + (1 + \tau + t)/\mu, \delta\}$, *and* $E|X|^\gamma < \infty$, *then*

$$\sum_{n=1}^{\infty} n^t P\left(\left|\sum_{i=1}^{\infty} a_{ni} X_{ni}\right| > \varepsilon\right) < \infty \text{ for all } \varepsilon > 0. \tag{1.7}$$

(ii) *If* $\tau + t + 1 = 0$ *and* $E(|X| \log |X|) < \infty$, *then* (1.7) *holds.*

Remark 1. There is a question in the proofs of $I_2^* < \infty$ of Theorem C(i) in Baek et al. ([1]). The Rosenthal inequality plays a key role in this proof, but it is still an open problem to obtain Rosenthal inequality for pairwise negatively dependent random variables. Clearly Theorem C(i) holds if $\{X_{ni}, i \geq 1, n \geq 1\}$ is an array of rowwise negatively dependent random variables.

Liu ([11]) introduced the following dependence structure.

Definition 1. Random variables $Y_1, Y_2, \ldots$ are said to be *extended negatively dependent* (END) if there exists a constant $M > 0$ such that for each $n \geq 2$, the following two inequalities hold:

$$P\{Y_1 \leq y_1, \ldots, Y_n \leq y_n\} \leq M \prod_{i=1}^{n} P\{Y_i \leq y_i\}$$

and

$$P\{Y_1 > y_1, \ldots, Y_n > y_n\} \leq M \prod_{i=1}^{n} P\{Y_i > y_i\}$$

for every sequence $\{y_1, \ldots, y_n\}$ of real numbers.

Random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ are said to be an *array of rowwise END random variables* if for each $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is END.

In the case $M = 1$ the notion of END random variables reduces to the well-known notion of so-called *negatively dependent* (ND) random variables which was introduced by Lehmann ([10]) (cf. also Joag-Dev and Proschan ([9])). As it is mentioned in Liu ([11]), the END structure is substantially more comprehensive than the ND structure in that it can reflect not only a negative dependence structure but also a positive one, to some extent. Liu ([11]) pointed out that the END random variables can be taken as negatively or positively dependent and provided some interesting examples to support this idea. Joag-Dev and Proschan ([9]) also pointed out that negatively associated (NA) random variables must be ND and ND is not necessarily NA, thus NA random variables are END. A great numbers of articles for NA random variables have appeared in literature. But very few papers are written for END random variables. For example, for END random variables with heavy tails Liu ([11]) obtained the precise large deviations and Liu ([12]) studied sufficient and necessary conditions for moderate deviations, and Wang *et al.* ([19]) studied complete convergence for weighted sums and arrays of rowwise END.

In this paper, we obtain a complete convergence for weighted sums of END random variables under general conditions inspiring by Sung ([15]) and Sung *et al.* ([16]). As its applications eight corollaries of the complete convergence of weighted sums for arrays of rowwise END random variables are given, which extend and improve Theorems A, B and C for $t > -1$ and some other known results.

Throughout this paper, C will represent positive constants which their value may change from one place to another. For $x \geq 0$ the symbol $[x]$ denotes the greatest integer in $x, \log x = \max\{1, \ln x\}$, where $\ln x$ denotes the natural logarithm.

2. Lemmata

In order to prove our main result, we need the following lemmas. The first lemma was obtained in Liu ([12]).

Lemma 1. *Let* $\{Y_n, n \geq 1\}$ *be a sequence of END random variables.*

1) *If* $\{f_n, n \geq 1\}$ *is a sequence of monotone increasing* (*or all monotone decreasing*) *functions, then* $\{f_n(Y_n), n \geq 1\}$ *is a sequence of END random variables.*

2) *There exists a constant* M *such that* $E(\prod_{j=1}^n Y_j^+) \leq M \prod_{j=1}^n EY_j^+, n \geq 2,$ *where* $EY^+ = E(\max\{Y, 0\})$.

Lemma 2. *Let* $X_1, X_2, \ldots, X_n$ *be END random variables such that*

$$|X_k| \leq b_k, 1 \leq k \leq n.$$

Then for any $t > 0$,

$$E\exp\left(t\sum_{k=1}^n X_k\right) \leq M\exp\left\{t\sum_{k=1}^n EX_k + \frac{t^2}{2}\sum_{k=1}^n e^{tb_k}EX_k^2\right\}.$$

Proof. By Lemma 1 for any $t > 0, \{\exp(tX_k), 1 \leq k \leq n\}$ is nonnegative END, thus, we have

$$E\exp\left(t\sum_{k=1}^n X_k\right) \leq M\prod_{k=1}^n Ee^{tX_k}.$$

Since

$$\begin{aligned} Ee^{tX_k} &= E\left(1 + tX_k + \frac{1}{2!}t^2X_k^2 + \frac{1}{3!}t^3X_k^3 + \cdots\right) \\ &\leq 1 + tEX_k + \frac{t^2EX_k^2}{2}\left(1 + \frac{tb_k}{3} + \frac{t^2b_k^2}{3\cdot 4} + \cdots\right) \\ &\leq 1 + tEX_k + \frac{t^2EX_k^2}{2}e^{tb_k} \\ &\leq \exp\left(tEX_k + \frac{t^2}{2}e^{tb_k}EX_k^2\right). \end{aligned}$$

Therefore

$$E\exp\left(t\sum_{k=1}^{n}X_k\right)\le M\exp\left\{t\sum_{k=1}^{n}EX_k+\frac{t^2}{2}\sum_{k=1}^{n}e^{tb_k}EX_k^2\right\}.$$ □

3. Main results

With the preliminaries accounted for, the main theorem can now be presented.

Theorem 1. *Let* $\{X_{ni},1\le i\le k_n,n\ge 1\}$ *be an array of rowwise END random variables, where* $\{k_n,n\ge 1\}$ *is a sequence of positive integers. Let* $\{a_n,n\ge 1\}$ *and* $\{d_n,n\ge 1\}$ *be sequences of positive constants with* $\lim_{n\to\infty}d_n=0$*. Suppose that*

(i) $\sum_{n=1}^{\infty}a_n\sum_{i=1}^{k_n}P(|X_{ni}|>\epsilon)<\infty$ *for all* $\varepsilon>0$,

(ii) $\sum_{n=1}^{\infty}a_n\left(\sum_{i=1}^{k_n}P(|X_{ni}|>d_n)\right)^{q_1}<\infty$ *for some* $q_1>0$,

(iii) $\frac{1}{d_n}\sum_{i=1}^{k_n}E|X_{ni}|^2I(|X_{ni}|\le d_n)\to 0$ *as* $n\to\infty$,

(iv) $\sum_{i=1}^{k_n}EX_{ni}I(|X_{ni}|\le d_n)\to 0$ *as* $n\to\infty$,

(v) $\sum_{n=1}^{\infty}a_n\exp(-q_2/d_n)<\infty$ *for some* $q_2>0$.

Then

$$\sum_{n=1}^{\infty}a_nP\left(\left|\sum_{i=1}^{k_n}X_{ni}\right|>\varepsilon\right)<\infty \text{ for all } \varepsilon>0. \tag{3.1}$$

Proof. Let $N_1=\{n:\sum_{i=1}^{k_n}P(|X_{ni}|>d_n)>1\}$ and $N_2=\{n:\sum_{i=1}^{k_n}P(|X_{ni}|>d_n)\le 1\}$. By (ii),

$$\begin{aligned}\sum_{n\in N_1}a_n&<\sum_{n\in N_1}a_n\left(\sum_{i=1}^{k_n}P(|X_{ni}|>d_n)\right)^{q_1}\\&\le\sum_{n=1}^{\infty}a_n\left(\sum_{i=1}^{k_n}P(|X_{ni}|>d_n)\right)^{q_1}<\infty.\end{aligned}$$

Note that for any $\epsilon>0$,

$$\sum_{n=1}^{\infty}a_nP\left(\left|\sum_{i=1}^{k_n}X_{ni}\right|>\varepsilon\right)\le\sum_{n\in N_1}a_n+\sum_{n\in N_2}a_nP\left(\left|\sum_{i=1}^{k_n}X_{ni}\right|>\varepsilon\right).$$

Hence in order to prove (3.1), it is enough to show that

$$\sum_{n\in N_2}a_nP\left(\left|\sum_{i=1}^{k_n}X_{ni}\right|>\varepsilon\right)<\infty.$$

So without loss of generality, we assume that

$$\sum_{i=1}^{k_n}P(|X_{ni}|>d_n)\le 1 \text{ for all } n\ge 1. \tag{3.2}$$

Define for all $\varepsilon > 0$ and $1 \le i \le k_n, n \ge 1$,

$$\begin{aligned} X_{ni}^{(1)} &= -d_n I(X_{ni} < -d_n) + X_{ni} I(|X_{ni}| \le d_n) + d_n I(X_{ni} > d_n), \\ X_{ni}^{(2)} &= (X_{ni} + d_n) I(X_{ni} < \frac{-\varepsilon}{3([q_1]+1)}) + (X_{ni} - d_n) I(X_{ni} > \frac{\varepsilon}{3([q_1]+1)}), \\ X_{ni}^{(3)} &= X_{ni} - X_{ni}^{(1)} - X_{ni}^{(2)}. \end{aligned}$$

To prove (3.1), it is enough to prove that

$$\sum_{n=1}^{\infty} a_n P\left(\left| \sum_{i=1}^{k_n} X_{ni}^{(l)} \right| > \varepsilon/3 \right) < \infty, \ l = 1, 2, 3. \tag{3.3}$$

First we prove that (3.3) holds for $l = 1$. Clearly $\{X_{ni}^{(1)}, 1 \le i \le k_n, n \ge 1\}$ is an array of rowwise END random variables by Lemma 1. Applying Markov's inequality and Lemma 2 to $\{X_{ni}^{(1)}, 1 \le i \le k_n, n \ge 1\}$ for each fixed $n \ge 1$ and $t > 0$, we have

$$\begin{aligned} P\left(\sum_{i=1}^{k_n} X_{ni}^{(1)} > \varepsilon/3 \right) &\le \exp(-\frac{t\varepsilon}{3}) E \exp(t \sum_{i=1}^{k_n} X_{ni}^{(1)}) \\ &\le M \exp\left\{ -\frac{t\varepsilon}{3} + t \sum_{i=1}^{k_n} E X_{ni}^{(1)} + \frac{t^2}{2} e^{t d_n} \sum_{i=1}^{k_n} E(X_{ni}^{(1)})^2 \right\}. \end{aligned}$$

From the definition of $X_{ni}^{(1)}$, we have, by (iv) and (3.2),

$$\left| \sum_{i=1}^{k_n} E X_{ni}^{(1)} \right| \le \left| \sum_{i=1}^{k_n} E X_{ni} I(|X_{ni}| \le d_n) \right| + d_n \sum_{i=1}^{k_n} P(|X_{ni}| > d_n) \to 0$$

as $n \to \infty$. By (iii) and (3.2), we obtain

$$\sum_{i=1}^{k_n} E(X_{ni}^{(1)})^2 \le \sum_{i=1}^{k_n} E X_{ni}^2 I(|X_{ni}| \le d_n) + d_n^2 \sum_{i=1}^{k_n} P(|X_{ni}| > d_n) = o(d_n).$$

Therefore, by putting $t = 6q_2/(d_n \varepsilon)$ and the above arguments, for sufficiently large n, we obtain

$$P\left(\sum_{i=1}^{k_n} X_{ni}^{(1)} > \varepsilon/3 \right) \le M \exp(-q_2/d_n).$$

Thus using (v), we get

$$\sum_{n=1}^{\infty} a_n P\left(\sum_{i=1}^{k_n} X_{ni}^{(1)} > \varepsilon/3 \right) \le C + M \sum_{n=1}^{\infty} a_n \exp(-q_2/d_n) < \infty.$$

If we consider $-X_{ni}^{(1)}$ instead of $X_{ni}^{(1)}$ in the arguments above, in a similar manner we obtain

$$\sum_{n=1}^{\infty} a_n P\left(\sum_{i=1}^{k_n} -X_{ni}^{(1)} > \varepsilon/3\right) \leq C + M \sum_{n=1}^{\infty} a_n \exp(-q_2/d_n) < \infty.$$

Therefore, (3.3) holds for $l = 1$.

Next we prove (3.3) holds for $l = 2$. Note that

$$P\left(\left|\sum_{i=1}^{k_n} X_{ni}^{(2)}\right| > \varepsilon/3\right) \leq P\left(\bigcup_{i=1}^{k_n}\left(X_{ni}^{(2)} \neq 0\right)\right) \leq \sum_{i=1}^{k_n} P\left(|X_{ni}| > \frac{\varepsilon}{3([q_1]+1)}\right).$$

Thus, (3.3) holds for $l = 2$ by (i).

Finally, we prove (3.3) holds for $l = 3$. For sufficiently large n such that $d_n < \frac{\epsilon}{3([q_1]+1)}$, from the definition of $X_{ni}^{(3)}$, we get: if $X_{ni} \leq d_n$, then $X_{ni}^{(3)} \leq 0$; if $X_{ni} > d_n$, then $X_{ni}^{(3)} \leq \frac{\epsilon}{3([q_1]+1)}$. So we have by (3.2) that

$$\begin{aligned}
&P\left(\sum_{i=1}^{k_n} X_{ni}^{(3)} > \varepsilon/3\right) \\
&\leq P(\text{there are at least } [q_1]+1 \text{ values of } i \in \{1, 2, \ldots, k_n\} \text{ such that } X_{ni} > d_n) \\
&\leq \sum_{1 \leq i_1 < \cdots < i_{[q_1]+1} \leq k_n} P(X_{ni_1} > d_n, \ldots, X_{ni_{[q_1]+1}} > d_n) \\
&\leq M \sum_{1 \leq i_1 < \cdots < i_{[q_1]+1} \leq k_n} P(X_{ni_1} > d_n) \cdots P(X_{ni_{[q_1]+1}} > d_n) \\
&\leq M\left(\sum_{i=1}^{k_n} P(X_{ni} > d_n)\right)^{[q_1]+1} \leq M\left(\sum_{i=1}^{k_n} P(X_{ni} > d_n)\right)^{q_1}.
\end{aligned}$$

Therefore $\sum_{n=1}^{\infty} a_n P\left(\sum_{i=1}^{k_n} X_{ni}^{(3)} > \varepsilon/3\right) < \infty$ by (ii). In a similar manner, we have $\sum_{n=1}^{\infty} a_n P\left(\sum_{i=1}^{k_n} -X_{ni}^{(3)} > \varepsilon/3\right) < \infty$. Thus (3.3) holds for $l = 3$. □

By Markov's inequality and Theorem 1, we have the following corollary at once.

Corollary 1. *Let* $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}, \{k_n, n \geq 1\}, \{a_n, n \geq 1\}$, *and* $\{d_n, n \geq 1\}$ *be as in Theorem 1 except that* (ii) *is replaced by* (ii)':

(ii)' $\sum_{n=1}^{\infty} a_n \left(d_n^{-q} \sum_{i=1}^{k_n} E|X_{ni}|^q\right)^{q_1}$ *for some* $q > 0$ *and* $q_1 > 0$.

Then (3.1) *holds.*

Now let $\{k_n, n \geq 1\}$ be a strictly increasing subsequence of positive integers and $\{b_n, n \geq 1\}$ a positive monotone increasing subsequence of real numbers with $0 < b_n \uparrow \infty$. Following Gut ([3]), we define

$$\psi(x) = Card\{n : b_{k_n} \leq x\} \text{ for } x > 0, \psi(0) = 0.$$

Set $M(x)=\sum_{n=1}^{[x]} k_n$ for $x \geq 0$. We have:

Corollary 2. *Let* $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ *be an array of rowwise zero-mean END random variables which are weakly mean dominated by a random variable* X, *i.e.,*

$$\frac{1}{k_n}\sum_{i=1}^{k_n} P(|X_{ni}|>x) \leq C_1 P(|X|>x) \text{ for all } x \geq 0 \text{ and } n \geq 1, \tag{3.4}$$

where C_1 *is a positive constant. Let* $b_n=\phi(n)$ *for* $n \geq 1$, *where* ϕ *is a positive nondecreasing function satisfying*

$$\frac{\phi(x)}{\sqrt{x \log x}} \to \infty \text{ as } x \to \infty. \tag{3.5}$$

Assume that for some $C_2>0, M(\psi(2x)) \leq C_2 M(\psi(x))$ *for all* $x \geq 0$. *If* $EM(\psi(|X|))<\infty, E|X|^p<\infty$ *for some* $p>2$, *then*

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon\phi(k_n)\right) < \infty \text{ for all } \varepsilon>0.$$

Proof. We will apply Corollary 1 with $a_n=1, d_n=1/\log n, n \geq 1$ and X_{ni} replaced by $X_{ni}/\phi(k_n), 1 \leq i \leq k_n, n \geq 1$. Clearly, Conditions (i), (ii)', (iii) and (v) can be shown to hold in the same way as in the proof of Corollary 1 of Sung ([15]). We shall prove that Condition (iv) holds. As $EX_{ni}=0$, and by (3.4) and (3.5) we have

$$\begin{aligned}
\left|\sum_{i=1}^{k_n} E\frac{X_{ni}}{\phi(k_n)} I\left(\left|\frac{X_{ni}}{\phi(k_n)}\right| \leq \frac{1}{\log n}\right)\right| &\leq \sum_{i=1}^{k_n} E\left|\frac{X_{ni}}{\phi(k_n)}\right| I\left(\left|\frac{X_{ni}}{\phi(k_n)}\right| > \frac{1}{\log n}\right)\\
&\leq \log n \sum_{i=1}^{k_n} E\left|\frac{X_{ni}}{\phi(k_n)}\right|^2\\
&\leq C_1\frac{k_n \log n}{\phi^2(k_n)} E|X|^2 \to 0
\end{aligned}$$

as $n \to \infty$. □

Corollary 3. *Let* $\{X_{ni}, i \geq 1, n \geq 1\}$ *be an array of rowwise zero-mean END random variables which are stochastically dominated by a random variable* X *satisfying* $E|X|^\gamma<\infty$, *where* $\gamma=p(t+\beta+1) \geq 1$ *and* $t>-1$ *and* $p>0$. *Let* $\{b_{ni}, i \geq 1, n \geq 1\}$ *be an array of real numbers satisfying* (1.1). *Assume that* $\sum_{i=1}^{\infty} b_{ni}X_{ni}$ *is finite a.s. for any* $n \geq 1$.

(i) *If* $1 \leq \gamma < 2$, *then* (1.2) *holds.*

(ii) *If* $\gamma \geq 2$ *and*

$$\sum_{i=1}^{\infty} b_{ni}^2 = o\left(\frac{n^{2/p}}{\log n}\right), \tag{3.6}$$

then (1.2) *holds.*

Proof. Since $\sum_{i=1}^{\infty} b_{ni}X_{ni}$ is finite a.s., there exists positive integer k_n such that $P(n^{-1/p}|\sum_{i=k_n+1}^{\infty} b_{ni}X_{ni}| > \epsilon/2) < 1/n^{(t+2)}$ for all $n \geq 1$ and for all $\varepsilon > 0$. Therefore in order to prove (1.2), we only need to prove that

$$\sum_{n=1}^{\infty} n^t P\left(n^{-1/p}\left|\sum_{i=1}^{k_n} b_{ni}X_i\right| > \varepsilon/2\right) < \infty. \tag{3.7}$$

Without loss of generality, we may assume that $|b_{ni}| \leq 1$ (for all $i \geq 1, n \geq 1$) and $\sum_{i=1}^{\infty} |b_{ni}|^q \leq n^{\beta}$ (for all $n \geq 1$) by (1.1) and $b_{ni} > 0$ (for all $i \geq 1, n \geq 1$). Hence

$$\sum_{i=1}^{\infty} b_{ni}^{q+\theta} \leq n^{\beta} \text{ for all } \theta \geq 0. \tag{3.8}$$

We will apply Corollary 1 with $a_n = n^t, n \geq 1$ and $d_n = (\log n)^{-1}$ and X_{ni} replaced by $n^{-1/p}b_{ni}X_{ni}$ $(1 \leq i \leq k_n, n \geq 1)$. Taking $\delta > 0$ such that $\gamma - \delta \geq q$ and $\gamma - \delta > 0$, we get by the stochastic domination hypothesis and (3.8) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^t \sum_{i=1}^{k_n} P\left(\left|\frac{b_{ni}X_{ni}}{n^{1/p}}\right| > \varepsilon\right) \\
&\leq C\sum_{n=1}^{\infty} n^t \sum_{i=1}^{\infty} E\left|\frac{b_{ni}X_{ni}}{n^{1/p}}\right|^{\gamma-\delta} I\left(\left|\frac{b_{ni}X_{ni}}{n^{1/p}}\right| > \varepsilon\right) \\
&\leq C\sum_{n=1}^{\infty} n^t \sum_{i=1}^{\infty} E\left|\frac{b_{ni}X_{ni}}{n^{1/p}}\right|^{\gamma-\delta} I(|X_{ni}| > \varepsilon n^{1/p}) \\
&\leq C\sum_{n=1}^{\infty} n^{-1+\delta/p} E|X|^{\gamma-\delta} I(|X| > \varepsilon n^{1/p}) \\
&= C\sum_{n=1}^{\infty} n^{-1+\delta/p} \sum_{j=n}^{\infty} E|X|^{\gamma-\delta} I(\varepsilon j^{1/p} < |X| \leq \varepsilon(j+1)^{1/p}) \\
&= C\sum_{j=1}^{\infty} E|X|^{\gamma-\delta} I(\varepsilon j^{1/p} < |X| \leq \varepsilon(j+1)^{1/p}) \sum_{n=1}^{j} n^{-1+\delta/p} \\
&\leq C\sum_{j=1}^{\infty} j^{\delta/p} E|X|^{\gamma-\delta} I(\varepsilon j^{1/p} < |X| \leq \varepsilon(j+1)^{1/p}) \\
&\leq CE|X|^{\gamma} < \infty.
\end{aligned}$$

Thus condition (i) of Corollary 1 holds.

Taking $q_1 \geq 2$, we also have by the stochastic domination hypothesis and (3.8) and $t > -1$ that

$$\sum_{n=1}^{\infty} n^t \left((\log n)^{\gamma} \sum_{i=1}^{k_n} E\left|\frac{b_{ni}X_{ni}}{n^{1/p}}\right|^{\gamma}\right)^{q_1} \leq C\sum_{n=1}^{\infty} n^t \left(E|X|^{\gamma}(\log n)^{\gamma} \sum_{i=1}^{k_n} \left|\frac{b_{ni}}{n^{1/p}}\right|^{\gamma}\right)^{q_1}$$

$$\begin{aligned}&\le C\sum_{n=1}^{\infty} n^t\Big((\log n)^{\gamma} n^{-(t+1)}\Big)^{q_1}\\ &\le C\sum_{n=1}^{\infty} n^{-(t+1)q_1+t}(\log n)^{\gamma q_1}<\infty.\end{aligned}$$

Thus condition (ii)' of Corollary 1 holds.

We have by the stochastic domination hypothesis, (3.6), (3.8) and $t>-1$ that

$$\begin{aligned}&\log n\sum_{i=1}^{k_n} E\left|\frac{b_{ni}X_{ni}}{n^{1/p}}\right|^2 I\left(\left|\frac{b_{ni}X_{ni}}{n^{1/p}}\right|\le\frac{1}{\log n}\right)\\ \le&\begin{cases}(\log n)^{-1+\gamma}n^{-\gamma/p}\sum_{i=1}^{k_n}E\left|b_{ni}X_{ni}\right|^{\gamma} & 1\le\gamma<2\\ n^{-2/p}\log n\sum_{i=1}^{k_n}E\left|b_{ni}X_{ni}\right|^2 & \gamma\ge 2\end{cases}\\ \le&\begin{cases}Cn^{-(t+1)}(\log n)^{-1+\gamma} & 1\le\gamma<2\\ Cn^{-2/p}\log n\sum_{i=1}^{\infty}b_{ni}^2 & \gamma\ge 2\end{cases}\to 0 \text{ as } n\to\infty.\end{aligned}$$

Therefore condition (iii) of Corollary 1 holds.

By $EX_{ni}=0,\gamma\ge 1,t>-1$, the stochastic domination hypothesis, and (3.8), we obtain

$$\begin{aligned}&\left|\sum_{i=1}^{k_n}E\frac{b_{ni}X_{ni}}{n^{1/p}}I\left(\left|\frac{b_{ni}X_{ni}}{n^{1/p}}\right|\le 1/\log n\right)\right|\\ \le&\sum_{i=1}^{k_n}E\left|\frac{b_{ni}X_{ni}}{n^{1/p}}\right|I\left(\left|\frac{b_{ni}X_{ni}}{n^{1/p}}\right|>1/\log n\right)\\ \le&(\log n)^{\gamma-1}n^{-\gamma/p}\sum_{i=1}^{k_n}E\left|b_{ni}X_{ni}\right|^{\gamma}\\ \le&C(\log n)^{\gamma-1}n^{-(t+1)}\to 0\text{ as } n\to\infty.\end{aligned}$$

Thus condition (iv) of Corollary 1 holds. Condition (v) of Corollary 1 holds if we take $q_2=t+2$. Therefore all conditions of Corollary 1 are satisfied and so (3.7) holds from Corollary 1. □

Corollary 4. *Let $\{X_{ni},i\ge 1,n\ge 1\}$ be an array of rowwise zero-mean END random variables which are stochastically dominated by a random variable X satisfying $E|X|^{\gamma}\log|X|<\infty$, where $\gamma=p(t+\beta+1)\ge 1$ and $t>-1$ and $p>0$. Let $\{b_{ni},i\ge 1,n\ge 1\}$ be an array of real numbers satisfying* (1.4). *Assume that $\sum_{i=1}^{\infty}b_{ni}X_{ni}$ is finite a.s. for any $n\ge 1$.*

(i) *If $1\le\gamma<2$, then* (1.2) *holds.*

(ii) *If $\gamma\ge 2$, and $\{b_{ni},i\ge 1,n\ge 1\}$ satisfies* (3.6), *then* (1.2) *holds.*

Proof. The proof is similar to that of Corollary 3 and is omitted. □

Remark 2. 1) If $0 < q \le 1$ in Corollary 3, then $\sum_{i=1}^{\infty} b_{ni}X_{ni}$ is finite a.s. for $\forall n \ge 1$.

2) For $t > -1$, Corollary 3 and Corollary 4 extends Theorem A and Theorem B from sequence of independent random variables on arrays of rowwise END random variables respectively. Moreover, condition (3.6) in Corollary 3 and Corollary 4 is weaker than the condition (1.3) of Theorem A and Theorem B when $\gamma \ge 2$.

3) Let $b_{ni} = a_{ni}n^{\mu}, \mu = 1/p$, thus, from (1.5) and (1.6), we obtain $\sup_{n,i} |b_{ni}| < \infty, \sum_{i=1}^{\infty} |b_{ni}| = O(n^{\mu+\tau})$. Let $\beta = \mu+\tau$, then $1+(1+\tau+t)/\mu = p(t+\beta+1) > 1, \sum_{i=1}^{\infty} b_{ni}^2 = O(n^{\mu+\tau}) = O(n^{\tau+1/p}), 0 \le \tau < 1/p$. Therefore all the conditions of Corollary 3 and Corollary 4 are satisfied. So Corollary 3 and Corollary 4 extend and improve Theorem C when $t > -1$.

Corollary 5. *Let* $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ *be an array of rowwise zero-mean END random variables which are stochastically dominated by a random variable* X *satisfying* $E|X|^{2p} < \infty$ *for some* $p \ge 1$. *Let* $\{c_{ni}, 1 \le i \le n, n \ge 1\}$ *be an array of constants satisfying*

$$\max_{1\le i\le n} |c_{ni}| = O\left(\frac{1}{n^{1/p}}\right) \tag{3.9}$$

and

$$\sum_{i=1}^{n} c_{ni}^2 = o\left(\frac{1}{\log n}\right). \tag{3.10}$$

Then

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} c_{ni}X_{ni}\right| > \varepsilon\right) < \infty \textit{ for any } \varepsilon > 0.$$

Proof. We apply Corollary 3 with $t = 0, \beta = 1, q = p$, and for $n \ge 1$

$$b_{ni} = \begin{cases} c_{ni}n^{1/p} & 1 \le i \le n \\ 0 & i > n. \end{cases}$$

By (3.9) and (3.10), we obtain

$$\sup_{n,i} |b_{ni}| < \infty, \sum_{i=1}^{\infty} |b_{ni}|^p = O(n),$$

and (3.6) holds. Therefore all conditions of Corollary 3 are satisfied and Corollary 5 follows from Corollary 3. □

Remark 3. Corollary 5 extends Theorem 4.1.3. of Stout ([13]) on sequence of independent random variables to arrays of rowwise END random variables. Furthermore, Corollary 5 generalizes and improves Theorem 2.1 of Zarei and Jabbari ([20]), and extends the result of Taylor et al. ([18]).

Corollary 6. *Let* $\{X_{ni}, i \geq 1, n \geq 1\}$ *be an array of rowwise zero-mean END random variables which are stochastically dominated by a random variable* X *satisfying* $E|X|^p < \infty$ *for some* $p > 2$*. Let* $\{b_{ni}, i \geq 1, n \geq 1\}$ *be an array of constants satisfying* (3.6) *and*

$$\sum_{i=1}^{\infty} |b_{ni}|^q = O(1) \text{ for some } 2 \leq q < p.$$

Assume that $\sum_{i=1}^{\infty} b_{ni} X_{ni}$ *is finite a.s. for any* $n \geq 1$*. Then*

$$\sum_{n=1}^{\infty} P\left(n^{-1/p} \left| \sum_{i=1}^{\infty} b_{ni} X_{ni} \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0.$$

Proof. Let $t = 0$ and $\beta = 0$. Clearly $\sup_{n,i} |b_{ni}| < \infty$. Thus the result follows from Corollary 3(ii). ☐

Corollary 7. *Let* $\{X_{ni}, i \geq 1, n \geq 1\}$ *be an array of rowwise zero-mean END random variables which are stochastically dominated by a random variable* X *satisfying* $E|X|^2 \log |X| < \infty$*. Let* $\{b_{ni}, i \geq 1, n \geq 1\}$ *be an array of constants satisfying*

$$\sum_{i=1}^{\infty} b_{ni}^2 = O(1).$$

Assume that $\sum_{i=1}^{\infty} b_{ni} X_{ni}$ *is finite a.s. for any* $n \geq 1$*. Then*

$$\sum_{n=1}^{\infty} P\left(n^{-1/2} \left| \sum_{i=1}^{\infty} b_{ni} X_{ni} \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0.$$

Proof. Let $t = 0, \beta = 0$, and $p = 2$. Clearly $\sup_{n,i} |b_{ni}| < \infty$. Thus the result follows from Corollary 4(ii). ☐

Remark 4. Corollary 6 and Corollary 7 extend Corollary 1 and Corollary 2 of Sung ([14]) for sequence of independent random variables to arrays of rowwise END random variables respectively. Moreover, (3.6) in Corollary 6 of this paper is weaker than the condition $\sum_{i=1}^{\infty} b_{ni}^2 = O(n^\alpha)$ for some $\alpha < 2/p$ from Corollary 1 of Sung ([14]).

Corollary 8. *Let* $\{X_n, -\infty < n < \infty\}$ *be a sequence of zero-mean END random variables which are stochastically dominated by a random variable* X *satisfying* $E|X|^{p(t+2)} < \infty$ *for some* $0 < p < 2, p(t+2) > 1,$ *and* $t > -1$*. Let* $\{a_n, -\infty < n < \infty\}$ *be a sequence of real numbers such that* $\sum_{n=-\infty}^{\infty} |a_n| < \infty$*. Set* $a_{ni} = \sum_{j=i+1}^{i+n} a_j$ *for each* i *and* n*. Then*

$$\sum_{n=1}^{\infty} n^t P\left(n^{-1/p} \left| \sum_{i=-\infty}^{\infty} a_{ni} X_i \right| > \epsilon \right) < \infty \text{ for all } \epsilon > 0.$$

Proof. The proof is similar to that of Corollary 3 of Sung ([14]) and is omitted. ☐

Remark 5. Corollary 8 extends Corollary 3 of Sung ([14]) for independent random variables to arrays of rowwise END random variables when $t > -1$.

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