Lithuanian Mathematical Journal, Vol. 52, No. 3, July, 2012, pp. 316–325

COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF ARRAYS OF BANACH-SPACE-VALUED RANDOM ELEMENTS*

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Received September 27, 2010; revised June 25, 2012

Abstract. We study the complete convergence for weighted sums of arrays of Banach-space-valued random elements and obtain some new results that extend and improve the related known works in the literature.

MSC: 60B12, 60F05, 60F25, 60G42

Keywords: array of Banach-space-valued random elements, weighted sums, complete convergence

1 INTRODUCTION

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [8] as follows: A sequence of random variables $\{U_n,\, n \in \mathbb{N}\}$ (where $\mathbb{N}$ is the set of positive integers) is said to converge completely to a constant C if $\sum_{n=1}^{\infty} \mathbf{P}(|U_n - C| > \epsilon) < \infty$ for all $\epsilon > 0$. In view of the Borel–Cantelli lemma, this implies that $U_n \to C$ almost surely (a.s.). The converse is true if the random variables $\{U_n,\, n \in \mathbb{N}\}$ are independent.

The way of measuring the rate of convergence considered in our paper originates from the results of [4, 7] and [12].

Theorem A. (*See* [4, 7].) *If* $\{X_i,\, i \geqslant 1\}$ *is a sequence of independent identically distributed random variables and* $\theta \geqslant 1$*, then the following two statements are equivalent*:

(a) $\mathbf{E}|X_1|^{\theta} < \infty$ *and* $\mathbf{E}(X_1) = 0$,

(b) $\sum_{n=1}^{\infty} n^{\theta-2}\mathbf{P}(|\sum_{i=1}^{n} X_i| > \epsilon n) < \infty$ *for all* $\epsilon > 0$.

* The research of T.-C. Hu is partially supported by the National Science Council, grant NSC 99-218-M-007-001-my2. The research of M. Ordóñez Cabrera is partially supported by the Plan Andaluz de Investigacion de la Junta de Andalucia FQM-127 and grant P08-FQM-03543, and by MEC grant MTM2009-10696-C02-01. The research of A. Volodin is partially supported by the National Science and Engineering Research Council of Canada.

This result was extended to Banach space setting by Norvaiša [12] as follows.

Theorem B. (*See* [12].) *If* $\{X_i,\, i \geqslant 1\}$ *is a sequence of independent identically distributed random elements taking values in a real separable Banach space* $(B, \|\cdot\|)$*, number* $\theta \geqslant 1$, $\mathbf{E}\|X_1\|^\theta < \infty$*, and* $\mathbf{E}(X_1) = 0$*. The following two statements are equivalent*:

(a) $\sum_{n=1}^{\infty} n^{\theta-2}\mathbf{P}(\|\sum_{i=1}^{n} X_i\| > \epsilon n) < \infty$ *for all* $\epsilon > 0$*,*
(b) $\lim_{n\to\infty} \mathbf{E}\|\sum_{i=1}^{n} X_i\|/n = 0$.

Also, a characterization of statement (b) in terms of probabilistic geometry of the Banach space B is provided by Norvaiša [12]. Many other authors have devoted their study to complete convergence (see [2, 3, 5, 9, 10, 13, 14, 15]).

In the following, we assume that $\{X_{nj},\, j \in \mathbb{N},\, n \in \mathbb{N}\}$ is an array of random elements in a separable real Banach space $(B, \|\cdot\|)$ and $\{a_{nj},\, j \in \mathbb{N},\, n \in \mathbb{N}\}$ is an array of real constants. Denote

$$S_n \equiv \sum_{j=1}^{\infty} a_{nj} X_{nj}.$$

In the following, we assume that the series S_n converges almost surely if the almost sure convergence does not automatically follow from the hypotheses.

Hu et al. [9] obtained the following result.

Theorem C. (*See* [9].) *Let* $\{X_{nj},\, j \in \mathbb{N},\, n \in \mathbb{N}\}$ *be an array of rowwise independent random elements stochastically dominated by a random variable* X (*the technical definitions are given in the next section*). *Assume that*

$$\sup_{j\geqslant 1} |a_{nj}| = O\bigl(n^{-\gamma}\bigr) \quad \textit{for some } \gamma > 0 \tag{1.1}$$

and

$$\sum_{j=1}^{\infty} |a_{nj}| = O\bigl(n^{\alpha}\bigr) \quad \textit{for some } \alpha < \gamma.$$

If

$$\mathbf{E}|X|^{1+(1+\alpha+\beta)/\gamma} < \infty \quad \textit{for some } \beta \in (-1, \gamma-\alpha-1]$$

and

$$S_n \xrightarrow{\mathbf{P}} 0,$$

then

$$\sum_{n=1}^{\infty} n^{\beta}\mathbf{P}\bigl(\|S_n\| > \epsilon\bigr) < \infty \quad \textit{for all } \epsilon > 0.$$

The proof of Theorem C in [9] is rather complicated since it uses the Stieltjes integral techniques, summation by parts lemma, and so on. When $\alpha + \beta > -1$, Ahmed et al. [2] established a more general result and with simpler proof than that of Hu et al. [9]. Volodin et al. [15] generalized the result of Ahmed et al. [2]; meanwhile, they studied the special case $\alpha + \beta = -1$ and obtained the following Theorem D. Sung et al. [14] and Chen et al. [5] studied the case of $\beta = -1$ and $\alpha > 0$, and Chen et al. [13] improved the result of Sung et al. [14]. Qiu [13] improved and generalized the corresponding results of Volodin et al. [15] and Chen et al. [5] in the case of $\alpha + \beta > -1$.

However, they did not study the relatively important special case $\alpha + \beta = -1$ (except Volodin et al. [15]). Baek et al. [3] established some results for arrays of rowwise negatively dependent random variables that complement the results of Ahmed et al. [2] in the case of real random variables (and not for random elements in Banach spaces). The results of Baek et al. [3] are in the same spirit as those established by Volodin et al. [15] for weighted sums of arrays of Banach-space-valued random elements.

Theorem D. (*See* [15].) *Let $\{X_{nj},\ j\in\mathbb{N},\ n\in\mathbb{N}\}$ be an array of rowwise independent random elements stochastically dominated by a random variable X. Assume that* (1.1) *holds and*

$$\sum_{j=1}^{\infty}|a_{nj}|^{\theta}=O\bigl(n^{\alpha}\bigr)\quad \text{for some } 0<\theta\leqslant 2 \text{ and any } \alpha \text{ such that } \theta+\frac{\alpha}{\gamma}<2.$$

Let $\beta=-1-\alpha$ and fix $\delta>\theta$ such that $\theta+\alpha/\gamma<\delta\leqslant 2$. If

$$\mathbf{E}|X|^{\delta}<\infty \quad \text{and} \quad S_n\xrightarrow{\mathbf{P}}0,$$

then

$$\sum_{n=1}^{\infty}n^{\beta}\mathbf{P}\bigl(\|S_n\|>\epsilon\bigr)<\infty\quad \text{for all } \epsilon>0.$$

We assume in Theorem D that the series S_n converges a.s. when $\theta>1$, since the a.s. convergence does not automatically follow from the hypotheses. In this paper, we assume without explicit mention that each series S_n converges a.s. if the almost sure convergence does not automatically follow from the hypotheses. Note also that if $\beta<-1$, then the conclusions of Theorems C and D, as well as the results of the present article, hold automatically, and hence, they are of interest only for $\beta\geqslant -1$. If $\beta\geqslant-1$, then $\beta=-1-\alpha$ implies that $\alpha\leqslant 0$.

In this paper, we improve Theorem D in three directions, namely:

(i) The moment condition in our results is strictly weaker than in Theorem D.
(ii) When $0<\theta<1$, the assumptions of rowwise independence of $\{X_{nj},\ j\in\mathbb{N},\ n\in\mathbb{N}\}$ and $S_n\xrightarrow{\mathbf{P}}0$ in Theorem D are removed.
(iii) In Theorem 2, we deal with the case $\theta>2$.

2 PRELIMINARIES

Let $\{\Omega,\mathcal{F},\mathbf{P}\}$ be a probability space, and let B be a separable real Banach space with norm $\|\cdot\|$. A random element is defined to be an $\mathcal{F}$-measurable mapping of Ω into B equipped with the Borel σ-algebra (that is, the σ-algebra generated by the open sets determined by $\|\cdot\|$). The expected value of a B-valued random element X is defined to be the Bochner integral and denoted by $\mathbf{E}X$.

Let $\{X_{nj},\ j\in\mathbb{N},\ n\in\mathbb{N}\}$ be an array of random elements (not necessarily rowwise independent and identically distributed) taking values in B. The array of random elements $\{X_{nj},\ j\in\mathbb{N},\ n\in\mathbb{N}\}$ is said to be stochastically dominated by a random variable X if there exists a constant D such that

$$\sup_{j\in\mathbb{N},\,n\in\mathbb{N}}\mathbf{P}\bigl(\|X_{nj}\|>x\bigr)\leqslant D\mathbf{P}\bigl(|X|>x\bigr)\quad \text{for all } x>0.$$

In this case, we write $\{X_{nj},\ j\in\mathbb{N},\ n\in\mathbb{N}\}\prec X$. Let $\{a_{nj},\ j\in\mathbb{N},\ n\in\mathbb{N}\}$ be an array of real constants (called weights). Consider the sequence of weighted sums $S_n\equiv\sum_{j=1}^{\infty}a_{nj}X_{nj}$, $n\in\mathbb{N}$.

Let $1\leqslant p\leqslant 2$, and let $\{\theta_n,\ n\in\mathbb{N}\}$ be independent and identically distributed stable random variables, each with characteristic function $\phi(t)=\exp(-|t|^p)$, $-\infty<t<\infty$. The separable real Banach space B is said to be of stable type p if $\sum_{n=1}^{\infty}\theta_n v_n$ converges almost surely whenever $\{v_n,\ n\in\mathbb{N}\}\subseteq B$ with $\sum_{n=1}^{\infty}\|v_n\|^p<\infty$. Equivalent characterizations of a Banach space being of stable type p, properties of stable type p Banach spaces, and various relationships between the conditions "Rademacher type p" and "stable type p" can be found in Adler et al. [1].

Next, we present some lemmas that will be used to prove our main results.

Lemma 1. (*See* [6].) *For every $p \geqslant 2$, there exists a positive constant C_p depending only on p such that, for any sequence $\{X_n,\, n \in \mathbb{N}\}$ of independent B-valued random elements with $X_n \in L^p$, $n \in \mathbb{N}$, the following inequality holds*:

$$\mathbf{E}\left|\left\|\sum_{j=1}^{n} X_j\right\| - \mathbf{E}\left\|\sum_{j=1}^{n} X_j\right\|\right|^p \leqslant C_p\left\{\left(\sum_{j=1}^{n}\mathbf{E}\|X_j\|^2\right)^{p/2} + \sum_{j=1}^{n}\mathbf{E}\|X_j\|^p\right\}.$$

The next lemma is well known, and its proof is left as an easy exercise for the interested reader.

Lemma 2. *Let $\{X_{nj},\, j \in \mathbb{N},\, n \in \mathbb{N}\}$ be an array of random variables with $\{X_{nj},\, j \in \mathbb{N},\, n \in \mathbb{N}\} \prec X$. Then there exists a constant C such that, for all $q > 0$ and $x > 0$,*

(i) $\mathbf{E}\|X_{nj}\|^q I(\|X_{nj}\| \leqslant x) \leqslant C\{\mathbf{E}|X|^q I(|X| \leqslant x) + x^q \mathbf{P}(|X| > x)\}$,
(ii) $\mathbf{E}\|X_{nj}\|^q I(\|X_{nj}\| > x) \leqslant C\mathbf{E}|X|^q I(|X| > x)$.

Lemma 3. (*See* [11, *Lemma* 6.5].) *Let $\{X_n,\, n \in \mathbb{N}\}$ be a sequence of symmetric B-valued random elements. Let $\{\xi_n,\, n \in \mathbb{N}\}$ and $\{\zeta_n,\, n \geqslant 1\}$ be real random variables such that $\xi_n = \phi_n(X_n)$, where $\phi_n : B \to \mathbb{R}$ are symmetric* (*even*), *and similarly for ζ_n. If $|\xi_n| \leqslant |\zeta_n|$ a.s. for every n, then*

$$\mathbf{P}\left(\left\|\sum_n \xi_n X_n\right\| > x\right) \leqslant 2\mathbf{P}\left(\left\|\sum_n \zeta_n X_n\right\| > x\right) \quad \textit{for all } x > 0.$$

In particular, this inequality applies to the case where $\xi_n = I(X_n \in A_n) \leqslant 1 \equiv \zeta_n$ with the sets A_n symmetric in B (for example, $A_n = \{\|X_n\| \leqslant a_n\}$).

Lemma 4. (*See* [10].) *Let $\{X_{nj},\, 1 \leqslant j \leqslant k_n,\, n \in \mathbb{N}\}$ be an array of rowwise independent symmetric random elements. Suppose that there exists $\delta > 0$ such that $\|X_{nj}\| \leqslant \delta$ a.s. for all $1 \leqslant j \leqslant k_n$, $n \in \mathbb{N}$. If $\sum_{j=1}^{k_n} X_{nj} \xrightarrow{\mathbf{P}} 0$, then $\mathbf{E}\|\sum_{j=1}^{k_n} X_{nj}\| \to 0$ as $n \to \infty$.*

Lemma 5. (*See* [1].) *Let $\{X_{nj},\, j \in \mathbb{N},\, n \in \mathbb{N}\}$ be an array of rowwise independent mean-zero random elements in a stable type p* (1 < p < 2) *Banach space B. Suppose that $\{X_{nj},\, j \in \mathbb{N},\, n \in \mathbb{N}\} \prec X$. Moreover, assume that*

$$\sup_{n\geqslant 1}\sum_{j=1}^{\infty}|a_{nj}|^p < \infty \quad \textit{and} \quad \sup_{j\geqslant 1}|a_{nj}| = o(1).$$

If $\lim_{t\to\infty} t^p \mathbf{P}(|X| > t) = 0$, then $S_n \xrightarrow{\mathbf{P}} 0$.

Throughout this paper, C always stands for a positive constant which may differ from one place to another, the symbol $[x]$ denotes the greatest integer less than or equal to x, and the symbol $\sharp A$ denotes the number of elements of a finite set A.

3 MAIN RESULTS AND PROOFS

With the preliminary results accounted for, we can formulate and prove the main results of this paper.

Theorem 1. *Let $\{X_{nj},\, j \in \mathbb{N},\, n \in \mathbb{N}\}$ be an array of random elements with $\{X_{nj},\, j \in \mathbb{N},\, n \in \mathbb{N}\} \prec X$. Assume that* (1.1) *holds and*

$$\sum_{j=1}^{\infty}|a_{nj}|^{\theta} = O\big(n^{\alpha}\big) \quad \textit{for some } 0 < \theta < 1 \textit{ and some } \alpha. \tag{3.1}$$

Let $\beta = -1 - \alpha$. *If*

$$\mathbf{E}\big(|X|^\theta \log\big(1 + |X|\big)\big) < \infty,$$

then

$$\sum_{n=1}^{\infty} n^\beta \mathbf{P}\big(\|S_n\| > \epsilon\big) < \infty \quad \textit{for all } \epsilon > 0. \tag{3.2}$$

Proof. From (1.1) and (3.1), without loss of generality, we can assume that

$$\sup_{j\geqslant 1} |a_{nj}| = n^{-\gamma}, \tag{3.3}$$

$$\sum_{j=1}^{\infty} |a_{nj}|^\theta = n^\alpha. \tag{3.4}$$

Let $Y_{nj} = a_{nj}X_{nj}I(\|a_{nj}X_{nj}\| \leqslant 1)$, $j \in \mathbb{N}$, $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} n^\beta \mathbf{P}\big(\|S_n\| > \epsilon\big) \leqslant \sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{\infty} \mathbf{P}\big(\|a_{nk}X_{nk}\| > 1\big) + \sum_{n=1}^{\infty} n^\beta \mathbf{P}\left(\left\|\sum_{k=1}^{\infty} Y_{nk}\right\| > \epsilon\right) := I_1 + I_2.$$

Therefore, in order to prove (3.2), it suffices to show that $I_1 < \infty$ and $I_2 < \infty$. Since $\alpha + \beta = -1$ and $\theta > 0$, by Lemma 2, (3.3), and (3.4) we have

$$\begin{aligned}
I_1 &\leqslant \sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{\infty} \mathbf{E}\|a_{nk}X_{nk}\|^\theta I\big(\|a_{nk}X_{nk}\| > 1\big) \leqslant C\sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{\infty} \mathbf{E}|a_{nk}X|^\theta I\big(|X| > |a_{nk}|^{-1}\big) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1}\mathbf{E}|X|^\theta I\big(|X| > n^\gamma\big) = C\sum_{n=1}^{\infty} n^{-1} \sum_{j=n}^{\infty} \mathbf{E}|X|^\theta I\big(j^\gamma < |X| \leqslant (j+1)^\gamma\big) \\
&= C\sum_{j=1}^{\infty} \mathbf{E}|X|^\theta I\big(j^\gamma < |X| \leqslant (j+1)^\gamma\big) \sum_{n=1}^{j} n^{-1} \leqslant C\sum_{j=1}^{\infty} \log j\mathbf{E}|X|^\theta I\big(j^\gamma < |X| \leqslant (j+1)^\gamma\big) \\
&\leqslant C\mathbf{E}\big(|X|^\theta \log\big(1+|X|\big)\big) < \infty.
\end{aligned} \tag{3.5}$$

Let $I_{nk} = \{i\colon (nk)^\gamma \leqslant |a_{ni}|^{-1} < (n(k+1))^\gamma\}$, $k \in \mathbb{N}$, $n \in \mathbb{N}$. Then $\bigcup_{k=1}^{\infty} I_{nk} = \mathbb{N}$ for all $n \in \mathbb{N}$. Choose t such that $\theta < t < 1$. By the Markov inequality, Lemma 2, and (3.5) we have

$$\begin{aligned}
I_2 &\leqslant C\sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{\infty} \mathbf{E}\|Y_{nk}\|^t \leqslant C\sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{\infty} \big\{\mathbf{E}|a_{nk}X|^t I\big(|X| \leqslant |a_{nk}|^{-1}\big) + \mathbf{P}\big(|X| > |a_{nk}|^{-1}\big)\big\} \\
&\leqslant C + C\sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{\infty} (\sharp I_{nk})(nk)^{-\gamma t}\mathbf{E}|X|^t I\big(|X| < \big(n(k+1)\big)^\gamma\big) \\
&\leqslant C + C\sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{\infty} (\sharp I_{nk})(nk)^{-\gamma t} \sum_{i=1}^{n(k+1)} \mathbf{E}|X|^t I\big((i-1)^\gamma \leqslant |X| < i^\gamma\big)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C + C\sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} (\sharp I_{nk})(nk)^{-\gamma t} \sum_{i=1}^{2n} \mathbf{E}|X|^t I\big((i-1)^\gamma \leqslant |X| < i^\gamma\big) \\
&\quad + C\sum_{n=1}^{\infty} n^{\beta} \sum_{k=2}^{\infty} (\sharp I_{nk})(nk)^{-\gamma t} \sum_{i=2n+1}^{n(k+1)} \mathbf{E}|X|^t I\big((i-1)^\gamma \leqslant |X| < i^\gamma\big) \\
&:= C + I_3 + I_4.
\end{aligned} \tag{3.6}
$$

Since $t > \theta$ and $\gamma > 0$, we have $k^{\gamma(t-\theta)} > j^{\gamma(t-\theta)}$ for all $k > j$, where $j, k \in \mathbb{N}$. By (3.4) we have

$$
\begin{aligned}
n^{\alpha} &= \sum_{i=1}^{\infty} |a_{ni}|^\theta = \sum_{k=1}^{\infty} \sum_{i \in I_{nk}} |a_{ni}|^\theta \geqslant \sum_{k=1}^{\infty} (\sharp I_{nk})\big(n(k+1)\big)^{-\gamma\theta} \\
&\geqslant \sum_{k=j}^{\infty} (\sharp I_{nk})\big(n(k+1)\big)^{-\gamma t}\big(n(j+1)\big)^{\gamma(t-\theta)} > 2^{-\gamma t} \sum_{k=j}^{\infty} (\sharp I_{nk})(nk)^{-\gamma t}(nj)^{\gamma(t-\theta)}.
\end{aligned}
$$

Hence,

$$
\sum_{k=j}^{\infty} (\sharp I_{nk})(nk)^{-\gamma t} \leqslant C n^{\alpha-\gamma(t-\theta)} j^{-\gamma(t-\theta)} \quad \text{for all } j \in \mathbb{N}. \tag{3.7}
$$

By (3.7) we can get that

$$
\begin{aligned}
I_3 &\leqslant C\sum_{n=1}^{\infty} n^{\beta} n^{\alpha-\gamma(t-\theta)} \sum_{i=1}^{2n} \mathbf{E}|X|^t I\big((i-1)^\gamma \leqslant |X| < i^\gamma\big) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1-\gamma(t-\theta)} + C\sum_{n=1}^{\infty} n^{-1-\gamma(t-\theta)} \sum_{i=2}^{2n} \mathbf{E}|X|^t I\big((i-1)^\gamma \leqslant |X| < i^\gamma\big) \\
&\leqslant C + C\sum_{i=2}^{\infty} \mathbf{E}|X|^t I\big((i-1)^\gamma \leqslant |X| < i^\gamma\big) \sum_{n=[i/2]}^{\infty} n^{-1-\gamma(t-\theta)} \\
&\leqslant C + C\sum_{i=2}^{\infty} i^{-\gamma(t-\theta)} \mathbf{E}|X|^t I\big((i-1)^\gamma \leqslant |X| < i^\gamma\big) \\
&\leqslant C + C\sum_{i=2}^{\infty} i^{\gamma\theta} \mathbf{E} I\big((i-1)^\gamma \leqslant |X| < i^\gamma\big) \leqslant C + C\mathbf{E}|X|^\theta < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &\leqslant C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=2n+1}^{\infty} \sum_{k=[i/n-1]}^{\infty} (\sharp I_{nk})(nk)^{-\gamma t} \mathbf{E}|X|^t I\big((i-1)^\gamma \leqslant |X| < i^\gamma\big) \\
&\leqslant C\sum_{n=1}^{\infty} n^{\beta} \sum_{i=2n+1}^{\infty} n^{\alpha-\gamma(t-\theta)} \left(\frac{i}{n}\right)^{-\gamma(t-\theta)} \mathbf{E}|X|^t I\big((i-1)^\gamma \leqslant |X| < i^\gamma\big)
\end{aligned}
$$

$$\leqslant C\sum_{i=2}^{\infty} i^{-\gamma(t-\theta)}\mathbf{E}|X|^t I\big((i-1)^\gamma \leqslant |X| < i^\gamma\big)\sum_{n=1}^{[i/2]} n^{-1}$$
$$\leqslant C\sum_{i=2}^{\infty} i^{-\gamma(t-\theta)}\log i\mathbf{E}|X|^t I\big((i-1)^\gamma \leqslant |X| < i^\gamma\big) \leqslant C\mathbf{E}\big(|X|^\theta \log\big(1+|X|\big)\big) < \infty.$$

Therefore, (3.2) holds. □

Remark 1. If we compare Theorem 1 with Theorem D in the case $0 < \theta \leq 1$, then we see that neither the assumption of rowwise independence of $\{X_{nj},\ j \in \mathbb{N},\ n \in \mathbb{N}\}$ nor $S_n \xrightarrow{\mathbf{P}} 0$ is required. In addition, the moment condition in Theorem 1 is strictly weaker than in Theorem D.

Theorem 2. *Let* $\{X_{nj},\ j \in \mathbb{N},\ n \in \mathbb{N}\}$ *be an array of rowwise independent random elements with* $\{X_{nj},\ j \in \mathbb{N},\ n \in \mathbb{N}\} \prec X$. *Assume that* (1.1) *holds and*

$$\sum_{j=1}^{\infty} |a_{nj}|^\theta = O\big(n^\alpha\big) \quad \textit{for some } \theta \geqslant 1 \textit{ and some } \alpha.$$

Moreover, assume that $\sum_{j=1}^{\infty} a_{nj}^2 = O(n^\eta)$ *for some* $\eta < 0$ *when* $\theta \geqslant 2$. *Let* $\beta = -1-\alpha$. *If*

$$\mathbf{E}\big(|X|^\theta \log\big(1+|X|\big)\big) < \infty \quad \textit{and} \quad S_n \xrightarrow{\mathbf{P}} 0,$$

then (3.2) *holds.*

Proof. Since $S_n \xrightarrow{\mathbf{P}} 0$, by the standard argument we may assume that random variables $\{X_{nj},\ j \in \mathbb{N},\ n \in \mathbb{N}\}$ are symmetric. The assumption of a.s. convergence of $\sum_{j=1}^{\infty} a_{nj}X_{nj}$ for every n implies that there exists a positive integer k_n such that

$$\mathbf{P}\left(\left\|\sum_{j=k_n+1}^{\infty} a_{nj}X_{nj}\right\| > \frac{\epsilon}{2}\right) < \frac{1}{n^{2+\beta}} \quad \text{for all } n \geqslant 1.$$

Therefore, in order to prove (3.2), we only need to prove that

$$\sum_{n=1}^{\infty} n^\beta \mathbf{P}\left(\left\|\sum_{j=1}^{k_n} a_{nj}X_{nj}\right\| > \frac{\epsilon}{2}\right) < \infty.$$

Let Y_{nj} be as in Theorem 1. Then

$$\sum_{n=1}^{\infty} n^\beta \mathbf{P}\left(\left\|\sum_{j=1}^{k_n} a_{nj}X_{nj}\right\| > \frac{\epsilon}{2}\right)$$
$$\leqslant \sum_{n=1}^{\infty} n^\beta \sum_{j=1}^{\infty} \mathbf{P}\big(\|a_{nj}X_{nj}\| > 1\big) + \sum_{n=1}^{\infty} n^\beta \mathbf{P}\left(\left\|\sum_{j=1}^{k_n} Y_{nj}\right\| > \frac{\epsilon}{2}\right) := I_5 + I_6.$$

Similarly to the proof of (3.5) in Theorem 1, we have $I_5 < \infty$. Therefore, in order to prove (3.2), we only need to prove that $I_6 < \infty$. Since $S_n \xrightarrow{\mathbf{P}} 0$, by Lemma 3 we can get that $\sum_{j=1}^{\infty} Y_{nj} \xrightarrow{\mathbf{P}} 0$. Hence, $\sum_{j=1}^{k_n} Y_{nj} \xrightarrow{\mathbf{P}} 0$.

Since $\|Y_{nj}\| \leqslant 1$ for all $j \in \mathbb{N}$, $n \in \mathbb{N}$, by Lemma 4 we have

$$\mathbf{E}\left\|\sum_{j=1}^{k_n} Y_{nj}\right\| \to 0.$$

Thus, in order to prove that $I_6 < \infty$, we only need to prove that

$$I_6^* = \sum_{n=1}^{\infty} n^{\beta}\mathbf{P}\left(\left|\left\|\sum_{j=1}^{k_n} Y_{nj}\right\| - \mathbf{E}\left\|\sum_{j=1}^{k_n} Y_{nj}\right\|\right| > \frac{\epsilon}{4}\right) < \infty.$$

Case 1: $1 \leqslant \theta < 2$. Letting $t = 2$ in (3.6) of Theorem 1, by Lemmas 1 and 2 we have

$$I_6^* \leqslant C\sum_{n=1}^{\infty} n^{\beta}\sum_{j=1}^{k_n}\mathbf{E}\|Y_{nj}\|^2 < \infty.$$

Thus, (3.2) holds.

Case 2: $\theta \geqslant 2$. Taking v such that $v > \max\{\theta, -2(1+\beta)/\eta\}$, by Lemma 1 we have

$$I_6^* \leqslant C\sum_{n=1}^{\infty} n^{\beta}\left\{\left(\sum_{j=1}^{k_n}\mathbf{E}\|Y_{nj}\|^2\right)^{v/2} + \sum_{j=1}^{k_n}\mathbf{E}\|Y_{nj}\|^{v}\right\} := I_7 + I_8.$$

By Lemma 2 we have

$$\begin{aligned} I_7 &\leqslant C\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{j=1}^{k_n}\mathbf{P}\big(|a_{nj}X| > 1\big) + \sum_{j=1}^{k_n}\mathbf{E}|a_{nj}X|^2 I\big(|a_{nj}X| \leqslant 1\big)\right)^{v/2} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{j=1}^{k_n}\mathbf{E}|a_{nj}X|^2\right)^{v/2} \leqslant C\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{j=1}^{\infty}|a_{nj}|^2\right)^{v/2} \leqslant C\sum_{n=1}^{\infty} n^{\beta+v\eta/2} < \infty. \end{aligned}$$

Similarly to the proof of $I_2 < \infty$ in Theorem 1, we have $I_8 < \infty$. Thus, (3.2) holds. □

Remark 2. (i) The moment condition in Theorem 2 is strictly weaker than in Theorem D for $1 \leqslant \theta < 2$.

(ii) If $\beta < -1$, then obviously $\sum_{n=1}^{\infty} n^{\beta}\mathbf{P}(\|S_n\| > \epsilon) < \infty$ for all $\epsilon > 0$. If $\beta \geqslant -1$, then $\beta = -1-\alpha$ implies that $\alpha \leqslant 0$, and thus, by the conditions $\theta = 2$ and $\theta + \alpha/\gamma < \delta \leqslant 2$ in Theorem D, we can get that $\alpha < 0$. Hence, we have $\sum_{j=1}^{\infty} a_{nj}^2 = O(n^{\alpha})$ for $\alpha < 0$. However, the case $\theta > 2$ is not considered in Theorem D.

Corollary 1. *Suppose that B is of stable type p for some $1 < p < 2$. Let $\{X_{nj},\, j \in \mathbb{N},\, n \in \mathbb{N}\}$ be an array of mean-zero rowwise independent random elements with $\{X_{nj},\, j \in \mathbb{N},\, n \in \mathbb{N}\} \prec X$. Assume that* (1.1) *holds and*

$$\sum_{j=1}^{\infty}|a_{nj}|^{\theta} = O\big(n^{\alpha}\big) \quad \text{for some } 1 < \theta \leqslant p \text{ and some } \alpha. \tag{3.8}$$

Let $\beta = -1-\alpha$. If

$$\mathbf{E}\big(|X|^{\theta}\log\big(1+|X|\big)\big) < \infty,$$

then (3.2) *holds.*

Proof. If $\beta < -1$, then (3.2) clearly holds, and hence, it is of interest only for $\beta \geqslant -1$. If $\beta \geqslant -1$, then $\beta = -1 - \alpha$ implies that $\alpha \leqslant 0$, and by (3.8) we can get that

$$\sup_{n\geqslant 1} \sum_{j=1}^{\infty} |a_{nj}|^{\theta} < \infty.$$

Since $\mathbf{E}(|X|^{\theta} \log(1 + |X|)) < \infty$, we have

$$\lim_{t\to\infty} t^{\theta} \mathbf{P}\big(|X| > t\big) = 0.$$

Therefore, in order to prove (3.2), by Theorem 2 we only need to check that $S_n \xrightarrow{\mathbf{P}} 0$. Since B is of stable type p for some $1 < p < 2$ and $\theta \leqslant p$, B is of stable type θ. By Lemma 5 the convergence in probability holds. $\square$

Remark 3. The moment condition in Corollary 1 is strictly weaker than in Theorem 3.3 of Volodin et al. [15].

Remark 4 and open problem. The authors believe that Theorems 1 and 2 can be further improved in the direction of relaxing the moment conditions. Namely, we guess that the assumption $\mathbf{E}(|X|^{\theta} \log(1+|X|)) < \infty$ can be weakened to $\mathbf{E}|X|^{\theta} < \infty$. Despite our efforts to solve this problem, it is still an *open problem*. We would also like to mention that this logarithmic term appears only in the somewhat peculiar case $\alpha + \beta = -1$.

Acknowledgment. The authors would like to thank the referee for helpful comments that helped to improve the presentation.

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