Available online at www.sciencedirect.com
SCIENCE DIRECT®
Statistics & Probability Letters 71 (2005) 193–202
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

On the almost sure growth rate of sums of lower negatively dependent nonnegative random variables

Oleg Klesov[a], Andrew Rosalsky[b,*], Andrei I. Volodin[c]

[a]Department of Mathematical Analysis and Probability Theory, National Technical University of Ukraine (KPI), Prospekt Peremogy 37, 02056 Kiev, Ukraine
[b]Department of Statistics, University of Florida, Gainesville, FL 32611, USA
[c]Department of Mathematics and Statistics, University of Regina, Regina, Sask., Canada S4S 0A2

Received 12 July 2004
Available online 20 November 2004

Abstract

For a sequence of lower negatively dependent nonnegative random variables $\{X_n, n \geqslant 1\}$, conditions are provided under which $\lim_{n\to\infty} \sum_{j=1}^{n} X_j / b_n = \infty$ almost surely where $\{b_n, n \geqslant 1\}$ is a nondecreasing sequence of positive constants. The results are new even when they are specialized to the case of nonnegative independent and identically distributed summands and $b_n = n^r$, $n \geqslant 1$ where $r > 0$.

MSC: primary 60F15

Keywords: Sums of lower negatively dependent random variables; Nonnegative random variables; Sums of independent and identically distributed random variables; Almost sure growth rate

1. Introduction

Throughout this paper, let $\{X_n, n \geqslant 1\}$ be a sequence of nonnegative random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. Their partial sums will be denoted, as usual, by $S_n = \sum_{j=1}^{n} X_j$, $n \geqslant 1$. It is well known that if the random variables $\{X_n, n \geqslant 1\}$ are independent and identically

*Corresponding author. Tel.: +1 352 392 1941 ext:225; fax: +1 352 392 5175.
E-mail address: rosalsky@stat.ufl.edu (A. Rosalsky).

0167-7152/$ - see front matter
doi:10.1016/j.spl.2004.10.027

distributed (i.i.d.) with $EX_1 = \infty$, then

$$\lim_{n\to\infty} \frac{S_n}{n} = \infty \quad \text{almost surely (a.s.).}$$

In this paper, we study the almost sure growth rate of S_n; more specifically, we provide conditions under which

$$\lim_{n\to\infty} \frac{S_n}{b_n} = \infty \quad \text{a.s.},$$

where $\{b_n, n \geqslant 1\}$ is a nondecreasing sequence of positive constants. In fact, we examine this problem in more generality than the case of i.i.d. summands. In the main results, Theorems 1 and 2, the summands $\{X_n, n \geqslant 1\}$ do not need to be independent or identically distributed but, nevertheless, they are new results in the i.i.d. case. In Theorems 1 and 2, it is assumed that $\{X_n, n \geqslant 1\}$ is a sequence of random variables which are *lower negatively dependent*. This is the assertion that for all $n \geqslant 1$ and all $x_1, \ldots, x_n \in \mathbb{R}$,

$$P\{X_1 \leqslant x_1, \ldots, X_n \leqslant x_n\} \leqslant \prod_{j=1}^{n} P\{X_j \leqslant x_j\}. \tag{1.1}$$

If for all $n > m \geqslant 1$ and all $x_1, x_2 \in \mathbb{R}$

$$P\{X_m \leqslant x_1, X_n \leqslant x_2\} \leqslant P\{X_m \leqslant x_1\} \cdot P\{X_n \leqslant x_2\},$$

then $\{X_n, n \geqslant 1\}$ is said to be a sequence of *pairwise lower negatively dependent* random variables. Of course, (1.1) is automatic if the $\{X_n, n \geqslant 1\}$ are independent. A sequence of lower negatively dependent random variables $\{Y_n, n \geqslant 1\}$ (not necessarily nonnegative) obeys the *strong law of large numbers* (SLLN) $\sum_{j=1}^{n} Y_j / b_n \to 0$ a.s. under suitable conditions; see for example Matuła (1992), Kim and Baek (1999) (wherein $b_n = n$, $n \geqslant 1$), Amini and Bozorgnia (2000) (wherein $b_n = n^r$, $n \geqslant 1$ with $r > \frac{1}{2}$), Kim and Kim (2001), and Taylor et al. (2002). In Kim and Kim (2001) the norming sequence $\{b_n, n \geqslant 1\}$ is very rapidly growing in that it satisfies the condition $\sum_{j=n}^{\infty} b_j^{-2} = \mathcal{O}(b_n^{-2})$ and for this reason their assertion that their Theorem 1 extends Theorem 6 of Adler et al. (1992) (from the independent case to the pairwise lower negatively dependent case) is incorrect.

2. Mainstream

The key lemma for proving Theorem 1 will now be established.

Lemma 1. *Let $\{S_n, n \geqslant 1\}$ be a nondecreasing sequence of nonnegative random variables, let $\{b_n, n \geqslant 1\}$ be a nondecreasing sequence of positive constants, and let $\{a_n, n \geqslant 1\}$ be a sequence of positive constants. Suppose that there exists a strictly increasing sequence of positive integers $\{m(k), k \geqslant 1\}$ and a constant $M < \infty$ such that*

$$b_{m(k+1)} \leqslant M b_{m(k)}, \quad k \geqslant 1 \tag{2.1}$$

and for some $\varepsilon > 0$

$$\min_{m(k)+1 \leqslant n \leqslant m(k+1)} a_n \geqslant \frac{\varepsilon}{m(k+1) - m(k)}, \quad k \geqslant 1. \tag{2.2}$$

If for some $0 < c_0 \leqslant \infty$ *and all* $0 < c < c_0$

$$\sum_{n=1}^{\infty} a_n P\{S_n \leqslant c b_n\} < \infty, \tag{2.3}$$

then

$$\liminf_{n\to\infty} \frac{S_n}{b_n} \geqslant \frac{c_0}{M^2} \quad \text{a.s.} \tag{2.4}$$

Proof. For arbitrary $0 < c < c_0$,

$$\begin{aligned}
\infty &> \sum_{k=1}^{\infty} \sum_{n=m(k)+1}^{m(k+1)} a_n P\{S_n \leqslant c b_n\} \quad \text{(by (2.3))} \\
&\geqslant \sum_{k=1}^{\infty} \frac{\varepsilon}{m(k+1) - m(k)} P\{S_{m(k+1)} \leqslant c b_{m(k)}\}(m(k+1) - m(k)) \\
&\quad \text{(by (2.2)}, S_n \uparrow, \text{ and } b_n \uparrow) \\
&\geqslant \varepsilon \sum_{k=1}^{\infty} P\left\{S_{m(k+1)} \leqslant \frac{c}{M} b_{m(k+1)}\right\} \quad \text{(by (2.1))}
\end{aligned}$$

and hence

$$\sum_{k=1}^{\infty} P\{S_{m(k)} \leqslant c b_{m(k)}\} < \infty$$

for all $0 < c < \frac{c_0}{M}$. Then by the Borel–Cantelli lemma

$$P\{S_{m(k)} \leqslant c b_{m(k)} \text{ i.o. } (k)\} = 0$$

for all $0 < c < \frac{c_0}{M}$. Hence

$$\liminf_{k\to\infty} \frac{S_{m(k)}}{b_{m(k)}} \geqslant c \quad \text{a.s.}$$

for all $0 < c < \frac{c_0}{M}$. Letting $c \uparrow \frac{c_0}{M}$ yields

$$\liminf_{k\to\infty} \frac{S_{m(k)}}{b_{m(k)}} \geqslant \frac{c_0}{M} \quad \text{a.s.}$$

Then for $n \geqslant m(1)+1$, writing $m(k)+1 \leqslant n \leqslant m(k+1)$ where $k=k(n)\geqslant 1$ and recalling that $0 \leqslant S_n \uparrow, 0 < b_n \uparrow$, and (2.1), we have

$$\begin{aligned}\liminf_{n\to\infty} \frac{S_n}{b_n} &\geqslant \liminf_{n\to\infty} \frac{S_{m(k(n))}}{b_{m(k(n)+1)}}\\ &\geqslant \liminf_{n\to\infty} \frac{S_{m(k(n))}}{Mb_{m(k(n))}}\\ &\geqslant \liminf_{k\to\infty} \frac{S_{m(k)}}{Mb_{m(k)}}\\ &\geqslant \frac{c_0}{M^2} \quad \text{a.s.} \quad \square\end{aligned}$$

Corollary 1. *Let* $\{S_n, n\geqslant 1\}$ *be a nondecreasing sequence of nonnegative random variables and let* $r>0$. *If for some* $0<c_0\leqslant\infty$ *and all* $0<c<c_0$

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{S_n \leqslant cn^r\} < \infty,$$

then

$$\liminf_{n\to\infty} \frac{S_n}{n^r} \geqslant c_0 \quad \textit{a.s.}$$

Proof. Let $\delta>0$ and $\theta>1$ be arbitrary and let $m(k)=[\theta^k], k\geqslant 1$. Let $b_n=n^r,\ n\geqslant 1$. Set $M=(1+\delta)\theta^r$. Then $b_{m(k+1)}\leqslant Mb_{m(k)}$ for all large k. Set $a_n=1/n,\ n\geqslant 1$. Now for some $\varepsilon>0$ and large k

$$\min_{m(k)+1\leqslant n\leqslant m(k+1)} a_n \geqslant \frac{\varepsilon}{m(k+1)-m(k)}.$$

It is clear that the sequences $\{b_n, n\geqslant 1\}, \{m(k), k\geqslant 1\}$, and $\{a_n, n\geqslant 1\}$ can be redefined for small values of n and k so that (2.1), (2.2), and (2.3) (for all $0<c<c_0$) hold with M and ε as above. Hence by Lemma 1

$$\liminf_{n\to\infty} \frac{S_n}{n^r} \geqslant \frac{c_0}{(1+\delta)^2\theta^{2r}} \quad \text{a.s.}$$

Letting $\delta\downarrow 0$ and $\theta\downarrow 1$ yields the conclusion (2.4). $\square$

The next corollary of Lemma 1 was originally due to Gut et al. (1997) when $S_n=\sum_{j=1}^n X_j, n\geqslant 1$ where $\{X_n, n\geqslant 1\}$ is a sequence of i.i.d. nonnegative random variables and $b_n=n^r, n\geqslant 1$ with $r\geqslant 1$ but the proof of it provided by them is incorrect. However, a valid proof of their result can be given using the method of proof of another result by Gut et al. (1997). It should be noted that the faster $b_n\uparrow\infty$ in Corollary 2, the stronger is the assumption (2.6) but so is the conclusion (2.7).

Corollary 2. *Let* $\{S_n, n\geqslant 1\}$ *be a nondecreasing sequence of nonnegative random variables and let* $\{b_n, n\geqslant 1\}$ *be a nondecreasing sequence of positive constants such that*

$$b_{2n}=\mathcal{O}(b_n). \tag{2.5}$$

If

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{S_n \leqslant cb_n\} < \infty \tag{2.6}$$

for all $0 < c < \infty$, *then*

$$\lim_{n\to\infty} \frac{S_n}{b_n} = \infty \quad a.s. \tag{2.7}$$

Proof. Let $m(k) = 2^k, k \geqslant 1, \varepsilon = \frac{1}{2}$, and $a_n = 1/n, n \geqslant 1$. It follows from (2.5) that (2.1) holds for some constant $M < \infty$. Moreover, (2.2) is immediate. Taking $c_0 = \infty$, the conclusion (2.7) follows directly from Lemma 1. □

Theorem 1 may now be presented. Its proof was inspired by that of a classical result of Derman and Robbins (1955) showing that for i.i.d. summands $\{Y_n, n \geqslant 1\}$ with $EY_1^+ = EY_1^- = \infty$ that $\lim_{n\to\infty} \sum_{j=1}^n Y_j/n = \infty$ a.s. can prevail.

Theorem 1. *Let* $S_n = \sum_{j=1}^n X_j$, $n \geqslant 1$ *where* $\{X_n, n \geqslant 1\}$ *is a sequence of lower negatively dependent nonnegative random variables. Let* $\{b_n, n \geqslant 1\}$ *be a nondecreasing sequence of positive constants and let* $\{a_n, n \geqslant 1\}$ *be a sequence of positive constants. Suppose that there exists a strictly increasing sequence of positive integers* $\{m(k), k \geqslant 1\}$ *such that* $b_{m(k+1)} = \mathcal{O}(b_{m(k)})$ *and* (2.2) *holds for some* $\varepsilon > 0$. *Furthermore, suppose that for all* $0 < c < \infty$

$$\sum_{n=1}^{\infty} a_n \prod_{j=1}^{n} P\{X_j \leqslant cb_n\} < \infty. \tag{2.8}$$

Then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = \infty \quad a.s. \tag{2.9}$$

Proof. Since $b_{m(k+1)} = \mathcal{O}(b_{m(k)})$, there exists a constant $M < \infty$ such that (2.1) holds. Let $0 < c < \infty$ be arbitrary and set $c_0 = \infty$. Then for $n \geqslant 1$,

$$\begin{aligned} P\{S_n \leqslant cb_n\} &\leqslant P\{X_1 \leqslant cb_n, \ldots, X_n \leqslant cb_n\} \text{ (since the } X_j \text{ are nonnegative)} \\ &\leqslant \prod_{j=1}^{n} P\{X_j \leqslant cb_n\} \quad \text{(by (1.1))} \end{aligned}$$

and hence

$$\sum_{n=1}^{\infty} a_n P\{S_n \leqslant cb_n\} \leqslant \sum_{n=1}^{\infty} a_n \prod_{j=1}^{n} P\{X_j \leqslant cb_n\} < \infty \quad \text{(by (2.8))}.$$

The conclusion (2.9) follows immediately from Lemma 1 noting that $0 \leqslant S_n \uparrow$. □

The next theorem is in effect a special case of Theorem 1. Theorem 2 can of course also be proved in a similar manner to that of Theorem 1 by using Corollary 2 instead of Lemma 1.

Theorem 2. *Let $S_n = \sum_{j=1}^{n} X_j$, $n \geqslant 1$ where $\{X_n, n \geqslant 1\}$ is a sequence of lower negatively dependent nonnegative random variables and suppose that the nonnegative function $h(x)$ on $[0, \infty)$ is such that*

$$P\{X_n > x\} \geqslant h(x) \quad \text{for all } n \geqslant 1 \text{ and } x \geqslant 0. \tag{2.10}$$

Let $\{b_n, n \geqslant 1\}$ be a nondecreasing sequence of positive constants satisfying (2.5). *Suppose that for all $0 < c < \infty$*

$$\sum_{n=1}^{\infty} \frac{(1 - h(cb_n))^n}{n} < \infty. \tag{2.11}$$

Then

$$\lim_{n \to \infty} \frac{S_n}{b_n} = \infty \quad a.s. \tag{2.12}$$

Proof. Let $m(k) = 2^k, k \geqslant 1, \varepsilon = \frac{1}{2}$, and $a_n = 1/n, n \geqslant 1$. Then $b_{m(k+1)} = \mathcal{O}(b_{m(k)})$ and (2.2) holds as was noted in the proof of Corollary 2. Note that for all $0 < c < \infty$

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \prod_{j=1}^{n} P\{X_j \leqslant cb_n\} &= \sum_{n=1}^{\infty} \frac{1}{n} \prod_{j=1}^{n} (1 - P\{X_j > cb_n\}) \\ &\leqslant \sum_{n=1}^{\infty} \frac{(1 - h(cb_n))^n}{n} \quad \text{(by (2.10))} \\ &< \infty \quad \text{(by} \quad \text{(2.11)).} \end{aligned}$$

The conclusion (2.12) follows directly from Theorem 1. □

We now obtain the following corollary of Theorem 2. It should be noted that (2.13) and (2.14) ensure that $EX_n^{1/r} = \infty$, $n \geqslant 1$. Moreover, the larger r is taken in Corollary 3, the more stringent are the hypotheses but the conclusion (2.15) is also stronger.

Corollary 3. *Let $S_n = \sum_{j=1}^{n} X_j$, $n \geqslant 1$ where $\{X_n, n \geqslant 1\}$ is a sequence of lower negatively dependent nonnegative random variables. Let $r > 0$ and suppose there exists a nonnegative function $g(x)$ on $[0, \infty)$ such that*

$$x^{1/r} P\{X_n > x\} \geqslant g(x) \quad \text{for all } n \geqslant 1 \text{ and } x \geqslant 0 \tag{2.13}$$

and

$$\log \log x = \mathrm{o}(g(x)) \quad \text{as } x \to \infty. \tag{2.14}$$

Then

$$\lim_{n \to \infty} \frac{S_n}{n^r} = \infty \quad a.s. \tag{2.15}$$

Proof. We will apply Theorem 2 with the function $h(x)$ defined by $h(0) = 0$, $h(x) = g(x)/x^{1/r}$, $x > 0$ and with $b_n = n^r$, $n \geqslant 1$. By (2.14) we can write $g(x) = G(x) \log \log x$ where $G(x) \to \infty$ as $x \to \infty$. For $0 < c < \infty$ and all large n

$$\begin{aligned}
\frac{(1 - h(cn^r))^n}{n} &\leqslant \frac{\left(\exp\left\{-\frac{g(cn^r)}{(cn^r)^{1/r}}\right\}\right)^n}{n} \quad \text{(by the elementary inequality } 1 - x \leqslant e^{-x}\text{)} \\
&= \frac{1}{n} \exp\left\{-\frac{g(cn^r)}{c^{1/r}}\right\} \\
&= \frac{1}{n} \exp\left\{-\frac{G(cn^r) \log \log(cn^r)}{c^{1/r}}\right\} \\
&\leqslant \frac{1}{n} \exp\{-2 \log \log n\} \\
&= \frac{1}{n(\log n)^2}
\end{aligned}$$

and hence

$$\sum_{n=1}^{\infty} \frac{(1 - h(cn^r))^n}{n} < \infty.$$

The conclusion (2.15) follows immediately from Theorem 2. □

The fourth corollary is in effect a special case of Corollary 3.

Corollary 4. *Let* $S_n = \sum_{j=1}^n X_j$, $n \geqslant 1$ *where* $\{X_n, n \geqslant 1\}$ *is a sequence of identically distributed lower negatively dependent* (*a fortiori, i.i.d.*) *nonnegative random variables. Let* $r > 0$ *and suppose that*

$$\frac{x^{1/r} P\{X_1 > x\}}{\log \log x} \to \infty \quad \text{as } x \to \infty. \tag{2.16}$$

Then

$$\lim_{n \to \infty} \frac{S_n}{n^r} = \infty \quad \text{a.s.} \tag{2.17}$$

Proof. Let $g(x) = x^{1/r} P\{X_1 > x\}$, $x \geqslant 0$. Then (2.13) holds since the $\{X_n, n \geqslant 1\}$ are identically distributed, and (2.14) holds by (2.16). The conclusion (2.17) then follows immediately from Corollary 3. □

The following example demonstrates that Corollary 4 is sharp and hence so are Corollary 3 and Theorems 1 and 2.

Example 1. Let $r \geqslant 1$ and let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed lower negatively dependent random variables where

$$P\{X_1 > x\} = \frac{e^{e/r} (\log \log x)^2}{x^{1/r}}, \quad x \geqslant e^e.$$

Then (2.16) holds whence (2.17) follows from Corollary 4. We will now show that for arbitrary $\varepsilon > 0$, the SLLN

$$\lim_{n\to\infty} \frac{S_n}{n^{r+\varepsilon}} = 0 \quad \text{a.s.} \tag{2.18}$$

holds. Note that

$$\begin{aligned} EX_1^{1/(r+\varepsilon)} &= \text{Const.} + \int_{e^e}^{\infty} e^{e/r} x^{\frac{1}{r+\varepsilon}-1} \frac{(\log\log x)^2}{x^{1/r}}\,\mathrm{d}x \\ &= \text{Const.} + \int_{e^e}^{\infty} e^{e/r} \frac{(\log\log x)^2}{x^{1+\frac{1}{r}-\frac{1}{r+\varepsilon}}}\,\mathrm{d}x \\ &< \infty. \end{aligned} \tag{2.19}$$

It is well known (see, e.g., Sawyer, 1966; Chatterji, 1969/1970; or Martikainen and Petrov, 1980) that the famous Marcinkiewicz–Zygmund SLLN holds irrespective of the joint distributions of the identically distributed summands $\{Y_n, n \geqslant 1\}$ when $E|Y_1|^p < \infty$, where $0 < p < 1$. Hence (2.18) follows from (2.19) and the Marcinkiewicz–Zygmund SLLN.

Remark 1. Suppose that $\{X_n, n \geqslant 1\}$ is a sequence of i.i.d. nonnegative random variables with

$$P\{X_1 > x\} \sim \frac{c \log\log x}{x^{1/r}} v(x) \quad \text{as } x \to \infty, \tag{2.20}$$

where c and r are positive constants and $v(x)$ is a positive function. If $v(x) \to \infty$ as $x \to \infty$, then $S_n/n^r \to \infty$ a.s. by Corollary 4. But if $v(x) \nrightarrow \infty$ as $x \to \infty$, then $\liminf_{n\to\infty} S_n/n^r < \infty$ a.s. can prevail. This follows by taking $v(x) = (\log\log x)^{-1/r}$, $x \geqslant e^e$ where $r > 1$ and applying Theorem 5 of Erickson (1976) which establishes that $\liminf_{n\to\infty} S_n/n^r = b$ a.s. for some constant $0 < b < \infty$.

Remark 2. Suppose that $\{X_n, n \geqslant 1\}$ is a sequence of i.i.d. nonnegative random variables satisfying (2.20) with $v(x) \equiv 1$. If $0 < r \leqslant 1$, then $\lim_{n\to\infty} S_n/n^r = \infty$ a.s. by the Kolmogorov SLLN. An interesting question which we are unable to resolve is whether or not $\lim_{n\to\infty} S_n/n^r = \infty$ a.s. when $r > 1$. However, it follows from the ensuing theorem that $\liminf_{n\to\infty} S_n/n^r \geqslant c^r$ a.s. It is interesting to notice that the lower bound c^r depends on the distribution of X_1. In addition, since $\sum_{n=1}^{\infty} P\{X_n > Mn^r\} = \infty$ for all $0 < M < \infty$, it follows from the Borel–Cantelli lemma that

$$\limsup_{n\to\infty} \frac{S_n}{n^r} \geqslant \limsup_{n\to\infty} \frac{X_n}{n^r} = \infty \quad \text{a.s.}$$

Theorem 3. *Let $S_n = \sum_{j=1}^{n} X_j$, $n \geqslant 1$ where $\{X_n, n \geqslant 1\}$ is a sequence of identically distributed lower negatively dependent (a fortiori, i.i.d.) nonnegative random variables. Suppose that*

$$P\{X_1 > x\} \geqslant \frac{C(x) \log\log x}{x^{1/r}}, \quad x > e, \tag{2.21}$$

where $\lim_{x\to\infty} C(x) = c \in (0,\infty)$ *and* $r>0$. *Then*

$$\liminf_{n\to\infty} \frac{S_n}{n^r} \geqslant c^r \quad a.s. \tag{2.22}$$

Proof. Let $c_0 = c^r$. Now for all $0<\gamma<c_0$,

$$\lim_{n\to\infty} \frac{C(\gamma n^r)}{\gamma^{1/r}} = \frac{c}{\gamma^{1/r}} > 1$$

and so $C(\gamma n^r)/\gamma^{1/r} \geqslant 1+\delta$ for some $\delta>0$ and large n. Thus for all $0<\gamma<c_0$ and all large n

$$\begin{aligned}
P\{S_n \leqslant \gamma n^r\} &\leqslant P\{X_1 \leqslant \gamma n^r, \ldots, X_n \leqslant \gamma n^r\} \\
&\quad \text{(since the } X_j \text{ are nonnegative)} \\
&\leqslant \left(1 - \frac{C(\gamma n^r)\log\log(\gamma n^r)}{\gamma^{1/r} n}\right)^n \quad \text{(by (1.1) and (2.21))} \\
&\leqslant \exp\left\{-\frac{C(\gamma n^r)}{\gamma^{1/r}} \log\log(\gamma n^r)\right\} \\
&\quad \left(\text{since} \left(1-\frac{x}{n}\right)^n \leqslant e^{-x} \text{ for } 0<x<n\right) \\
&= (\log(\gamma n^r))^{-C(\gamma n^r)/\gamma^{1/r}} \\
&\leqslant (\log(\gamma n^r))^{-1-\delta} \leqslant \frac{2}{r^{1+\delta}(\log n)^{1+\delta}}.
\end{aligned}$$

Consequently,

$$\sum_{n=1}^{\infty} a_n P\{S_n \leqslant \gamma n^r\} < \infty.$$

The conclusion (2.22) follows immediately from Corollary 1. □

Remark 3. Corollary 4 may be contrasted with the following result of Rosalsky (1993) concerning the growth rate of sums of identically distributed strictly positive random variables irrespective of their joint distributions.

Theorem 4 (*Rosalsky, 1993*). *Let* $S_n = \sum_{j=1}^n X_j$, $n \geqslant 1$ *where* $\{X_n, n\geqslant 1\}$ *is a sequence of identically distributed random variables with* $X_1 > 0$ *a.s. and let* $\{b_n, n\geqslant 1\}$ *be a sequence of positive constants with* $\liminf_{n\to\infty}(b_n/n) = 0$. *Then*

$$\limsup_{n\to\infty} \frac{S_n}{b_n} = \infty \quad a.s.$$

irrespective of the joint distributions of the $\{X_n, n\geqslant 1\}$.

References

Adler, A., Rosalsky, A., Taylor, R.L., 1992. Some strong laws of large numbers for sums of random elements. Bull. Inst. Math. Acad. Sinica 20, 335–357.

Amini, M., Bozorgnia, A., 2000. Negatively dependent bounded random variable probability inequalities and the strong law of large numbers. J. Appl. Math. Stochastic Anal. 13, 261–267.

Chatterji, S.D., 1969/1970. A general strong law. Invent. Math. 9, 235–245.

Derman, C., Robbins, H., 1955. The strong law of large numbers when the first moment does not exist. Proc. Nat. Acad. Sci. U.S.A. 41, 586–587.

Erickson, K.B., 1976. Recurrence sets of normed random walk in R^d. Ann. Probab. 4, 802–828.

Gut, A., Klesov, O., Steinebach, J., 1997. Equivalences in strong limit theorems for renewal counting processes. Statist. Probab. Lett. 35, 381–394.

Kim, T.-S., Baek, J.I., 1999. The strong laws of large numbers for weighted sums of pairwise quadrant dependent random variables. J. Korean Math. Soc. 36, 37–49.

Kim, T.-S., Kim, H.-C., 2001. On the law of large numbers for weighted sums of pairwise negative quadrant dependent random variables. Bull. Korean Math. Soc. 38, 55–63.

Martikainen, A.I., Petrov, V.V., 1980. On a theorem of Feller. Teor. Veroyatnost. i Primenen. 25, 194–197 (in Russian) (English translation: Theory Probab. Appl. 25, 191–193).

Matuła, P., 1992. A note on the almost sure convergence of sums of negatively dependent random variables. Statist. Probab. Lett. 15, 209–213.

Rosalsky, A., 1993. On the almost certain limiting behavior of normed sums of identically distributed positive random variables. Statist. Probab. Lett. 16, 65–70.

Sawyer, S., 1966. Maximal inequalities of weak type. Ann. of Math. (2) 84, 157–174.

Taylor, R.L., Patterson, R.F., Bozorgnia, A., 2002. A strong law of large numbers for arrays of rowwise negatively dependent random variables. Stochastic Anal. Appl. 20, 643–656.