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On convergence properties of sums of dependent random variables under second moment and covariance restrictions

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Abstract

For a sequence of dependent square-integrable random variables and a sequence of positive constants $\{b_n, n \geq 1\}$, conditions are provided under which the series $\sum_{i=1}^{n}(X_i - EX_i)/b_i$ converges almost surely as $n \to \infty$ and $\{X_n, n \geq 1\}$ obeys the strong law of large numbers $\lim_{n\to\infty} \sum_{i=1}^{n}(X_i - EX_i)/b_n = 0$ almost surely. The hypotheses stipulate that two series converge, where the convergence of the first series involves the growth rates of $\{\text{Var } X_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ and the convergence of the second series involves the growth rate of $\{\sup_{n\geq 1} |\text{Cov }(X_n, X_{n+k})|, k \geq 1\}$.

MSC: 60F15

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of square-integrable random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and let $\{b_n, n \geq 1\}$ be a sequence of positive constants. The random variables $\{X_n, n \geq 1\}$ are not assumed to be independent. We assume that there exists a sequence of constants $\{\rho_k, k \geq 1\}$ such that

$$\sup_{n\geq 1} |\text{Cov }(X_n, X_{n+k})| \leq \rho_k, \quad k \geq 1 \tag{1.1}$$

and we provide conditions on the growth rates of $\{\text{Var } X_n, n \geq 1\}$, $\{b_n, n \geq 1\}$ and $\{\rho_k, k \geq 1\}$ under which (i) the series

$$\sum_{i=1}^{n}(X_i - EX_i)/b_i \quad \text{converges almost surely (a.s.) as } n \to \infty \tag{1.2}$$

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and (ii) $\{X_n, n \geq 1\}$ obeys the *strong law of large numbers* (SLLN)

$$\lim_{n\to\infty} \sum_{i=1}^{n} (X_i - EX_i)/b_n = 0 \quad \text{a.s.} \tag{1.3}$$

The history and literature on the SLLN problem for dependent summands is not nearly as extensive and complete as it is for the case of independent summands. It appears that the best result establishing a SLLN for a sequence of correlated random variables satisfying (1.1) and $\text{Var } X_n = \mathcal{O}(1)$ is that of Lyons (1988) wherein $b_n \equiv n$. We point out that Lyons (1988) did not provide conditions for a series of correlated random variables to converge a.s.; he only treated the SLLN problem. The only SLLN result for correlated summands that we are aware of satisfying (1.1) without assuming that $\text{Var } X_n = \mathcal{O}(1)$ is due to Hu et al. (2005) where again $b_n \equiv n$. This result established a link between the Golden Ratio φ and the SLLN. Other results on the SLLN problem for a sequence of correlated random variables are those of Chandra (1991), Gapoškin (1975), Móricz (1977, 1985), Serfling (1970b, 1980).

In the current work, the main result, Theorem 1, provides conditions under which (1.2) and (1.3) hold with $\{b_n, n \geq 1\}$ being more general than only $b_n \equiv n$ in that $n = \mathcal{O}(b_n)$ where again it is not assumed that $\text{Var } X_n = \mathcal{O}(1)$. The proof of Theorem 1 is classical in nature and is based on the general "method of subsequences". This method was apparently developed initially by Rajchman (1932) (see Chung (1974), p. 103) and has since been used by numerous other authors. However, the key inequality used in our proof is a much more recent result due to Serfling (1970a).

The plan of the paper is as follows. In Section 2, a lemma which is used in the proof of Theorem 1 is presented. Theorem 1 is stated and proved in Section 3. In Section 4, Theorem 1 is compared with a well-known result.

2. Preliminaries

Throughout this paper, the symbol C denotes a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance. For $x \geq 1$, the natural (base e) logarithm of x and the logarithm of x to the base 2 will be denoted, respectively, by $\log x$ and $\text{Log } x$. We note that for all $x \geq 1$, $\text{Log } x = C \log x$ where $C = 1/\log 2$.

The following lemma is used in the proof of Theorem 1.

Lemma 1. *Let $\{X_n, n \geq 1\}$ be a sequence of square-integrable random variables and suppose that there exists a sequence of constants $\{\rho_k, k \geq 1\}$ such that*

$$\sup_{n\geq 1} |\text{Cov } (X_n, X_{n+k})| \leq \rho_k, \quad k \geq 1. \tag{2.1}$$

Let $\{b_n, n \geq 1\}$ be a sequence of positive constants such that

$$n = \mathcal{O}(b_n). \tag{2.2}$$

Then for all $n \geq 0$, $m \geq n + 2$, and $0 \leq q < 1$,

$$E\left(\sum_{i=n+1}^{m} \frac{X_i - EX_i}{b_i}\right)^2 \leq \sum_{i=n+1}^{m} \frac{\text{Var } X_i}{b_i^2} + \frac{C}{n^{1-q}} \sum_{k=1}^{m-n-1} \frac{\rho_k}{k^q}$$

where C is a constant independent of n and m.

Proof. For all $n \geq 0$, $m \geq n + 2$, and $0 \leq q < 1$,

$$\begin{aligned} E\left(\sum_{i=n+1}^{m} \frac{X_i - EX_i}{b_i}\right)^2 &= \sum_{i=n+1}^{m} \frac{\text{Var } (X_i)}{b_i^2} + 2\sum_{i=n+1}^{m-1}\sum_{j=i+1}^{m} \frac{\text{Cov } (X_i, X_j)}{b_i b_j} \\ &\leq \sum_{i=n+1}^{m} \frac{\text{Var } X_i}{b_i^2} + C\sum_{i=n+1}^{m-1}\sum_{j=i+1}^{m} \frac{\rho_{j-i}}{ij} \quad \text{(by (2.1) and (2.2))} \\ &= \sum_{i=n+1}^{m} \frac{\text{Var } X_i}{b_i^2} + C\sum_{i=n+1}^{m-1}\sum_{j=i+1}^{m} \frac{\rho_{j-i}}{j-i}\left(\frac{1}{i} - \frac{1}{j}\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=n+1}^{m} \frac{\operatorname{Var} X_i}{b_i^2} + C \sum_{k=1}^{m-n-1} \frac{\rho_k}{k} \left(\sum_{l=n+1}^{m-k} \frac{1}{l} - \sum_{l=n+k+1}^{m} \frac{1}{l} \right) \\
&\le \sum_{i=n+1}^{m} \frac{\operatorname{Var} X_i}{b_i^2} + C \sum_{k=1}^{m-n-1} \frac{\rho_k}{k} \left(\sum_{l=n+1}^{n+k} \frac{1}{l} \right) \\
&\le \sum_{i=n+1}^{m} \frac{\operatorname{Var} X_i}{b_i^2} + C \sum_{k=1}^{m-n-1} \frac{\rho_k}{k} \log \left(\frac{n+k}{n} \right) \\
&\le \sum_{i=n+1}^{m} \frac{\operatorname{Var} X_i}{b_i^2} + \frac{C}{n^{1-q}} \sum_{k=1}^{m-n-1} \frac{\rho_k}{k^q} \qquad (2.3)
\end{aligned}
$$

since $\log(1+x) \le x^{1-q}/(1-q)$ for all $x \ge 0$.

3. The main result

With the preliminaries accounted for, the main result may be stated and proved. We note that the condition (3.2) is indeed stronger than that the condition $\sum_{k=1}^{\infty} \rho_k/k < \infty$ of Lyons (1988). However, as we remarked in Section 1, Lyons (1988) treated the case $\operatorname{Var} X_n = \mathcal{O}(1)$ and $b_n \equiv n$ to prove a SLLN and he did not establish a result along the lines of (3.3). We also note that the condition (3.1) is automatic if

$$
\frac{\operatorname{Var} X_n}{b_n^2} = \mathcal{O}\left(\frac{1}{n(\log n)^3 (\log \log n)^{1+\varepsilon}} \right) \quad \text{for some } \varepsilon > 0.
$$

For example, suppose that $\operatorname{Var} X_n \sim n/(\log n)^2$. If $b_n \sim n \log n$, then (2.2) and (3.1) hold whereas if $b_n \sim n$, then (2.2) holds but (3.1) fails.

Theorem 1. *Let $\{X_n, n \ge 1\}$ be a sequence of square-integrable random variables and suppose that there exists a sequence of constants $\{\rho_k, k \ge 1\}$ such that (2.1) holds. Let $\{b_n, n \ge 1\}$ be a sequence of positive constants satisfying (2.2). Suppose that*

$$
\sum_{n=1}^{\infty} \frac{(\operatorname{Var}\ X_n)(\log n)^2}{b_n^2} < \infty \qquad (3.1)
$$

and

$$
\sum_{k=1}^{\infty} \frac{\rho_k}{k^q} < \infty \quad \text{for some } 0 \le q < 1. \qquad (3.2)
$$

Then

$$
\sum_{i=1}^{n} \frac{X_i - EX_i}{b_i} \quad \text{converges a.s. as } n \to \infty \qquad (3.3)
$$

and if $b_n \uparrow$, the SLLN

$$
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} (X_i - EX_i)}{b_n} = 0 \quad \text{a.s.} \qquad (3.4)
$$

obtains.

Proof. Note at the outset that (2.2) ensures that $\lim_{n\to\infty} b_n = \infty$. Thus if $b_n \uparrow$, then the conclusion (3.4) follows immediately from (3.3) and the Kronecker lemma. To prove (3.3), note that for all $n \ge 1$,

$$
\sup_{m>n} E\left(\sum_{i=1}^{m} \frac{X_i - EX_i}{b_i} - \sum_{i=1}^{n} \frac{X_i - EX_i}{b_i} \right)^2 = \sup_{m>n} E\left(\sum_{i=n+1}^{m} \frac{X_i - EX_i}{b_i} \right)^2
$$

$$\begin{aligned}
&\le \sup_{m>n}\left(\sum_{i=n+1}^{m}\frac{\mathrm{Var}\,X_i}{b_i^2}+\frac{C}{n^{1-q}}\sum_{k=1}^{m-n-1}\frac{\rho_k}{k^q}\right) \quad \text{(by Lemma 1)}\\
&= \sum_{i=n+1}^{\infty}\frac{\mathrm{Var}\,X_i}{b_i^2}+\frac{C}{n^{1-q}}\sum_{k=1}^{\infty}\frac{\rho_k}{k^q}\\
&= o(1) \quad \text{as } n\to\infty \text{ (by (3.1) and (3.2), and } q<1).
\end{aligned}$$

Then by the Cauchy Convergence Criterion (see, e.g., Chow and Teicher (1997, p. 99)), there exists a random variable S on $(\Omega, \mathcal{F}, P)$ with $ES^2 < \infty$ such that

$$\sum_{i=1}^{n}\frac{X_i - EX_i}{b_i} \xrightarrow{\mathcal{L}_2} S.$$

Thus for all $n \ge 1$,

$$\sum_{i=1}^{m}\frac{X_i - EX_i}{b_i} - \sum_{i=1}^{2^n}\frac{X_i - EX_i}{b_i} \xrightarrow{\mathcal{L}_2} S - \sum_{i=1}^{2^n}\frac{X_i - EX_i}{b_i} \quad \text{as } m\to\infty$$

whence (see, e.g., Chow and Teicher (1997, p. 101))

$$\begin{aligned}
E\left(S-\sum_{i=1}^{2^n}\frac{X_i-EX_i}{b_i}\right)^2 &= \lim_{m\to\infty} E\left(\sum_{i=1}^{m}\frac{X_i-EX_i}{b_i}-\sum_{i=1}^{2^n}\frac{X_i-EX_i}{b_i}\right)^2\\
&= \lim_{m\to\infty} E\left(\sum_{i=2^n+1}^{m}\frac{X_i-EX_i}{b_i}\right)^2\\
&\le \lim_{m\to\infty}\left(\sum_{i=2^n+1}^{m}\frac{\mathrm{Var}\,X_i}{b_i^2}+\frac{C}{2^{n(1-q)}}\sum_{k=1}^{m-2^n-1}\frac{\rho_k}{k^q}\right) \quad \text{(by Lemma 1)}\\
&\le \sum_{i=2^n}^{\infty}\frac{\mathrm{Var}\,X_i}{b_i^2}+\frac{C}{2^{n(1-q)}}\sum_{k=1}^{\infty}\frac{\rho_k}{k^q}. \qquad (3.5)
\end{aligned}$$

Next, it will be shown that

$$\sum_{i=1}^{2^n}\frac{X_i-EX_i}{b_i} \to S \quad \text{a.s.} \qquad (3.6)$$

For arbitrary $\varepsilon > 0$,

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left\{\left|\sum_{i=1}^{2^n}\frac{X_i-EX_i}{b_i}-S\right|>\varepsilon\right\} &\le \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty} E\left(\sum_{i=1}^{2^n}\frac{X_i-EX_i}{b_i}-S\right)^2 \quad \text{(by the Markov inequality)}\\
&\le \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\left(\sum_{i=2^n}^{\infty}\frac{\mathrm{Var}\,X_i}{b_i^2}+\frac{C}{2^{n(1-q)}}\sum_{k=1}^{\infty}\frac{\rho_k}{k^q}\right) \quad \text{(by (3.5))}\\
&= C\sum_{i=2}^{\infty}\sum_{n=1}^{[\mathrm{Log}\ i]}\frac{\mathrm{Var}\,X_i}{b_i^2}+C\left(\sum_{n=1}^{\infty}\frac{1}{2^{n(1-q)}}\right)\left(\sum_{k=1}^{\infty}\frac{\rho_k}{k^q}\right)\\
&\le C\sum_{i=2}^{\infty}\frac{(\mathrm{Var}\,X_i)\log i}{b_i^2}+C\sum_{k=1}^{\infty}\frac{\rho_k}{k^q} \quad \text{(since } q<1)\\
&< \infty \quad \text{(by (3.1) and (3.2))}.
\end{aligned}$$

Then by Borel–Cantelli lemma and the arbitrariness of $\varepsilon > 0$, (3.6) follows.

We will now verify that

$$\max_{2^{n-1}<k\le 2^n}\left|\sum_{i=1}^{k}\frac{X_i-EX_i}{b_i}-\sum_{i=1}^{2^{n-1}}\frac{X_i-EX_i}{b_i}\right|\to 0\quad \text{a.s. as } n\to\infty. \tag{3.7}$$

For $a\ge 0$ and $n\ge 1$, let the joint distribution function of $X_{a+1},\ldots,X_{a+n}$ be denoted by $F_{a,n}$. Define a functional g on $\{F_{a,n}: a\ge 0, n\ge 1\}$ by

$$g(F_{a,n})=\sum_{i=a+1}^{a+n}\frac{\operatorname{Var}X_i}{b_i^2}+2\sum_{i=a+1}^{a+n-1}\sum_{j=i+1}^{a+n}\frac{|\operatorname{Cov}(X_i,X_j)|}{b_ib_j},\quad a\ge 0, n\ge 1$$

where the second term is interpreted as 0 if $n=1$. Then for $a\ge 0$, $k\ge 1$, and $m\ge 1$,

$$\begin{aligned}g(F_{a,k})+g(F_{a+k,m}) &= \sum_{i=a+1}^{a+k}\frac{\operatorname{Var}X_i}{b_i^2}+2\sum_{i=a+1}^{a+k-1}\sum_{j=i+1}^{a+k}\frac{|\operatorname{Cov}(X_i,X_j)|}{b_ib_j}\\ &\quad+\sum_{i=a+k+1}^{a+k+m}\frac{\operatorname{Var}X_i}{b_i^2}+2\sum_{i=a+k+1}^{a+k+m-1}\sum_{j=i+1}^{a+k+m}\frac{|\operatorname{Cov}(X_i,X_j)|}{b_ib_j}\\ &\le \sum_{i=a+1}^{a+k+m}\frac{\operatorname{Var}X_i}{b_i^2}+2\sum_{i=a+1}^{a+k+m-1}\sum_{j=i+1}^{a+k+m}\frac{|\operatorname{Cov}(X_i,X_j)|}{b_ib_j}\\ &= g(F_{a,k+m}).\end{aligned}$$

Moreover, it follows from (2.3) that for all $a\ge 0$ and $n\ge 1$,

$$E\left(\sum_{i=a+1}^{a+n}\frac{X_i-EX_i}{b_i}\right)^2\le g(F_{a,n}).$$

Then by Serfling's (1970a) generalization of the Rademacher–Menchoff fundamental maximal inequality for the partial sums of orthogonal random variables (see also Stout (1974), Sections 2.3 and 2.4), for all $a\ge 0$ and $n\ge 1$,

$$E\left(\max_{1\le k\le n}\left|\sum_{i=a+1}^{a+k}\frac{X_i-EX_i}{b_i}\right|\right)^2\le(\operatorname{Log}2n)^2 g(F_{a,n}). \tag{3.8}$$

Thus for arbitrary $\varepsilon>0$,

$$\begin{aligned}&\sum_{n=1}^{\infty}P\left\{\max_{2^{n-1}<k\le 2^n}\left|\sum_{i=1}^{k}\frac{X_i-EX_i}{b_i}-\sum_{i=1}^{2^{n-1}}\frac{X_i-EX_i}{b_i}\right|>\varepsilon\right\}\\ &\le 1+\frac{1}{\varepsilon^2}\sum_{n=2}^{\infty}E\left(\max_{2^{n-1}<k\le 2^n}\left|\sum_{i=1}^{k}\frac{X_i-EX_i}{b_i}-\sum_{i=1}^{2^{n-1}}\frac{X_i-EX_i}{b_i}\right|\right)^2\quad\text{(by the Markov inequality)}\\ &= 1+\frac{1}{\varepsilon^2}\sum_{n=2}^{\infty}E\left(\max_{2^{n-1}<k\le 2^n}\left|\sum_{i=2^{n-1}+1}^{k}\frac{X_i-EX_i}{b_i}\right|\right)^2\\ &= 1+\frac{1}{\varepsilon^2}\sum_{n=2}^{\infty}E\left(\max_{1\le k\le 2^{n-1}}\left|\sum_{i=2^{n-1}+1}^{2^{n-1}+k}\frac{X_i-EX_i}{b_i}\right|\right)^2\\ &\le 1+\frac{1}{\varepsilon^2}\sum_{n=2}^{\infty}(\operatorname{Log}(2\cdot 2^{n-1}))^2 g(F_{2^{n-1},2^{n-1}})\quad\text{(by (3.8))}\end{aligned}$$

$$
\begin{aligned}
&\leq 1 + C\sum_{n=2}^{\infty}(\text{Log}2^{n-1})^2\left(\sum_{i=2^{n-1}+1}^{2^n}\frac{\text{Var }X_i}{b_i^2} + 2\sum_{i=2^{n-1}+1}^{2^n-1}\sum_{j=i+1}^{2^n}\frac{|\text{Cov }(X_i, X_j)|}{b_i b_j}\right)\\
&\leq 1 + C\sum_{n=2}^{\infty}\sum_{i=2^{n-1}+1}^{2^n}\frac{(\text{Var }X_i)(\text{Log }i)^2}{b_i^2} + C\sum_{n=2}^{\infty}n^2\sum_{i=2^{n-1}+1}^{2^n-1}\sum_{j=i+1}^{2^n}\frac{\rho_{j-i}}{ij} \quad \text{(by (2.1) and (2.2))}\\
&\leq 1 + C\sum_{i=1}^{\infty}\frac{(\text{Var }X_i)(\log i)^2}{b_i^2} + C\sum_{n=2}^{\infty}\frac{n^2}{(2^{n-1})^{1-q}}\sum_{k=1}^{2^{n-1}-1}\frac{\rho_k}{k^q} \quad \text{(by arguing as in the proof of Lemma 1)}\\
&\leq 1 + C\sum_{i=1}^{\infty}\frac{(\text{Var }X_i)(\log i)^2}{b_i^2} + C\left(\sum_{n=2}^{\infty}\frac{n^2}{(2^{n-1})^{1-q}}\right)\left(\sum_{k=1}^{\infty}\frac{\rho_k}{k^q}\right)\\
&< \infty
\end{aligned}
$$

by (3.1) and (3.2), $q < 1$, and

$$\frac{n^2}{(2^{n-1})^{1-q}} = \mathcal{O}\left(\left(\frac{1}{2^{\frac{1-q}{2}}}\right)^n\right).$$

Then by the Borel–Cantelli lemma and the arbitrariness of $\varepsilon > 0$, (3.7) follows.

Next, for $k \geq 2$, let $n \geq 1$ be such that $2^{n-1} < k \leq 2^n$. Then

$$
\begin{aligned}
\left|\sum_{i=1}^{k}\frac{X_i - EX_i}{b_i} - S\right| &\leq \left|\sum_{i=1}^{k}\frac{X_i - EX_i}{b_i} - \sum_{i=1}^{2^{n-1}}\frac{X_i - EX_i}{b_i}\right| + \left|\sum_{i=1}^{2^{n-1}}\frac{X_i - EX_i}{b_i} - S\right|\\
&\leq \max_{2^{n-1}<j\leq 2^n}\left|\sum_{i=1}^{j}\frac{X_i - EX_i}{b_i} - \sum_{i=1}^{2^{n-1}}\frac{X_i - EX_i}{b_i}\right| + \left|\sum_{i=1}^{2^{n-1}}\frac{X_i - EX_i}{b_i} - S\right|\\
&\to 0 \quad \text{a.s. as } k \to \infty
\end{aligned}
$$

by (3.6) and (3.7) thereby proving that

$$\lim_{k\to\infty}\sum_{i=1}^{k}\frac{X_i - EX_i}{b_i} = S \quad \text{a.s.}$$

and hence (3.3) is established.

4. Concluding comments

To conclude, we compare Theorem 1 with a well-known result. The following proposition is essentially Corollary 2.4.1 of Stout (1974, p. 28).

Proposition 1. *Let $\{X_n, n \geq 1\}$ be a sequence of square-integrable random variables and suppose that there exists a sequence of constants $\{r_k, k \geq 1\}$ such that*

$$\sup_{n\geq 1}\frac{|\text{Cov }(X_n, X_{n+k})|}{(\text{Var }\ X_n \cdot \text{Var }\ X_{n+k})^{1/2}} \leq r_k, \quad k \geq 1.$$

Let $\{b_n, n \geq 1\}$ be a sequence of positive constants. If

$$\sum_{n=1}^{\infty}\frac{(\text{Var }\ X_n)(\log n)^2}{b_n^2} < \infty$$

and

$$\sum_{k=1}^{\infty} r_k < \infty, \tag{4.1}$$

then

$$\sum_{i=1}^{n} \frac{X_i - EX_i}{b_i} \quad \textit{converges a.s. as } n \to \infty.$$

To compare Theorem 1 with Proposition 1, observe that condition (2.2) is not needed in Proposition 1. In general, the conditions (3.2) and (4.1) are not comparable; however, if Var $X_n = \mathcal{O}(1)$, then (3.2) is weaker than (4.1).

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