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Strong Laws for Blockwise Martingale Difference Arrays in Banach Spaces

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Abstract—In this paper, we introduce the concepts of blockwise adapted array, blockwise martingale difference array, and establish some strong laws of large numbers for blockwise adapted arrays and blockwise martingale difference arrays. We also provide some characterizations of p-smoothable Banach spaces.

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1. INTRODUCTION

The concept of uniformly smooth space was introduced by Day [2]. Afterwards, many authors characterized p-smoothness of Banach spaces in various ways. Hoffmann-Jørgensen and Pisier [5] established an important relation between p-smoothness of a Banach space $\mathbf{E}$ and the strong law of large numbers of $\mathbf{E}$-valued martingales. In Cheng and Gan [1], p-smoothable Banach spaces were characterized in terms of atomic decompositions for Banach space valued martingales. Gan and Qiu [3] established a relation between p-smoothness of a Banach space $\mathbf{E}$ and the Hájek-Rényi inequality for $\mathbf{E}$-valued martingales. Recently, Quang and Huan [9] extended the result of Hoffmann-Jørgensen and Pisier [5] to strong martingale difference arrays.

Móricz [6] introduced the concepts of blockwise independence and blockwise quasiorthogonality. Móricz [6], Gaposhkin [4], Rosalsky and Thanh [12] showed that some properties of sequences of independent random variables can be applied to sequences consisting of independent blocks. Quang and Thanh [11] extended the Kolmogorov strong law of large numbers to blockwise martingale difference sequences. Thanh [13], Móricz et al. [7] considered dyadic blocks and established some strong laws of large numbers for random fields.

In this paper, we introduce the concepts of blockwise adapted array, blockwise martingale difference array, and establish some strong laws of large numbers for blockwise adapted arrays and blockwise martingale difference arrays, in which the results of Quang and Thanh [10, 11], Quang and Huan [9] will be generalized. The rest of the paper is organized as follows. Notation, technical definitions, and lemmas are presented in Section 2. Section 3 is devoted to our main results and their proofs.

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2. PRELIMINARIES

Throughout this paper, the symbol C will denote a generic positive constant which is not necessarily the same one in each appearance. Let a, b be real numbers, $\min\{a,b\}$ and $\max\{a,b\}$ will be denoted, respectively, by $a \wedge b$ and $a \vee b$, and $\log(a \vee 1)$ will be denoted by $\log^+ a$ (the logarithms are to the base 2). The set of all positive integers will be denoted by $\mathbb{N}$, and the set of all non-negative integers will be denoted by $\mathbb{N}_0$. The notation $(k,l) \preceq (m,n)$ means that $k \leqslant m$ and $l \leqslant n$, and the notation $(k,l) \prec (m,n)$ means that $(k,l) \preceq (m,n)$ and $(k,l) \neq (m,n)$.

Let $\{\omega(k), k \geqslant 1\}$ and $\{\nu(l), l \geqslant 1\}$ be strictly increasing sequences of positive integers with $\omega(1) = \nu(1) = 1$. For $(m,n) \in \mathbb{N}_0^2, (k,l) \in \mathbb{N}^2$, we introduce the following notations:

$$\Delta_{kl} = \{(i,j) : \big(\omega(k), \nu(l)\big) \preceq (i,j) \prec \big(\omega(k+1), \nu(l+1)\big)\},$$

$$\Delta^{(mn)} = \{(i,j) : (2^m, 2^n) \preceq (i,j) \prec (2^{m+1}, 2^{n+1})\},$$

$$\Delta_{kl}^{(mn)} = \Delta_{kl} \cap \Delta^{(mn)}, \quad \Lambda_{mn} = \{(k,l) : \Delta_{kl}^{(mn)} \neq \emptyset\},$$

$$\varphi(k,l) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \operatorname{card}(\Lambda_{ij}) I_{\Delta^{(ij)}}(k,l), \quad \psi(k,l) = \max_{(1,1) \preceq (i,j) \preceq (k,l)} \varphi(i,j),$$

where $\operatorname{card}(\Lambda_{ij})$ denotes the cardinality of the set Λ_{ij} and $I_{\Delta^{(ij)}}$ denotes the indicator function of the set $\Delta^{(ij)}$. It is easy to verify that if $\omega(k) = 2^{k-1}$, $\nu(l) = 2^{l-1}$ $\big((k,l) \in \mathbb{N}^2\big)$, then $\varphi(i,j) = \psi(i,j) = 1$ for all $(i,j) \in \mathbb{N}^2$. Further comments can be found in Quang and Thanh [10].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathbf{E}$ be a real separable Banach space, and let $\mathcal{B}(\mathbf{E})$ be the σ-algebra of all Borel sets in $\mathbf{E}$. Let $\{X_{ij}, (1,1) \preceq (m,n) \preceq (i,j) \preceq (M,N) \preceq (\infty,\infty)\}$ be a double array of $\mathbf{E}$-valued random elements, and let $\{\mathcal{F}_{ij}, (m,n) \preceq (i,j) \preceq (M,N)\}$ be a double array of non-decreasing sub-σ-algebras of $\mathcal{F}$ related to the partial order $\preceq$ on $\mathbb{N}^2$ such that X_{ij} is $\mathcal{F}_{ij}/\mathcal{B}(\mathbf{E})$-measurable for all $(m,n) \preceq (i,j) \preceq (M,N)$. Then $\{X_{ij}, \mathcal{F}_{ij}, (m,n) \preceq (i,j) \preceq (M,N)\}$ is said to be an *adapted array*.

Let $\{X_{ij}, \mathcal{F}_{ij}, (m,n) \preceq (i,j) \preceq (M,N)\}$ be an adapted array. For (i,j), $(m-1,n-1) \preceq (i,j) \preceq (M-1,N-1)$, we adopt the convention that $\mathcal{F}_{ij} = \{\emptyset, \Omega\}$ if $i = m-1$ or $j = n-1$ and set

$$\mathcal{F}_i^1 = \bigvee_{l=n}^{N} \mathcal{F}_{il} := \sigma\Big(\bigcup_{l=n}^{N} \mathcal{F}_{il}\Big), \quad \mathcal{F}_j^2 = \bigvee_{k=m}^{M} \mathcal{F}_{kj}, \quad \mathcal{F}_{ij}^- = \mathcal{F}_i^1 \bigvee \mathcal{F}_j^2.$$

The adapted array $\{X_{ij}, \mathcal{F}_{ij}, (m,n) \preceq (i,j) \preceq (M,N)\}$ is said to be a *martingale difference array* (respectively, *strong martingale difference array*) if for all $(m,n) \preceq (i,j) \preceq (M,N)$,

$$\mathbb{E}(X_{ij}|\mathcal{F}_{i-1}^1) = \mathbb{E}(X_{ij}|\mathcal{F}_{j-1}^2) = 0 \quad \text{(respectively,} \quad \mathbb{E}(X_{ij}|\mathcal{F}_{i-1,j-1}^-) = 0).$$

Clearly, a strong martingale difference array is a martingale difference array, and if $\{X_{ij}, (m,n) \preceq (i,j) \preceq (M,N)\}$ is a double array of independent zero mean random elements, then $\{X_{ij}, \mathcal{F}_{ij}, (m,n) \preceq (i,j) \preceq (M,N)\}$ is a strong martingale difference array, where $\mathcal{F}_{ij}$ is the σ-algebra generated by the family of random elements $\{X_{kl}, (m,n) \preceq (k,l) \preceq (i,j)\}$.

Let $\{X_{ij}, (i,j) \in \mathbb{N}^2\}$ be a double array of $\mathbf{E}$-valued random elements, and let $\{\mathcal{F}_{ij}, (i,j) \in \mathbb{N}^2\}$ be a double array of sub-σ-algebras of $\mathcal{F}$. The double array $\{X_{ij}, \mathcal{F}_{ij}, (i,j) \in \mathbb{N}^2\}$ is said to be a *blockwise adapted array* (respectively, *blockwise martingale difference array*, *blockwise strong martingale difference array*) with respect to the blocks $\{\Delta_{kl}, (k,l) \in \mathbb{N}^2\}$ if for each $(k,l) \in \mathbb{N}^2$, $\{X_{ij}, \mathcal{F}_{ij}, (i,j) \in \Delta_{kl}\}$ is an adapted array (respectively, martingale difference array, strong martingale difference array).

As in Pisier [8], a Banach space $\mathbf{E}$ is said to be *p-uniformly smooth* ($1 \leqslant p \leqslant 2$) if

$$\rho(\tau) = \sup\left\{\frac{||x+y|| + ||x-y||}{2} - 1, x, y \in \mathbf{E}, ||x|| = 1, ||y|| = \tau\right\} = \mathcal{O}(\tau^p).$$

A Banach space $\mathbf{E}$ is said to be *p-smoothable* if there exists an equivalent norm under which $\mathbf{E}$ is p-uniformly smooth.

It is well known that every real separable Banach space is 1-smoothable and the real line (the same as any Hilbert space) is 2-smoothable. If a real separable Banach space is p-smoothable for some $1 \leqslant p \leqslant 2$, then it is r-smoothable for all $r \in [1, p]$.

A double array of random elements $\{X_{ij}, (i,j) \in \mathbb{N}^2\}$ is said to be *stochastically dominated* by a random element X if there exists a constant C $(0 < C < \infty)$ such that

$$\mathbb{P}\{||X_{ij}|| > t\} \leqslant C\mathbb{P}\{||X|| > t\}, \quad t \geqslant 0, (i,j) \in \mathbb{N}^2. \tag{2.1}$$

This condition is, of course, automatic with $X = X_{11}$ and $C = 1$ if $\{X_{ij}, (i,j) \in \mathbb{N}^2\}$ is a double array of identically distributed random elements.

Lemma 2.1. (Quang and Huan [9]). *Let $\{X_{ij}, (i,j) \in \mathbb{N}^2\}$ be a double array of (real-valued) random variables which are stochastically dominated by a random variable X. If $\mathbb{E}(|X|^q \log^+ |X|) < \infty$, for some $q > 0$, then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\mathbb{E}\big(|X_{ij}|^r I(|X_{ij}| > (ij)^{\frac{1}{q}})\big)}{(ij)^{\frac{r}{q}}} < \infty \quad \text{for all} \quad 0 < r < q, \tag{2.2}$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\mathbb{E}\big(|X_{ij}|^p I(|X_{ij}| \leqslant (ij)^{\frac{1}{q}})\big)}{(ij)^{\frac{p}{q}}} < \infty \quad \text{for all} \quad p > q. \tag{2.3}$$

The next lemma is a generalization of Lemma 2.1 of Quang and Thanh [10].

Lemma 2.2. *Let $\Phi_1(.)$, $\Phi_2(.)$ be positive nondecreasing unbounded functions on $(0, \infty)$, and let $\{x_{ij}, (i,j) \in \mathbb{N}_0^2\}$ be an array of real numbers such that*

$$\lim_{i \vee j \to \infty} x_{ij} = 0.$$

Then the condition

$$\sup_{(m,n)\in\mathbb{N}_0^2} \frac{1}{\Phi_1(2^m)\Phi_2(2^n)} \sum_{i=0}^{m}\sum_{j=0}^{n} \Phi_1(2^{i+1})\Phi_2(2^{j+1}) \leqslant C < \infty \tag{2.4}$$

implies

$$\lim_{m \vee n \to \infty} \frac{1}{\Phi_1(2^m)\Phi_2(2^n)} \sum_{i=0}^{m}\sum_{j=0}^{n} \Phi_1(2^{i+1})\Phi_2(2^{j+1}) x_{ij} = 0.$$

Proof. For every $\varepsilon > 0$, there exists a positive integer n_1 such that for all $i \vee j \geqslant n_1$,

$$|x_{ij}| \leqslant \frac{\varepsilon}{2C}. \tag{2.5}$$

On the other hand, since $\Phi_1(.)$ and $\Phi_2(.)$ are positive nondecreasing unbounded functions on $(0, \infty)$, there exists a positive integer $m_1 > n_1$ such that for all $m \vee n \geqslant m_1$,

$$\left| \frac{1}{\Phi_1(2^m)\Phi_2(2^n)} \sum_{i\vee j < n_1, (0,0) \preceq (i,j) \preceq (m,n)} \Phi_1(2^{i+1})\Phi_2(2^{j+1}) x_{ij} \right| < \varepsilon/2. \tag{2.6}$$

Then by (2.4)–(2.6), we get

$$\left| \frac{1}{\Phi_1(2^m)\Phi_2(2^n)} \sum_{i=0}^{m}\sum_{j=0}^{n} \Phi_1(2^{i+1})\Phi_2(2^{j+1}) x_{ij} \right|$$

$$\leqslant \left| \frac{1}{\Phi_1(2^m)\Phi_2(2^n)} \sum_{i\vee j < n_1, (0,0) \preceq (i,j) \preceq (m,n)} \Phi_1(2^{i+1})\Phi_2(2^{j+1}) x_{ij} \right|$$

$$+\left|\frac{1}{\Phi_1(2^m)\Phi_2(2^n)}\sum_{i\vee j\geqslant n_1,(0,0)\preceq(i,j)\preceq(m,n)}\Phi_1(2^{i+1})\Phi_2(2^{j+1})x_{ij}\right|\leqslant\frac{\varepsilon}{2}+C\frac{\varepsilon}{2C}=\varepsilon.$$

This completes the proof of the lemma. □

Lemma 2.3 (Pisier [8]). *Let* $\mathbf{E}$ *be a real separable* p*-smoothable Banach space* $(1\leqslant p\leqslant 2)$. *Then, for all* $r\geqslant 1$, *there exists a positive constant* C *such that for all martingales* $\Big\{\sum_{i=1}^n X_i,\mathcal{F}_n,$ $n\geqslant 1\Big\}$ *with values in* $\mathbf{E}$, *we have*

$$\mathbb{E}\sup_{n\geqslant 1}\left|\left|\sum_{i=1}^n X_i\right|\right|^r\leqslant C\mathbb{E}\left(\sum_{n=1}^\infty||X_n||^p\right)^{\frac{r}{p}}.$$

The next corollary is an improvement of Lemma 1.1 of Quang and Huan [9].

Lemma 2.4. *Let* $\mathbf{E}$ *be a real separable* p*-smoothable Banach space* $(1\leqslant p\leqslant 2)$. *Then there exists a positive constant* C *such that for all martingale difference arrays* $\{X_{ij},\mathcal{F}_{ij},(1,1)\preceq(i,j)\preceq(m,n)\}$,

$$\mathbb{E}\max_{(1,1)\preceq(k,l)\preceq(m,n)}\left|\left|\sum_{i=1}^k\sum_{j=1}^l X_{ij}\right|\right|^p\leqslant C\mathbb{E}\sum_{i=1}^m\sum_{j=1}^n||X_{ij}||^p. \tag{2.7}$$

Proof. We easily obtain (2.7) in the case $p=1$. Now we consider the case $1<p\leqslant 2$ and $m\wedge n\geqslant 2$. Set

$$S_{kl}=\sum_{i=1}^k\sum_{j=1}^l X_{ij},\quad Y_l=\max_{1\leqslant k\leqslant m}||S_{kl}||.$$

Then for $k,l,(1,1)\preceq(k,l)\preceq(m,n)$, we have

$$\mathbb{E}(S_{kl}|\mathcal{F}^2_{l-1})=\mathbb{E}(S_{k,l-1}|\mathcal{F}^2_{l-1})+\sum_{i=1}^k\mathbb{E}(X_{il}|\mathcal{F}^2_{l-1})=S_{k,l-1}.$$

This means that for each k $(1\leqslant k\leqslant m)$, $\{S_{kl},\mathcal{F}^2_l,1\leqslant l\leqslant n\}$ is a martingale, and so $\{||S_{kl}||,\mathcal{F}^2_l,1\leqslant l\leqslant n\}$ is a nonnegative submartingale. It is easy to show that $\{Y_l,\mathcal{F}^2_l,1\leqslant l\leqslant n\}$ is a nonnegative submartingale. Applying Doob's inequality, we obtain

$$\mathbb{E}\max_{(1,1)\preceq(k,l)\preceq(m,n)}||S_{kl}||^p=\mathbb{E}\big(\max_{1\leqslant l\leqslant n}Y_l\big)^p\leqslant C\mathbb{E}Y_n^p. \tag{2.8}$$

On the other hand, we have that $\{S_{kn},\mathcal{F}^1_k,1\leqslant k\leqslant m\}$ is a martingale. It follows from Lemma 2.3 that

$$\mathbb{E}Y_n^p=\mathbb{E}\max_{1\leqslant k\leqslant m}||S_{kn}||^p\leqslant C\sum_{k=1}^m\mathbb{E}\left|\left|\sum_{j=1}^n X_{kj}\right|\right|^p. \tag{2.9}$$

For each k $(1\leqslant k\leqslant m)$, we again have that $\Big\{\sum_{j=1}^l X_{kj},\mathcal{F}^2_l,1\leqslant l\leqslant n\Big\}$ is a martingale. Thus,

$$\mathbb{E}\left|\left|\sum_{j=1}^n X_{kj}\right|\right|^p\leqslant C\sum_{l=1}^n\mathbb{E}||X_{kl}||^p. \tag{2.10}$$

Combining (2.8)–(2.10) yields (2.7).

Next, if $1<p\leqslant 2$ and $m\wedge n=1$, then (2.7) follows as in the case $1<p\leqslant 2$ and $m\wedge n\geqslant 2$. □

3. MAIN RESULTS

With the preliminaries accounted for, the first main result may be established. The following theorem extends the Kolmogorov strong law of large numbers to blockwise martingale difference arrays. It also generalizes some results of Quang and Thanh [10, 11]. The assertion (iii) of Theorem 3.1 is inspired by Theorem 1 of Móricz et al. [7].

Theorem 3.1. *Let* $\mathbf{E}$ *be a real separable Banach space and* $1 \leqslant p \leqslant 2$. *Then the following four statements are equivalent:*

(i) The Banach space $\mathbf{E}$ *is* p*-smoothable.*

(ii) For every blockwise martingale difference array $\{X_{ij}, \mathcal{F}_{ij}, (i,j) \in \mathbb{N}^2\}$ *with respect to the blocks* $\{\Delta_{kl}, (k,l) \in \mathbb{N}^2\}$ *and for any two functions* $\Phi_1(.)$, $\Phi_2(.)$ *which are positive nondecreasing unbounded functions on* $(0,\infty)$ *satisfying (2.4), the condition*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\big(\varphi(i,j)\big)^{p-1}}{\big(\Phi_1(i)\Phi_2(j)\big)^{p}}\mathbb{E}||X_{ij}||^p < \infty \tag{3.1}$$

implies

$$\frac{1}{\Phi_1(m)\Phi_2(n)}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij} \to 0 \quad a.s. \quad \text{as} \quad m \vee n \to \infty. \tag{3.2}$$

(iii) For every blockwise strong martingale difference array $\{X_{ij}, \mathcal{F}_{ij}, (i,j) \in \mathbb{N}^2\}$ *with respect to the blocks* $\{\Delta_{kl}, (k,l) \in \mathbb{N}^2\}$ *and for any two functions* $\Phi_1(.)$, $\Phi_2(.)$ *which are positive nondecreasing unbounded functions on* $(0,\infty)$ *such that*

$$\limsup_{m\to\infty}\frac{\Phi_1(2^{m+1})}{\Phi_1(2^m)} < \infty, \quad \liminf_{m\to\infty}\frac{\Phi_1(2^{m+1})}{\Phi_1(2^m)} > 1, \tag{3.3}$$

$$\limsup_{n\to\infty}\frac{\Phi_2(2^{n+1})}{\Phi_2(2^n)} < \infty, \quad \liminf_{n\to\infty}\frac{\Phi_2(2^{n+1})}{\Phi_2(2^n)} > 1, \tag{3.4}$$

the condition (3.1) implies (3.2).

(iv) For every strong martingale difference array $\{X_{ij}, \mathcal{F}_{ij}, (i,j) \in \mathbb{N}^2\}$, *the condition*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\mathbb{E}||X_{ij}||^p}{(ij)^p} < \infty$$

implies

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=i}^{n}X_{ij} \to 0 \quad a.s. \quad \text{as} \quad m \vee n \to \infty.$$

Proof. (i) $\Rightarrow$ (ii): Assume that the Banach space $\mathbf{E}$ is p-smoothable. For $(m,n) \in \mathbb{N}_0^2$, $(k,l) \in \Lambda_{mn}$, we set

$$r_k^{(m)} = \min\{r : r \in [\omega(k), \omega(k+1)] \cap [2^m, 2^{m+1}]\},$$
$$s_l^{(n)} = \min\{s : s \in [\nu(l), \nu(l+1)] \cap [2^n, 2^{n+1}]\},$$
$$\gamma_{kl}^{(mn)} = \max_{(u,v)\in\Delta_{kl}^{(mn)}}\left|\left|\sum_{i=r_k^{(m)}}^{u}\sum_{j=s_l^{(n)}}^{v}X_{ij}\right|\right|,$$
$$\gamma_{mn} = \frac{1}{\Phi_1(2^{m+1})\Phi_2(2^{n+1})}\sum_{(k,l)\in\Lambda_{mn}}\gamma_{kl}^{(mn)}.$$

Since $\{X_{ij}, \mathcal{F}_{ij}, (i,j) \in \Delta_{kl}\}$ is a martingale difference array, we may use the "tower property" of conditional expectation to show that $\{X_{ij}, \mathcal{F}_{ij}, (i,j) \in \Delta_{kl}^{(mn)}\}$ is also a martingale difference array. Then by the C_r inequality and Lemma 2.4, we have

$$\begin{aligned}
\mathbb{E}\gamma_{mn}^p &= \frac{1}{\big(\Phi_1(2^{m+1})\Phi_2(2^{n+1})\big)^p}\mathbb{E}\left(\sum_{(k,l)\in\Lambda_{mn}} \gamma_{kl}^{(mn)}\right)^p \\
&\leqslant \frac{\big(\mathrm{card}(\Lambda_{mn})\big)^{p-1}}{\big(\Phi_1(2^{m+1})\Phi_2(2^{n+1})\big)^p} \sum_{(k,l)\in\Lambda_{mn}} \mathbb{E}\big(\gamma_{kl}^{(mn)}\big)^p \\
&\leqslant C\frac{\big(\mathrm{card}(\Lambda_{mn})\big)^{p-1}}{\big(\Phi_1(2^{m+1})\Phi_2(2^{n+1})\big)^p} \sum_{(k,l)\in\Lambda_{mn}} \sum_{(i,j)\in\Delta_{kl}^{(mn)}} \mathbb{E}||X_{ij}||^p \\
&\leqslant C \sum_{(2^m,2^n)\preceq(i,j)\prec(2^{m+1},2^{n+1})} \frac{\big(\varphi(i,j)\big)^{p-1}}{\big(\Phi_1(i)\Phi_2(j)\big)^p}\mathbb{E}||X_{ij}||^p.
\end{aligned}$$

It thus follows from (3.1) that $\sum_{m=0}^\infty \sum_{n=0}^\infty \mathbb{E}\gamma_{mn}^p < \infty$. Applying the Markov inequality and the Borel-Cantelli lemma, we obtain

$$\gamma_{mn} \to 0 \quad a.s. \quad \text{as} \quad m \vee n \to \infty.$$

Then by Lemma 2.2, we see

$$\frac{1}{\Phi_1(2^m)\Phi_2(2^n)} \sum_{k=0}^m \sum_{l=0}^n \sum_{(i,j)\in I_{kl}} \gamma_{ij}^{(kl)} \to 0 \quad a.s. \quad \text{as } m \vee n \to \infty. \tag{3.5}$$

Next, for $(k,l) \in \mathbb{N}^2$, let $(m,n) \in \mathbb{N}_0^2$ be such that $(k,l) \in \Delta^{(mn)}$. Then

$$\frac{1}{\Phi_1(k)\Phi_2(l)} \left|\left|\sum_{i=1}^k \sum_{j=1}^l X_{ij}\right|\right| \leqslant \frac{1}{\Phi_1(2^m)\Phi_2(2^n)} \sum_{i=0}^m \sum_{j=0}^n \sum_{(\lambda,\mu)\in I_{ij}} \gamma_{\lambda\mu}^{(ij)}. \tag{3.6}$$

Combining (3.5) and (3.6) yields (3.2).

(ii) $\Rightarrow$ (iii): It suffices to show that the conditions (3.3) and (3.4) imply (2.4). Assume that (3.3) and (3.4) hold. First, we prove that there exists a positive constant C such that for all $m \geqslant 0$,

$$\Phi_1(2^{m+1}) - \Phi_1(2^m) \geqslant C\Phi_1(2^{m+1}). \tag{3.7}$$

This will be done by reductio ad absurdum. Let us assume that (3.7) fails, then for any $k \geqslant 1$, there exists a non-negative integer m_k such that

$$\Phi_1(2^{m_k+1}) - \Phi_1(2^{m_k}) < \frac{1}{k}\Phi_1(2^{m_k+1})$$

implying

$$\liminf_{k\to\infty} \frac{\Phi_1(2^{m_k+1})}{\Phi_1(2^{m_k})} \leqslant 1 + \liminf_{k\to\infty} \frac{1}{k}\frac{\Phi_1(2^{m_k+1})}{\Phi_1(2^{m_k})}. \tag{3.8}$$

On the other hand, by (3.3), we obtain

$$\liminf_{k\to\infty} \Big(\frac{1}{k}\frac{\Phi_1(2^{m_k+1})}{\Phi_1(2^{m_k})}\Big) = 0.$$

It follows from (3.8) that

$$\liminf_{m\to\infty} \frac{\Phi_1(2^{m+1})}{\Phi_1(2^m)} \leqslant 1,$$

which contradicts (3.3), and hence (3.7) holds. Thus,

$$\frac{1}{\Phi_1(2^m)}\sum_{i=0}^{m}\Phi_1(2^{i+1}) \leqslant \frac{1}{C}\frac{\Phi_1(2^{m+1})}{\Phi_1(2^m)},$$

and so (3.3) ensures that

$$\sup_{m\geqslant 0}\frac{1}{\Phi_1(2^m)}\sum_{i=0}^{m}\Phi_1(2^{i+1}) < \infty. \tag{3.9}$$

By the same method, we can prove the following inequality

$$\sup_{n\geqslant 0}\frac{1}{\Phi_2(2^n)}\sum_{j=0}^{n}\Phi_2(2^{j+1}) < \infty. \tag{3.10}$$

Combining (3.9) and (3.10) yields (2.4).

(iii) $\Rightarrow$ (iv): Clearly, if $\{X_{ij}, \mathcal{F}_{ij}, (i,j) \in \mathbb{N}^2\}$ is a strong martingale difference array, then it is a blockwise strong martingale difference array with respect to the blocks $\{\Delta^{(mn)}, (m,n) \in \mathbb{N}_0^2\}$. Therefore, the assertion (iv) follows immediately from the assertion (iii) by choosing $\Phi_1(x) = \Phi_2(x) = x$.

(iv) $\Rightarrow$ (i): Let $(X_j, \mathcal{F}_j, j \geqslant 1)$ be an arbitrary martingale difference sequence such that

$$\sum_{j=1}^{\infty}\frac{\mathbb{E}||X_j||^p}{j^p} < \infty.$$

For $(i,j) \in \mathbb{N}^2$, set

$$X_{ij} = X_j \quad \text{if} \quad i = 1, \quad X_{ij} = 0 \quad \text{if} \quad i > 1, \quad \mathcal{F}_{ij} = \mathcal{F}_j.$$

Then $\{X_{ij}, \mathcal{F}_{ij}, (i,j) \in \mathbb{N}^2\}$ is a strong martingale difference array. By the same argument as in the proof of Theorem 2.1 of Quang and Huan [9], we can show that $\mathbf{E}$ is p-smoothable. □

The following theorem is a variation of Theorem 3.1. The proof technique is similar to that of Theorem 3.1.

Theorem 3.2. *Let $\mathbf{E}$ be a real separable Banach space and $1 \leqslant p \leqslant 2$. Then the following two statements are equivalent:*

(i) The Banach space $\mathbf{E}$ is p-smoothable.

(ii) For every blockwise martingale difference array $\{X_{ij}, \mathcal{F}_{ij}, (i,j) \in \mathbb{N}^2\}$ with respect to the blocks $\{\Delta_{kl}, (k,l) \in \mathbb{N}^2\}$ and for any two functions $\Phi_1(.)$, $\Phi_2(.)$ which are positive nondecreasing unbounded functions on $(0,\infty)$ satisfying (2.4), the condition

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\mathbb{E}||X_{ij}||^p}{\big(\Phi_1(i)\Phi_2(j)\big)^p} < \infty \tag{3.11}$$

implies

$$\frac{1}{\Phi_1(m)\Phi_2(n)\big(\psi(m,n)\big)^{\frac{p-1}{p}}}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij} \to 0 \quad a.s. \quad \text{as} \quad m\vee n \to \infty. \tag{3.12}$$

Proof. (i) $\Rightarrow$ (ii): Assume that the Banach space $\mathbf{E}$ is p-smoothable. For $(m,n) \in \mathbb{N}_0^2$, $(k,l) \in \Lambda_{mn}$, we define $\gamma_{kl}^{(mn)}$ as in the proof of Theorem 3.1 and set

$$\gamma_{mn} = \frac{1}{\Phi_1(2^{m+1})\Phi_2(2^{n+1})\big(\psi(2^m,2^n)\big)^{\frac{p-1}{p}}}\sum_{(k,l)\in\Lambda_{mn}}\gamma_{kl}^{(mn)}.$$

By the C_r inequality and Lemma 2.4, we have

$$\mathbb{E}\gamma_{mn}^p = \frac{1}{\big(\Phi_1(2^{m+1})\Phi_2(2^{n+1})\big)^p\big(\psi(2^m,2^n)\big)^{p-1}}\mathbb{E}\left(\sum_{(k,l)\in\Lambda_{mn}}\gamma_{kl}^{(mn)}\right)^p$$

$$\leqslant \frac{\big(\mathrm{card}(\Lambda_{mn})\big)^{p-1}}{\big(\Phi_1(2^{m+1})\Phi_2(2^{n+1})\big)^p\big(\psi(2^m,2^n)\big)^{p-1}}\sum_{(k,l)\in\Lambda_{mn}}\mathbb{E}\big(\gamma_{kl}^{(mn)}\big)^p$$
$$\leqslant C\sum_{(2^m,2^n)\preceq(i,j)\prec(2^{m+1},2^{n+1})}\frac{\mathbb{E}||X_{ij}||^p}{\big(\Phi_1(i)\Phi_2(j)\big)^p}.$$

Then $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\mathbb{E}\gamma_{mn}^p<\infty$, and so $\gamma_{mn}\to 0 a.s.$ as $m\vee n\to\infty$. By the same argument as in the proof of the implication ((i) $\Rightarrow$ (ii)) of Theorem 3.1, we have (3.12).

(ii) $\Rightarrow$ (i): This implication follows immediately from the implication ((iv) $\Rightarrow$ (i)) of Theorem 3.1. □

The following corollary follows immediately from Theorem 3.2 and is a generalization of the implication ((i) $\Rightarrow$ (ii)) of Theorem 2.1 of Quang and Huan [9].

Corollary 3.3. *Let* $\mathbf{E}$ *be a real separable* p*-smoothable Banach space* $(1\leqslant p\leqslant 2)$*, let* $\{X_{ij},\mathcal{F}_{ij},(i,j)\in\mathbb{N}^2\}$ *be blockwise martingale difference array with respect to the blocks* $\{\Delta_{kl},(k,l)\in\mathbb{N}^2\}$*, and let* α*,* β *be positive real numbers. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\mathbb{E}||X_{ij}||^p}{i^{\alpha p}j^{\beta p}}<\infty,$$

then

$$\frac{1}{m^\alpha n^\beta\big(\psi(m,n)\big)^{\frac{p-1}{p}}}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}\to 0 \quad a.s. \quad \text{as} \quad m\vee n\to\infty.$$

We now extend the Marcinkiewicz–Zygmund strong law of large numbers to blockwise adapted arrays. The proof is inspired by the previous work of Quang and Huan [9].

Theorem 3.4. Let $\mathbf{E}$ be a real separable p-smoothable Banach space $(1<p\leqslant 2)$, and let $\{X_{ij},\mathcal{F}_{ij},(i,j)\in\mathbb{N}^2\}$ be a blockwise adapted array with respect to the blocks $\{\Delta_{kl},(k,l)\in\mathbb{N}^2\}$. Suppose that $\{X_{ij},(i,j)\in\mathbb{N}^2\}$ is stochastically dominated by a random element X with

$$\mathbb{E}(||X||^q\log^+||X||)<\infty \tag{3.13}$$

for some $q\in(1,p)$. Then

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\big(X_{ij}-\mathbb{E}(X_{ij}|\mathcal{F}^-_{i-1,j-1})\big)}{(mn)^{1/q}\big(\psi(m,n)\big)^{\frac{p-1}{p}}}\to 0 \quad a.s. \quad \text{as} \quad m\vee n\to\infty. \tag{3.14}$$

Proof. For $(i,j)\in\mathbb{N}^2$, set

$$X'_{ij}=X_{ij}I(||X_{ij}||\leqslant(ij)^{\frac{1}{q}}),\quad X''_{ij}=X_{ij}I(||X_{ij}||>(ij)^{\frac{1}{q}}).$$

Then

$$\mathbb{E}\big(X'_{ij}-\mathbb{E}(X'_{ij}|\mathcal{F}^-_{i-1,j-1})|\mathcal{F}^1_{i-1}\big)=0,\qquad \mathbb{E}\big(X'_{ij}-\mathbb{E}(X'_{ij}|\mathcal{F}^-_{i-1,j-1})|\mathcal{F}^2_{j-1}\big)=0,$$

and so $\{X'_{ij}-\mathbb{E}(X'_{ij}|\mathcal{F}^-_{i-1,j-1}),\mathcal{F}_{ij},(i,j)\in\mathbb{N}^2\}$ is a blockwise martingale difference array with respect to the blocks $\{\Delta_{kl},(k,l)\in\mathbb{N}^2\}$, and

$$\mathbb{E}||X'_{ij}-\mathbb{E}(X'_{ij}|\mathcal{F}^-_{i-1,j-1})||^p\leqslant\mathbb{E}\big(||X'_{ij}||+\mathbb{E}(||X'_{ij}|||\mathcal{F}^-_{i-1,j-1})\big)^p\leqslant 2^p\mathbb{E}||X'_{ij}||^p. \tag{3.15}$$

By (3.15) and (2.3), we get

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\mathbb{E}||X'_{ij}-\mathbb{E}(X'_{ij}|\mathcal{F}^-_{i-1,j-1})||^p}{(ij)^{\frac{p}{q}}}\leqslant 2^p\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\mathbb{E}||X'_{ij}||^p}{(ij)^{\frac{p}{q}}}<\infty.$$

By using Corollary 3.3, we get

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\big(X'_{ij}-\mathbb{E}(X'_{ij}|\mathcal{F}^{-}_{i-1,j-1})\big)}{(mn)^{1/q}\big(\psi(m,n)\big)^{\frac{p-1}{p}}}\to 0 \quad a.s. \quad \text{as} \quad m\vee n\to\infty. \tag{3.16}$$

Next, we again have that $\{X''_{ij}-\mathbb{E}(X''_{ij}|\mathcal{F}^{-}_{i-1,j-1}),\mathcal{F}_{ij},(i,j)\in\mathbb{N}^2\}$ is a blockwise martingale difference array with respect to the blocks $\{\Delta_{kl},(k,l)\in\mathbb{N}^2\}$. Similarly, by (2.2), for all $r\in[1,q)$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\mathbb{E}||X''_{ij}-\mathbb{E}(X''_{ij}|\mathcal{F}^{-}_{i-1,j-1})||^r}{(ij)^{\frac{r}{q}}}\leqslant 2^r\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\mathbb{E}||X''_{ij}||^r}{(ij)^{\frac{r}{q}}}<\infty.$$

Thus,

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\big(X''_{ij}-\mathbb{E}(X''_{ij}|\mathcal{F}^{-}_{i-1,j-1})\big)}{(mn)^{1/q}\big(\psi(m,n)\big)^{\frac{r-1}{r}}}\to 0 \quad a.s. \quad \text{as} \quad m\vee n\to\infty,$$

and so

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\big(X''_{ij}-\mathbb{E}(X''_{ij}|\mathcal{F}^{-}_{i-1,j-1})\big)}{(mn)^{1/q}\big(\psi(m,n)\big)^{\frac{p-1}{p}}}\to 0 \quad a.s. \quad \text{as} \quad m\vee n\to\infty. \tag{3.17}$$

Combining (3.16) and (3.17) yields (3.14). The proof is completed. □

Corollary 3.5. *Let* $\mathbf{E}$ *be a real separable* p*-smoothable Banach space* $(1<p\leqslant 2)$*, and let* $\{X_{ij},\mathcal{F}_{ij},(i,j)\in\mathbb{N}^2\}$ *be a blockwise strong martingale difference array with respect to the blocks* $\{\Delta_{kl},(k,l)\in\mathbb{N}^2\}$ *such that* $\{X_{ij},(i,j)\in\mathbb{N}^2\}$ *is stochastically dominated by a random element* X*. If (3.13) holds for some* $q\in(1,p)$*, then*

$$\frac{1}{(mn)^{1/q}\big(\psi(m,n)\big)^{\frac{p-1}{p}}}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}\to 0 \quad a.s. \quad \text{as} \quad m\vee n\to\infty.$$

The next corollary follows immediately from Corollary 3.5 and is an extension of Theorem 2.4 (i) of Quang and Huan [9].

Corollary 3.6. *Let* $\mathbf{E}$ *be a real separable* p*-smoothable Banach space* $(1<p\leqslant 2)$*, and let* $\{X_{ij},\mathcal{F}_{ij},(i,j)\in\mathbb{N}^2\}$ *be a blockwise strong martingale difference array with respect to the blocks* $\{\Delta^{(mn)},(m,n)\in\mathbb{N}_0^2\}$ *such that* $\{X_{ij},(i,j)\in\mathbb{N}^2\}$ *is stochastically dominated by a random element* X*. If (3.13) holds for some* $q\in(1,p)$*, then*

$$\frac{1}{(mn)^{1/q}}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}\to 0 \quad a.s. \quad \text{as} \quad m\vee n\to\infty.$$

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