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On Negatively Associated Random Variables[1]

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Abstract—In this note we show how Chebyshev's other inequality can be applied to construct negatively associated random variables and to lead to a simplification of proofs for some known results on such random variables. In addition some improvements of basic properties of negatively associated random variables are provided.

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1. INTRODUCTION AND SOME GENERALIZATIONS OF THE BASIC PROPERTIES OF NEGATIVELY ASSOCIATED RANDOM VARIABLES

Negatively associated random variables play a significant role in the theory of probability. Such random variables inherit some properties of independent random variables. It is important to note that independent random variables are negatively associated. In fact there exists a tight connection between these two notions. This connection helps to extend various asymptotic results on sums of independent random variables to negatively associated random variables. A survey on negatively associated random variables can be found in the recent monograph by Bulinski and Shashkin [1]. Despite substantial progress in the theory of negatively associated random variables, there are still many unsolved problems. For example, the construction of such random variables is an important problem. In the second part on the paper we demonstrate the usefulness of *Chebyshev's other inequality* [2] p. 716–719, in a construction of negatively associated random variables. We also show that the usage of this inequality can significantly simplify proofs of many known results. In the first part on the paper we discuss basic properties of negatively associated random variables. Note that some of these properties are mentioned without proofs in the literature. We provide rigorous proofs of these results, sometimes in a more general form.

In what follows we assume that all random variables under consideration are defined on a probability space $(\Omega, \mathcal{F}, P)$. We use standard notation; in particular, I_A denotes the indicator function of a set $A \subseteq \Omega$, and by symbols $\mathbb{R}$ and $\mathbb{N}$ we denote the set of all real numbers and the set of positive integers. The notion of negatively associated random variables was introduced by Alam and Saxena [3] and then thoroughly studied by Joag-Dev and Proschan [4].

1. Definition. Random variables $X_1, \ldots, X_n, n \geq 2$, are said to be *negatively associated*, if the inequality

$$\operatorname{cov}(f(X_{i_1}, \ldots, X_{i_k}), g(X_{j_1}, \ldots, X_{j_m})) \leq 0 \qquad (1)$$

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holds for all nonempty disjoint subsets $\{i_1,\dots,i_k\}$ and $\{j_1,\dots,j_m\}$ of the set $\{1,\dots,n\}$ and for all bounded coordinatewise increasing Borel functions $f(x_{i_1},\dots,x_{i_k})$ and $g(x_{j_1},\dots,x_{j_m})$, $x_1,\dots,x_n \in \mathbb{R}, k,m\in\mathbb{N},\ k+m\le n$. Random variables $X_t, t\in T$, with an arbitrary set T of real numbers are called negatively associated, if for any numbers $t_1,\dots,t_n\in T$, $n\ge 2$, random variables $X_{t_1},\dots,X_{t_n}$ are negatively associated.

We extend the definition by replacing the condition of coordinatewise increasing by coordinatewise decreasing: If f and g are coordinatewise decreasing functions, then the functions $-f$ and $-g$ are coordinatewise increasing and covariance (1) coincides with the covariance for $-f$ and $-g$. It follows directly from the definition that any subset (of two or more random variables) of negatively associated random variables is a set of negatively associated random variables itself.

The next important theorem was formulated without proof in Joag-Dev and Proschan [4].

2. Theorem. *Increasing functions defined on disjoint subsets of a set of negatively associated random variables are negatively associated.*

Proof. Let $f_r(x_{r,1},\dots x_{r,m_r}), r=1,\dots,n$, be real coordinatewise increasing functions of real arguments $x_{r,1},\dots,x_{r,m_k}\in\mathbb{R}$, $m_1,\dots,m_n\in\mathbb{N}$. Let moreover $X_{r,1},\dots,X_{r,m_r}$, $r=1,\dots,n$, be disjoint subsets of a set of negatively associated random variables. Further, let f and g be coordinatewise nondecreasing functions mentioned in the Definition 1. The composite functions

$$f'(x_{1,1},\dots,x_{k,m_k}) = f(f_1(x_{1,1},\dots,x_{1,r_1}),\dots,f_k(x_{k,1},\dots,x_{k,r_k})),$$

$$g'(x_{k+1,1},\dots,x_{n,m_n}) = g(f_{k+1}(x_{k+1,1},\dots,x_{k+1,m_{k+1}}),\dots,f_n(x_{n,1},\dots,x_{n,m_n}))$$

are bounded. Moreover, as compositions of coordinatewise increasing functions, they are coordinatewise increasing. Denote

$$Y_r = f_r(X_{r,1},\dots,X_{r,m_r}), r=1,\dots,n.$$

Because the random variables $X_{1,1},\dots,X_{n,m_n}$ are negatively associated, we obtain

$$\begin{aligned}&\operatorname{cov}(f(Y_1,\dots,Y_k), g(Y_{k+1},\dots,Y_n))\\ &= \operatorname{cov}(f'(X_{1,1}\dots,X_{k,m_k}), g'(X_{k+1,1},\dots,X_{n,m_n}))\le 0.\end{aligned}$$

This means that $Y_1,\dots,Y_n$ are negatively associated. The theorem is proved.

Note that if we truncate negatively associated random variables, then we may loose the negative association property because indicator function may be not monotone. The next corollary provides an important technique of *monotone truncation* that preserves the negatively association property.

3. Corollary. *Let $X_n, n\in\mathbb{N}$, be negatively associated random variables. Then for any $a_n,b_n\in\mathbb{R}$, $a_n\le b_n$, random variables $Y_n=f_n(X_n)$, where*

$$f_n(x) = a_n I_{(-\infty,a_n)}(x) + xI_{[a_n,b_n]}(x) + b_n I_{(b_n,\infty)}(x), x\in\mathbb{R}, n\in\mathbb{N},$$

are negatively associated.

Proof. The function $f_n(x), x\in\mathbb{R}$, is increasing. By Theorem 2 random variables $Y_n=f_n(X_n), n\in\mathbb{N}$, are negatively associated. The corollary is proved.

4. Corollary. *Let $X_n, n\in\mathbb{N}$, be negatively associated random variables. Then each of the sequences $\{X_n^+\}_{n\ge1}$ and $\{X_n^-\}_{n\ge1}$ consists of negatively associated random variables.*

Proof. Functions $f(x)=xI_{[0,\infty)}(x)$ and $g(x)=-xI_{(-\infty,0]}(x)$, $x\in\mathbb{R}$, are non-decreasing and non-increasing respectively. By Theorem 2 and by the remark after Definition 1 random variables $X_n^+ = f(X_n), n\in\mathbb{N}$, as well as $X_n^-=g(X_n), n\in\mathbb{N}$, are negatively associated. The corollary is proved.

The next theorem provides important properties of negatively associated random variables. In some sense these properties resemble the property of independence. With the help of this theorem many known weak and strong laws of large numbers for independent random variables can be extended to negatively associated random variables. But first we need to introduce the following notion.

5. Definition. Random variables $X_1,\dots,X_n$, $n\ge2$, are said to be *negatively dependent*, if for any numbers $x_1,\dots,x_n\in\mathbb{R}$ we have that

$$P\{\cap_{k=1}^n\{X_k\le x_k\}\}\le\prod_{k=1}^n P\{X_k\le x_k\},$$

$$P\{\cap_{k=1}^{n}\{X_k > x_k\}\} \le \prod_{k=1}^{n} P\{X_k > x_k\}.$$

Random variables $X_t, t \in T$, with an arbitrary set T of real numbers are called negatively dependent, if for any numbers $t_1, \dots, t_n \in T, n \ge 2$, random variables $X_{t_1}, \dots, X_{t_n}$ are negatively dependent.

Previously this notion was investigated by Lehman [6] under the name of *negatively quadrant dependent* random variables. Joag-Dev and Proschan [4] showed that negatively associated random variables are negatively dependent, but not vice versa.

We deduce the next Theorem 6 from Theorem 2. Nevertheless it should be regarded as a new result. Particular cases (3) and (4) of the general inequalities (2) are discussed in Joag-Dev and Proschan [4], and in Taylor et al. [5].

6. Theorem. *Let $X_1, \dots, X_n$ be negatively associated random variables. Then for any numbers $x_1, \dots, x_n \in \mathbb{R}$ and for any set $A \subseteq \{1, \dots, n\}$ the following inequalities hold*

$$P\{\cap_{k\in A}\{X_k \le x_k\} \cap \cap_{k\notin A}\{X_k < x_k\}\} \le \prod_{k\in A} P\{X_k \le x_k\} \prod_{k\notin A} P\{X_k < x_k\} \qquad (2)$$

$$P\{\cap_{k\in A}\{X_k \ge x_k\} \cap \cap_{k\notin A}\{X_k > x_k\}\} \le \prod_{k\in A} P\{X_k \ge x_k\} \prod_{k\notin A} P\{X_k > x_k\}.$$

Proof. We start with the proof of two particular cases of the first inequality in (2):

$$P\{X_k \le x_k, k = 1, \dots, n\} \le \prod_{k=1}^{n} P\{X_k \le x_k\}, \qquad (3)$$

$$P\{X_k < x_k, k = 1, \dots, n\} \le \prod_{k=1}^{n} P\{X_k < x_k\}.$$

Bounded functions

$$f(y_1, \dots, y_{n-1}) = -\prod_{k=1}^{n-1} I_{(-\infty,x_k]}(y_k) \text{ and } g(y_n) = -I_{(-\infty,x_n]}(y_n)$$

of real arguments $y_1, \dots, y_n$ are non-decreasing in each argument. By Definition 1 the inequality

$$E(f(X_1, \dots, X_{n-1}), g(X_n)) \le E(f(X_1, \dots, X_{n-1})Eg(X_n),$$

holds which may be rewritten in the following form

$$P\{X_1 \le x_1, \dots, X_n \le x_n\} \le P\{X_1 \le x_1, \dots, X_{n-1} \le x_{n-1}\}P\{X_n \le x_n\}.$$

In the same way we obtain

$$P\{X_1 \le x_1, \dots, X_{n-1} \le x_{n-1}\} \le P\{X_1 \le x_1, \dots, X_{n-2} \le x_{n-2}\}P\{X_{n-1} \le x_{n-1}\}.$$

It follows that

$$P\{X_1 \le x_1, \dots, X_n \le x_n\} \le P\{X_1 \le x_1\} \cdots P\{X_n \le x_n\}.$$

The first inequality in (3) is proved. The second one can be proved in the same way by the change of functions $I_{(-\infty,x_k]}$ and $I_{(-\infty,x_k)}, k = 1, \dots, n$.

Now we prove the first inequality in (2). Note that bounded functions $-\prod_{k\in A} I_{(-\infty,x_k]}(y_k)$ and $-\prod_{k\notin A} I_{(-\infty,x_k)}(y_k)$ of real arguments $y_1, \dots, y_n$ are non-decreasing in each argument. By Definition 1 the following inequality holds

$$E\left(\prod_{k\in A} I_{(-\infty,x_k]}(X_k) \prod_{k\notin A} I_{(-\infty,x_k)}(X_k)\right) \le E\left(\prod_{k\in A} I_{(-\infty,x_k]}(X_k)\right) E\left(\prod_{k\notin A} I_{(-\infty,x_k)}(X_k)\right),$$

which may be rewritten as

$$P\Big\{\cap_{k\in A}\{X_k \le x_k\} \cap \cap_{k\notin A}\{X_k < x_k\}\Big\} \le P\Big\{\cap_{k\in A}\{X_k \le x_k\}\Big\} P\Big\{\cap_{k\notin A}\{X_k < x_k\}\Big\}.$$

By inequalities (3) we obtain

$$P\{\cap_{k\in A}\{X_k \le x_k\}\} \le \prod_{k\in A} P\{X_k \le x_k\}, \quad P\{\cap_{k\notin A}\{X_k < x_k\}\} \le \prod_{k\notin A} P\{X_k < x_k\}.$$

The first inequality in (2) follows.

The second inequality in (2) can be proved in the similar way. It follows from the particular inequalities

$$P\{X_k \ge x_k, k = 1,\dots,n\} \le \prod_{k=1}^{n} P\{X_k \ge x_k\}, \tag{4}$$

$$P\{X_k > x_k, k = 1,\dots,n\} \le \prod_{k=1}^{n} P\{X_k > x_k\}.$$

We prove, for instance, the first one. Bounded functions

$$f'(y_1,\dots,y_n) = I_{[x_1,\infty)}(y_1)\cdots I_{[x_{n-1},\infty)}(y_{n-1}) \text{ and } g'(y_n) = I_{[x_n,\infty)}(y_n)$$

of real arguments $y_1,\dots,y_n$ are non-decreasing in each argument. By Definition 1 the following inequality holds

$$E(f'(X_1,\dots,X_{n-1}), g'(X_n)) \le Ef'(X_1,\dots,X_{n-1})Eg'(X_n).$$

It follows from this that

$$P\{X_1 \ge x_1,\dots,X_n \ge x_n\} \le P\{X_1 \ge x_1,\dots,X_{n-1} \ge x_{n-1}\}P\{X_n \ge x_n\}$$
$$\le \cdots \le P\{X_1 \ge x_1\}\cdots P\{X_n \ge x_n\}.$$

The theorem is proved.

In the next theorem two inequalities are proved. They are mentioned without proofs in Joag-Dev and Proschan [4], and Taylor et al [5].

7. Theorem *For any negatively associated random variables $X_1,\dots,X_n$ and any non-negative non-decreasing functions $f_1(x),\dots,f_n(x), x\in\mathbb{R}$, the following inequality holds*

$$E(f_1(X_1)\cdots f_n(X_n)) \le Ef_1(X_1)\cdots Ef_n(X_n). \tag{5}$$

In particular, if the random variables are non-negative, then

$$E(X_1^{p_1}\cdots X_n^{p_n}) \le EX_1^{p_1}\cdots EX_n^{p_n} \tag{6}$$

for any positive numbers $p_1,\dots,p_n$.

Proof. We may assume that mathematical expectations on the right hand sides are finite. Otherwise there is nothing to prove. For any $m\in\mathbb{N}$ functions $g_{k,m} = f_k \wedge m$, $k = 1,\dots,n$, are bounded and non-decreasing. By Definition 1 the following inequality holds

$$E(g_{1,m}(X_1)\cdots g_{n,m}(X_n)) \le E(g_{1,m}(X_1)\cdots g_{n-1,m}(X_{n-1}))Eg_{n,m}(X_n)$$
$$\le \cdots \le Eg_{1,m}(X_1)\cdots Eg_{n,m}(X_n).$$

Now inequality (5) follows from the monotone convergence theorem as $m\uparrow\infty$. Inequality (6) follows from inequality (5) with

$$f_k(x) = I_{[0,\infty)}(x)|x|^{p_k}, \quad k = 1,\dots,n.$$

The theorem is proved.

As it is shown by examples in Joag-Dev and Proschan [4], if we are given a collection of three or more random variables, they may be negatively dependent, but not negatively associated. Moreover, both inequalities in Definition 5 are required for them to be negatively dependent. But if we consider only two random variables the situation changes completely. The definition of negatively association of two random variables can be expressed in terms of their joint and marginal distribution functions.

8. Theorem. *Random variables X and Y are negatively associated if and only if one of the following inequalities holds*

$$P\{X \le x, Y \le y\} \le P\{X \le x\}P\{Y \le y\}, \quad P\{X \le x, Y < y\} \le P\{X \le x\}P\{Y < y\}, \tag{7}$$

$$P\{X < x, Y \le y\} \le P\{X < x\}P\{Y \le y\}, \quad P\{X < x, Y < y\} \le P\{X < x\}P\{Y < y\}$$

for any $x, y \in \mathbb{R}$.

Proof. All inequalities (7) hold for negatively associated random variables X and Y by Theorem 6. One can prove that each inequality in (7) implies the others. Assume that the first inequality in (7) holds. Then $P\{f(X) \le x,\ g(Y) \le y\} \le P\{f(X) \le x\}P\{g(Y) \le y\}$ for all $x, y \in \mathbb{R}$ and for any non-decreasing bounded functions $f, g : \mathbb{R} \to \mathbb{R}$. It follows from the Hoeffding identity [7] that

$$E(f(X)g(Y)) - Ef(X)Eg(Y)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [P\{f(X) \le x, g(Y) \le y\} - P\{f(X) \le x\}P\{g(Y) \le y\}]dxdy \le 0.$$

The random variables X and Y are negatively associated by Definition 1. The theorem is proved.

2. ON CHEBYSHEV'S OTHER INEQUALITY

In this section we present a construction of negatively associated random variables. The following can be considered as a classical method of a construction of a sequence of negatively associated random variables. Let $X_1, \ldots, X_n$ be a sequence of negatively correlated normally distributed random variables. Then they are negatively associated, as shown in Joag-Dev and Proschan [4]. Further we can obtain sequences of random variables with different (not normal) distributions by applying monotone increasing functions to each of the random variables.

A different construction of negatively associated random variables that we suggest in this section is based on Chebyshev's other inequality [2]. We start with an analog of this inequality for finite sums, which is presented in monograph [8].

9. Theorem. *Let* $\{a\} = \{a_1, \ldots, a_n\}$ *and* $\{b\} = \{b_1, \ldots, b_n\}$ *be two finite sequences of real numbers. If both sequences* $\{a\}$ *and* $\{b\}$ *are non-increasing or both sequences are non-decreasing, then*

$$\sum_{k=1}^{n} p_k a_k b_k \ge \sum_{k=1}^{n} p_k a_k \sum_{k=1}^{n} p_k b_k \tag{8}$$

for any numbers $p_k \ge 0, p_1 + \ldots + p_n = 1$. *If one the sequences* $\{a\}$ *or* $\{b\}$ *is non-decreasing and other sequence non-increasing, then*

$$\sum_{k=1}^{n} p_k a_k b_k \le \sum_{k=1}^{n} p_k a_k \sum_{k=1}^{n} p_k b_k. \tag{9}$$

10. Definition. Functions $f, g : \mathbb{R}^n \to \mathbb{R}$ are called *accordantly monotone by k-th argument*, if the functions $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n), x_k \in \mathbb{R}$, for any fixed $x_j, j = 1, \ldots, n,\ j \ne k$, are simultaneously increasing or decreasing. Functions f and g are called *discordantly monotone by k-th argument*, if functions $-f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n), x_k \in \mathbb{R}$, are accordantly monotone.

The famous *Chebyshev's other inequality* can be reformulated in terms of random variables as follows.

11. Theorem. *Let* X *be a random variable and* $f, g : \mathbb{R} \to \mathbb{R}$ *be functions such that* $E|f(X)g(X)| < \infty$, $E|f(X)| < \infty$, $E|g(X)| < \infty$. *If the functions* f *and* g *are accordantly monotone then* $E(f(X)g(X)) \ge Ef(X)Eg(X)$. *If the functions* f *and* g *are discordantly monotone, then* $E(f(X)g(X)) \le Ef(X)Eg(X)$.

It is remarkable to note that the original proof of the Chebyshev's other inequality is so general that there is nothing to add to it.

Both statements in the next theorem follow from the second part of Chebyshev's other inequality. The second statement of the next theorem seems to be new.

12. Theorem. *a) If functions* $f, g : \mathbb{R} \to \mathbb{R}$ *are discordant monotone, then for any random variable* X, *random variables* $f(X)$ *and* $g(X)$ *are negatively associated.*

b) If functions $f, g : \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, are discordant monotone by each argument, then for any independent random variables $X_1, \dots, X_n$, random variables $f(X_1, \dots, X_n)$ and $g(X_1, \dots, X_n)$ are negatively associated.

Proof. First we prove statement (a). The bounded function $\phi(f(x))$ and $\psi(g(x))$, $x \in \mathbb{R}$, are accordant (discordant, respectively) monotone for any bounded increasing functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$. By Chebyshev's other inequality we get that random variables $f(X)$ and $g(X)$ are negatively associated.

Now we prove statement (b). It is required to prove the inequality

$$E\left(\phi(f(X_1, \dots, X_n))\psi(g(X_1, \dots, X_n))\right) \leq E\phi(f(X_1, \dots, X_n))E\psi(g(X_1, \dots, X_n)) \quad (10)$$

for the functions ϕ and ψ mentioned above. For any $x_k \in \mathbb{R}, k = 2, \dots, n$, the following inequality holds

$$E(\phi(f(X_1, x_2, \dots, x_n))\psi(g(X_1, x_2, \dots, x_n))) \leq E\phi(f(X_1, x_2, \dots, x_n))E\psi(g(X_1, x_2, \dots, x_n)).$$

Note that the functions

$$E\phi(f(X_1, x_2, \dots, x_n)) \text{ and } E\psi(g(X_1, x_2, \dots, x_n)), x_2, \dots, x_n \in \mathbb{R},$$

satisfy the condition of the theorem. By Chebyshev's other inequality and by independence of X_1 and X_2 we obtain that

$$\begin{aligned} &E(\phi(f(X_1, X_2, x_3, \dots, x_n))\psi(g(X_1, X_2, x_3, \dots, x_n))) \\ &= \int_{-\infty}^{\infty} E(\phi(f(X_1, x_2, \dots, x_n))\psi(g(X_1, x_2, \dots, x_n)))dP\{X_2 < x_2\} \\ &\leq \int_{-\infty}^{\infty} E(\phi(f(X_1, x_2, \dots, x_n)))E(\psi(g(X_1, x_2, \dots, x_n)))dP\{X_2 < x_2\} \\ &\leq \int_{-\infty}^{\infty} E(\phi(f(X_1, x_2, \dots, x_n)))dP\{X_2 < x_2\} \int_{-\infty}^{\infty} E(\psi(g(X_1, x_2, \dots, x_n)))dP\{X_2 < x_2\} \\ &= E(\phi(f(X_1, X_2, x_3, \dots, x_n)))E(\psi(g(X_1, X_2, x_3, \dots, x_n))). \end{aligned}$$

These arguments can be applied to $X_3, \dots, X_n$ consequently. As a result we obtain inequality (10). The theorem is proved.

The following theorem appears in Lehman [6]. We present a new proof based on Chebyshev's other inequality.

13. Theorem. *Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be independent two-dimensional random vectors and functions $f, g : \mathbb{R}^n \to \mathbb{R}$. Assume that for each $k = 1, \dots, n$ random variables X_k and Y_k are negatively associated and for any $k = 1, \dots, n$ functions f and g are accordant monotone by k-th argument, or random variables $-X_k$ and Y_k or X_k and $-Y_k$ are negatively associated and functions f and g discordant monotone by k-th argument. Then the random variables $X = f(X_1, \dots, X_n)$ and $Y = g(Y_1, \dots, Y_n)$ are negatively associated.*

Proof. It is required to prove the inequality $E(\phi(X)\psi(Y)) \leq E\phi(X)E\psi(Y)$ for any bounded increasing functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$. If functions f and g are accordant (discordant) monotone by k-th argument, then composite functions $\phi(f)\psi(g)$ have this property, too. Without loss of generality we may assume that $\phi(x) = \psi(x) = x, x \in \mathbb{R}$, and functions f and g are bounded and continuous. It is sufficient to prove the theorem for bounded random variables. Really, for any $r \in \mathbb{N}$, bounded function

$$f_r(x) = (x \vee (-r)) \wedge r, x \in \mathbb{R}$$

is increasing. For any $k = 1, \dots, n$, random variables $f_r(X_k)$ and $f_r(Y_k)$ are negative associated. Note that random vectors $(f_r(X_k), f_r(Y_k))$, $k = 1, \dots, n$, are independent. Since the sequences $\{f_r(X_k)\}_{r \geq 1}$ and $\{f_r(Y_k)\}_{r \geq 1}$ converge pointwise to X_k and Y_k, then by the Dominated Convergence Theorem

$$\begin{aligned} &\lim_{r \to \infty} \operatorname{cov}(f(f_r(X_1), \dots, f_r(X_n)), g(f_r(Y_1), \dots, f_t(Y_n))) \quad (11) \\ &= \operatorname{cov}(f(X_1, \dots, X_n), g(Y_1, \dots, Y_n)). \end{aligned}$$

The covariance on the right hand side is non-positive if all covariances under the limit sign are non-positive. It is sufficient to prove the theorem for random variables with finite number of values. Really, let all random variables X_k and Y_k, $k = 1, \ldots, n$, be bounded by a positive number c. Define the function $\phi_r : \mathbb{R} \to \mathbb{R}$, by letting

$$\phi_r(x) = \begin{cases} -c, \text{ if } x < -c \\ -c + 2\frac{mc}{r}, \text{ if } \; -c + 2\frac{mc}{r} \le x < -c + 2\frac{(m+1)c}{r}, \quad m = 0, \ldots, r-1 \\ c, \text{ if } x > c. \end{cases}$$

Function ϕ_r is increasing and hence for each $k = 1, \ldots, n$ random variables $\phi_r(X_k)$ and $\phi_r(Y_k)$ are negatively associated. Note that the random vectors $(\phi_r(X_k), \phi_r(Y_k))$, $k = 1, \ldots, n$, are independent. From (11) with exchange of f_r on ϕ_r it follows that the covariance in the right hand side is nonnegative, if all covariances under the limit sign are non-negative.

In the following we assume that random variables $X_1, Y_1, \ldots, X_n, Y_n$ have finite number of values. Let X_k take values $a_{j,k}$, $j = 1, \ldots, m_k$, and Y_k take values $b_{j,k}$, $j = 1, \ldots, r_k$. Note that $p_{i,j,k} = P\{X_k = a_{i,k}, Y_k = b_{j,k}\}$ satisfy conditions

$$p_{i,1,k} + \cdots + p_{i,r_k,k} = p_{i,k} = P\{X_k = a_{i,k}\}, \quad p_{1,j,k} + \cdots + p_{m_k,j,k} = q_{j,k} = P\{Y_k = b_{j,k}\}.$$

In their turn, $p_{i,k}, i = 1, \ldots, m_k$, $q_{j,k}, j = 1, \ldots, r_k$, connected by equalities $p_{1,k} + \cdots + p_{m_k,k} = 1$ and $q_{1,k} + \cdots + q_{r_k,k} = 1$. If random variables X_n and Y_n are negatively associated and functions f and g are accordantly monotone by the argument $x_n \in \mathbb{R}$, then

$$\begin{aligned} &Ef(a_{i_1,1}, \ldots, a_{i_{n-1},n-1}, X_n) g(b_{j_1,1}, \ldots, b_{j_{n-1},n-1}, Y_n) \\ &= \sum_{i_n=1}^{m_n} \sum_{j_n=1}^{r_n} p_{i_n,j_n,n} f(a_{i_1,1}, \ldots, a_{i_n,n}) g(b_{j_1,1}, \ldots, b_{j_n,n}) \\ &\le \sum_{i_n=1}^{m_n} p_{i_n,j_n,n} f(a_{i_1,1}, \ldots, a_{i_n,n}) \sum_{j_n=1}^{r_n} p_{i_n,j_n,n} g(b_{i_1,1}, \ldots, b_{i_n,n}) \\ &= Ef(a_{i_1,1}, \ldots, a_{i_{n-1},n-1}, X_n) Eg(b_{i_1,1}, \ldots, b_{i_{n-1},n-1}, X_n). \end{aligned}$$

This inequality remains true if random variables X_n and $-Y_n$ are negatively associated and functions f and g are discordantly monotone by the argument $x_n \in \mathbb{R}$. With the help of the previous inequality we obtain

$$\begin{aligned} Ef(X_1, \ldots, X_n) g(Y_1, \ldots, Y_n) \le &\sum_{i_1=1}^{m_1} \sum_{j_1=1}^{r_1} p_{i_1,j_1,k} \cdots \sum_{i_{n-1}=1}^{m_{n-1}} \sum_{j_{n-1}=1}^{r_{n-1}} p_{i_{n-1},j_{n-1},n-1} \\ &\times Ef(a_{i_1,1}, \ldots, a_{i_{n-1},n-1}, X_n) Eg(b_{j_1,1}, \ldots, b_{j_{n-1},n-1}, Y_n). \end{aligned}$$

Taking into consideration that random vectors (X_{n-1}, Y_{n-1}) and (X_n, Y_n) are independent, we obtain that

$$\begin{aligned} &E\left(f(a_{i_1,1}, \ldots, a_{i_{n-2},n-2}, X_{n-1}, X_n) g(b_{j_1,1}, \ldots, b_{j_{n-2},n-2}, Y_{n-1}, Y_n)\right) \\ &= \sum_{i_{n-1}=1}^{m_{n-1}} \sum_{j_{n-1}=1}^{r_{n-1}} p_{i_{n-1},j_{n-1},n-1} Ef(a_{i_1,1}, \ldots, a_{i_{n-1},n-1}, X_n) Eg(b_{i_1,1}, \ldots, b_{i_{n-1},n-1}, Y_n). \end{aligned}$$

Note that functions $f(\cdots, X_n), g(\cdots, Y_n) : \mathbb{R}^{n-1} \to \mathbb{R}$ and random vectors $(X_1, Y_1), \ldots, (X_{n-1}, Y_{n-1})$ satisfy the assumptions of the theorem with the only difference that instead of n we use $n-1$. Hence the following inequality is true

$$\begin{aligned} Ef(X_1, \ldots, X_n) g(Y_1, \ldots, Y_n) \le &\sum_{i_1=1}^{m_1} \sum_{j_1=1}^{r_1} p_{i_1,j_1,k} \cdots \sum_{i_{n-2}=1}^{m_{n-2}} \sum_{j_{n-2}=1}^{r_{n-2}} p_{i_{n-2},j_{n-2},n-2} \\ &\times Ef(a_{i_1,1}, \ldots, a_{i_{n-2},n-2}, X_{n-1}, X_n) \, Eg(b_{j_1,1}, \ldots, b_{j_{n-2},n-2}, Y_{n-1}, Y_n). \end{aligned}$$

In a finite number of steps we obtain the required inequality

$$E\left(f(X_1,\ldots,X_n)g(Y_1,\ldots,Y_n)\right) \leq Ef(X_1,\ldots,X_n)Eg(Y_1,\ldots,Y_n).$$

The theorem is proved.

The following theorem is proved in Joag-Dev and Proschan [4]. We present a proof based on Chebyshev's other inequality.

14. Theorem. *Let $\mathcal{X}_t, t \in T$, be an arbitrary family of independent sets, each set consists of negatively associated random variables. Then the union $\bigcup_{t\in T}\mathcal{X}_t$ consists of negatively associated random variables.*

Proof. It is enough to prove the theorem for two sets. Let sets $\mathcal{X}'$ and $\mathcal{X}''$ be independent and each set consists of negatively associated random variables. By Definition 1 it is required to prove that any finite set $\mathcal{X} \subseteq \mathcal{X}' \cup \mathcal{X}''$ consists of negatively associated random variables. It is possible to assume that $\mathcal{X}$ contains representatives of both sets $\mathcal{X}'$ and $\mathcal{X}''$. Otherwise, the statement is true by the assumption.

Numerate in arbitrary order the random variables $X_1,\ldots,X_n$ from the set $\mathcal{X}$, $n \geq 2$. Let two nonempty disjoint subsets $\{i_1,\ldots,i_k\}$ and $\{j_1,\ldots,j_m\}$ of the set $\{1,\ldots,n\}$ be given and $f : \mathbb{R}^k \to \mathbb{R}$ $g : \mathbb{R}^m \to \mathbb{R}$ be bounded coordinatewise increasing Borel functions. Chose random variables from $\mathcal{X}$ with numbers $i_1,\ldots,i_k$. From these random variables we construct a random vector (X_1, Y_1), where the sub-vector X_1 consists of representatives of the set $\mathcal{X} \cap \mathcal{X}'$ and sub-vector Y_1 consists of representatives of the set $\mathcal{X} \cap \mathcal{X}''$.

Next we chose random variables from $\mathcal{X}$ with numbers $j_1,\ldots,j_m$ and use them to construct a random vector (X_2, Y_2), where sub-vector X_2 consists of representatives of the set $\mathcal{X} \cap \mathcal{X}'$ and sub-vector Y_2 consists from representatives of the set $\mathcal{X} \cap \mathcal{X}''$. Substitute random vectors (X_1, Y_1) and (X_2, Y_2) as arguments of the functions f and g. As a result we obtain random variables $f(X_1, Y_1)$ and $g(X_2, Y_2)$. It is required to prove the inequality

$$E(f(X_1,Y_1), g(X_2,Y_2)) \leq Ef(X_1,Y_1)Eg(X_2,Y_2). \tag{12}$$

It is enough (see the proof of Theorem 13) to establish Theorem 14 for bounded continuous functions f and g and for random vector (X_1, Y_1) and (X_2, Y_2) with finite number of values. Denote sets A_1, A_2, B_1, B_2 of possible values of random vectors X_1, X_2, Y_1, Y_2, respectively. Representatives of these sets will be denoted by symbols a_1, a_2, b_1, b_2. The mathematical expectation in the left hand side of (12) can be rewritten in the following form

$$E(f(X_1,Y_1),\ g(X_2,Y_2)) = \sum P\{X_1 = a_1, X_2 = a_2, Y_1 = b_1, Y_2 = b_2\} f(a_1,b_1)g(a_2,b_2),$$

where the summation goes over $a_1 \in A_1$, $a_2 \in A_2$, $b_1 \in B_1$, $b_2 \in B_2$. Since the sets $\mathcal{X}'$ and $\mathcal{X}''$ are independent, the following inequality holds

$$P\{X_1 = a_1,\ X_2 = a_2, Y_1 = b_1,\ Y_1 = b_2\} = P\{X_1 = a_1, X_2 = a_2\}P\{Y_1 = b_1, Y_2 = b_2\}.$$

Note that

$$E(f(a_1,Y_1)g(a_2,Y_2)) = \sum_{\substack{b_1\in B_1,\\ b_2\in B_2}} P\{Y_1 = b_1, Y_2 = b_2\} f(a_1,b_1)g(a_2,b_2).$$

Taking into consideration all these remarks we obtain

$$\begin{aligned} E(f(X_1,Y_1)g(X_2,Y_2)) &= \sum_{\substack{a_1\in A_1,\\ a_2\in A_2}} P\{X_1 = a_1, X_2 = a_2\} E(f(a_1,Y_1)g(a_2,Y_2)) \\ &\leq \sum_{\substack{a_1\in A_1,\\ a_2\in A_2}} P\{X_1 = a_1, X_2 = a_2\} Ef(a_1,Y_1)Eg(a_2,Y_2). \end{aligned} \tag{13}$$

The last inequality is true because of the negative association of random variables from the set $\mathcal{X}''$. Next we note that

$$\sum_{\substack{a_1\in A_1,\\ a_2\in A_2}} P\{X_1 = a_1,\ X_2 = a_2\} Ef(a_1,Y_1)Eg(a_2,Y_2) \tag{14}$$

$$= \sum_{b_1 \in B_1} \sum_{b_2 \in B_2} P\{Y_1 = b_1, Y_2 = b_2\} E(f(X_1, b_1) g(X_2, b_2))$$
$$\leq \sum_{b_1 \in B_1} \sum_{b_2 \in B_2} P\{Y_1 = b_1, Y_2 = b_2\} Ef(X_1, b_1) Eg(X_2, b_2).$$

The last inequality is true since random variables from the set $\mathcal{X}'$ are negatively associated. The last double sum can be rewritten as $E(f'(Y_1)g'(Y_2))$, where $f'(b_1) = Ef(X_1, b_1)$, $g'(b_2) = Eg(X_2, b_2)$. Functions f' and g' are coordinatewise increasing. Since the random variables from the set $\mathcal{X}''$ are negatively associated, then the following inequality is true

$$E(f'(Y_1)g'(Y_2)) \leq Ef'(Y_1)Eg'(Y_2).$$

Because X_1 and Y_2 are independent random vectors,

$$Ef'(Y_1) = \sum_{b_1 \in B_1} P(Y_1 = b_1) Ef(X_1, b_1) = \sum_{b_1 \in B_1} P(Y_1 = b_1) \sum_{a_1 \in A_1} P(X_1 = a_1) f(a_1, b_1)$$
$$= \sum_{a_1 \in A_1} \sum_{b_1 \in B_1} P(X_1 = a_1, Y_1 = b_1) f(a_1, b_1) = Ef(X_1, Y_2).$$

Similarly $Eg'(Y_2) = Eg(X_2, Y_2)$. Taking into consideration (13) and (14), we get the required inequality (12). The theorem is proved.

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