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Limiting Behavior of Moving Average Processes Based on a Sequence of ρ^- Mixing Random Variables

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Abstract

Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed ρ^--mixing random variables, $\{a_i, -\infty < i < \infty\}$ an absolutely summable sequence of real numbers. In this paper, we prove the complete convergence and Marcinkiewicz-Zygmund strong law of large numbers for the partial sums of moving average processes $\left\{\sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\right\}$ under the same conditions as the case of the usual partial sums.

Keyword: complete convergence, Marcinkiewicz-Zygmund strong laws of large numbers, moving average process, ρ^--mixing, $\rho*$-mixing, negative association.

1. Preliminaries

Let $\{Y_i, -\infty < i < +\infty\}$ be a doubly infinite sequence of identically distributed random variables and $\{a_i, -\infty < i < +\infty\}$ be an absolutely summable sequence of real numbers. Next, let

$$X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1$$

be the *moving average process based on the sequence* $\{Y_i, -\infty < i < +\infty\}$. As usual, we denote $S_n = \sum_{k=1}^{n} X_k, n \geq 1$, the sequence of partial sums.

Under the assumption that $\{Y_i, -\infty < i < +\infty\}$ is a sequence of independent identically distributed random variables, many limiting results have been obtained for the moving average process $\{X_n, n \geq 1\}$ .For example, Ibragimov [5] established the central limit theorem, Burton and Dehling [3] obtained a large deviation principle, and Li *et al*. [7] obtained the complete convergence result for $\{X_n, n \geq 1\}$.

Certainly, even if $\{Y_i, -\infty < i < +\infty\}$ is the sequence of independent identically distributed random variables, the moving average random variables $\{X_n, n \geq 1\}$ are dependent. This kind of dependence is called weak dependence. The partial sums of weakly dependent random variables $\{X_n, n \geq 1\}$ have similar limiting behaviour properties in comparison with the limiting properties of independent identically distributed random variables.

For example, we present some previous results connected with complete convergence. The following was proved in Hsu and Robbins [4].

Theorem A. *Suppose that* $\{X_n, n \geq 1\}$ *is a sequence of independent identically distributed random variables. If* $EX_1 = 0, E|X_1|^2 < 0, then \sum_{n=1}^{\infty} P\{|S_n| \geq \varepsilon n\} < \infty$ *for all* $\varepsilon > 0.$

Hsu-Robbins result was extended by Li et al. [7] for moving average processes.

Theorem B. *Suppose that* $\{X_n, n \geq 1\}$ *is the moving average processes based on a sequence* $\{Y_i, -\infty < i < \infty\}$ *of independent identically distributed random variables with* $EY_1 = 0, E|Y_1|^2 < 0, then \sum_{n=1}^{\infty} P\{|S_n| \geq \varepsilon n\} < \infty$ *for all* $\varepsilon > 0.$

Very few results for a moving average process based on a dependent sequence are known. In this paper, we provide a result on the limiting behaviour of a moving average process based on a ρ^--mixing sequence.

Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of random variables defined on a probability space (Ω, F, P) .For a set of integer numbers *T* denote u-algebra F (*T*) = $\sigma(Y_i, i \in T)$ and as usual, for a u-algebra F we denote by L^2 (F) the class of all F -measurable random variables with the finite second moment.

For two sets *S* and *T* of real numbers we denote

dist(*S, T*) = inf $\{|s - t|.s \in S, t \in T\}$.

The following definition was introduced in Wang and Lu [10] . A sequence of random variables $\{Y_i, -\infty < i < \infty\}$ is called ρ^--mixing if

$p^-(s)$ = sup$\{p(S,T); S,T$ are sets of integers,

dist$(S,T) \geq s\} \rightarrow 0$ $(S,T) \geq s\} \rightarrow 0$ as $s \rightarrow \infty$,

where

$$p^-(S,T) = \max \{0, \sup(Corr[f(Y_i, i \in S), g(Y_i j \in T)])\},$$

where supremum is taken over all coordinatewise increasing real functions f on R^S and *g* on R^T .

Next, a sequence of random variables $\{Y_i, -\infty < i < \infty\}$ is called ρ^*- mixing if for some integer $s \geq 1$

$$p(s)^* = \sup \sup \left\{ Corr(X,Y) : X \in L^2(F_T) \right\} < 1,$$

where the first sup is taken over all pairs of nonempty finite sets S, T of integers, such that dist $(S,T) \geq s$. The notion of $\rho*$-mixing seems to be similar to the notion of ρ-mixing, but Bryc and Smolenski [2] showed that they are quite different from each other.

Recall that a finite family of random variables $\{Y_i, 1 < i < n\}$ is said to be negatively associated, if for any disjoint finite subsets *S* and *T* of integers and any real coordinatewise nondecreasing functions f on R^S and g on R^T.

$$Cov(f(Y_i, i \in S), g(Y_j, j \in T)) \leq 0$$

whenever the covariance exists. This concept was introduced by Joag-Dev and Proschan [6].

It is easy to see that $\{Y_i, -\infty < i < \infty\}$ is negatively associated if and only if $\rho^-(s) = 0$ for all $s \geq 1$ and $\rho^-(s) \leq \rho^*(s)$. Hence the notion of ρ^- - mixing is weaker than both notions of negative association and $\rho*$-mixing.

The following inequality plays the crucial role in the proof of the main result and can be found in Wang and Lu [10], Theorem 2.1.

Rosenthal-type Maximal Inequality. *For a positive integer s, positive real numbers* $p \geq 2$ *and* $0 \leq t < (6p)^{-p/2}$, *if* $\{Y_i, -\infty < i < \infty\}$ *is a sequence of random variables with* $p^-(s) \leq t$, *with* $EY_i = 0$ *and* $E|Y_i|^p < \infty$ *for every* $i \geq 1$, *then for all* $n \geq 1$, *there is a positive constant C = C(p, s, t) such that*

$$E \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} Y_i \right|^q \leq C \left(\sum_{i=1}^{n} E|Y_k|^p + \left(\sum_{i=1}^{n} E|Y_k|^2 \right)^{p/2} \right)$$

We also need the following simple statement (cf. Property P2 in Wang and Lu [10]).

Property of ρ^--mixing random variables. *Let* $\{Y_n, n \geq 1\}$ *be a sequence of* ρ^- *-mixing random variables. If* $\{f_n, n \geq 1\}$ *is a sequence of real functions all of which are monotone nondecreasing (or all monotone nonincreasing), then* $\{f_n(Y_n), n \geq 1\}$ *is a sequence of* ρ^- *-mixing random variables.*

Note that Property P2 in Wang and Lu [10] is stated only for increasing functions. It is simple to see that this property remains true for nondecreasing functions, too. The statement for nonincreasing functions follows for the observation that if a function f_n is nonincreasing, then the function $-f_n$ is nondecreasing.

Recall that a measurable function h is said to be slowly varying if for each $\lambda > 0$

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = 1$$

We refer to Seneta [9] for other equivalent definitions and for detailed and comprehensive study of properties of such functions.

In the following, *C* will represent a positive constants although its value may change from one appearance to the next.

We need the following pure technical lemma.

Lemma. *Let h be a positive slowly varying function and Y be a random variable with* $E|Y|^{rp} h(|Y|^p) < \infty$, *where* $r \geq 1, p \geq 1$.

$$(i) \sum_{n=1}^{\infty} n^{r-1} h(n) P\{|Y| > n^{1/p}\} \leq CE|Y|^p h(|Y|^p).$$

(ii) *If* $s \geq 1, v > 0, sp > v$ *and* $E|Y|^{sp} h(|Y|^p) < \infty$ *, then*

$$\sum_{n=1}^{\infty} n^{s-1-v/p} h(n) E|Y|^v I\{|Y| > n^{1/p}\} \leq CE|Y|^{sp} h(|Y|^p).$$

Proof. First of all, we mention that statement (i) is well known, so we will prove only (ii). We have

$$\sum_{n=1}^{\infty} n^{s-1-v/p} h(n) E|Y|^{v} I\{|Y| > n^{1/p}\}$$

$$= C\sum_{n=1}^{\infty} n^{s-1-v/p} h(n) \sum_{m=n}^{\infty} E|Y|^{v} I\{m < |Y|^{p} \le m+1\}$$

$$= C\sum_{m=1}^{\infty} E\,|\,Y\,|^{v}\; I\{m < |Y|^{p} \le m+1\} \sum_{n=1}^{\infty} n^{x-1-v/p} h(n)$$

$$\le C\sum_{m=1}^{\infty} m^{s-1-v/p} h(m) E|Y|^{v} I\{m < |Y|^{p} \le m+1\}$$

$$= C\sum_{m=1}^{\infty} Em^{s-1-v/p} h(m)|Y|^{v} I\{m < |Y|^{p} \le m+1\}$$

$$\le C\sum_{m=1}^{\infty} E(|Y|^{p})^{s-1-v/p} |Y|^{p} h(|Y|^{p}) I\{m < |\,Y\,|^{p} \le m+1\}$$

$$\le CE|Y|^{rp} h(|Y|^{p}).$$

2. Mainstream

With the preliminaries accounted for, the main theorem can now be presented and proved.

Theorem. *Let* $h(x)$ *be a positive slowly varying function and* $1 \le p < 2, r \ge 1,\ pr \ne 1$. *Suppose* $\{Y_i, -\infty < i < \infty\}$ *is a sequence of identically distributed and* ρ^{-} *-mixing random variables and* $\{X_n, n \ge 1\}$ *is defined as above. Then* $EY_1 = 0$ *and* $E|Y_1|^{rp} h(|Y_1|^{p}) < \infty$ *imply that for all* $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\left(\max_{k \le n} \left| \sum_{j=1}^{k} X_j \right| \ge \varepsilon n^{1/p} \right) < \infty.$$

In particular, the assumptions $EY_1 = 0$ *and* $E|Y_1|^p < \infty, 1 < p < 2$ *imply Marcinkiewicz-Zygmund strong law of large numbers*

$$n^{-1/p} \sum_{n=1}^{\infty} X_k \to 0 \ \ a.s. \text{ as } n \to \infty$$

Proof: Let $Y_{nj}^{(1)} = -n^{1/p} I\{Y_j < -n^{1/p}\} + Y_j I\{|Y_j| \le n^{1/p}\} + n^{1/p} I\{Y_j > n^{1/p}\}$,

and $Y_{nj}^{(2)} = Y_j - Y_{nj}^{(1)}$ be the *monotone truncations* of $\{Y_j, -\infty < j < \infty\}$ Then by the property of ρ^- -mixing random variables, for any
$n \ge 1, \{Y_{nj}^{(1)} - EY_{rj}^{(1)}, -\infty < j < \infty\}$ and $\{Y_{rj}^{(2)}, -\infty < j < \infty\}$ are two sequences of ρ^- -mixing random variables. Note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j$$

and

$$\begin{aligned}
n^{-1/p} \max \left| E \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{rj}^{(1)} \right| &\le n^{-1/p} \sum_{i=-\infty}^{\infty} |a_i| \max_{1 \le k \le n} \left| E \sum_{j=i+1}^{i+k} Y_{rj}^{(1)} \right| \\
&\le Cn^{-1/p} \left(\left| EY_1 I\{|Y_1| \le n^{1/p}\} \right| + n^{1/p} P\{|Y_1| > n^{1/p}\} \right) \\
&\le Cn^{1-1/p} E|Y_1| I\{|Y_1| > n^{1/p}\} + CnP\{|Y_1| > n^{1/p}\} \\
&\le CE|Y_1|^p I\{|Y_1| > n^{1/p}\} + CnP\{|Y_1| > n^{1/p}\} \to 0,
\end{aligned}$$

as $n \to \infty$. Hence, for any $\varepsilon > 0$ where exists m large enough such that

$$n^{-1/p} \max_{1 \le k \le n} \left| E \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{rj}^{(1)} \right| < \varepsilon / 4.$$

Therefore, in order to prove Theorem, it is enough to prove that

$$I:\sum_{n=1}^{\infty} n^{r-2}h(n)P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}\left(Y_{rj}^{(1)}-EY_{rj}^{(1)}\right)\right|<\varepsilon n^{1/p}/4\right\}<\infty$$

and

$$J:\sum_{n=1}^{\infty} n^{r-2}h(n)P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_{rj}^{(2)}\right|>\varepsilon n^{1/p}/2\right\}<\infty.$$

For J by Markov inequality we have

$$J\le C\sum_{n=1}^{\infty} n^{r-2}n^{-1/p}h(n)E\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_{rj}^{(2)}\right|$$

$$\le C\sum_{n=1}^{\infty} n^{r-1-1/p}h(n)\left(E|Y_1|I\{|Y_1|>n^{1/p}\}+n^{1/p}P\{|Y_1|>n^{1/p}\}\right)$$

$$\le C\sum_{n=1}^{\infty} n^{r-1-1/p}h(n)E|Y_1|I\{|Y_1|>n^{1/p}\}+CE|Y_1|^{rp}h\left(|Y_1|^{p}\right)$$ (by Lemma (i))

$$\le CE|Y_1|^{rp}h(|Y|^{p})<\infty$$ (by Lemma (ii) with $v=1$, and $s=r$).

For I, fix any $q\ge 2$ (to be specified later). Then

$$I\le C\sum_{n=1}^{\infty} n^{r-2-q/p}h(n)E\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}\left(Y_{rj}^{(1)}-EY_{rj}^{(1)}\right)\right|^{q}$$

(by Markov inequality)

$$\le C\sum_{n=1}^{\infty} n^{r-2-q/p}h(n)E\left(\sum_{i=-\infty}^{\infty}|a_i|\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}\left(Y_{nj}^{(1)}-EY_{nj}^{(1)}\right)\right|\right)^{q}$$

$$\le C\sum_{n=1}^{\infty} n^{r-2-q/p}h(n)E\left(\sum_{i=-\infty}^{\infty}\left(|a_i|^{1-1/q}\right)\left(|a_i|^{1/p}\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}\left(Y_{nj}^{(1)}-EY_{nj}^{(1)}\right)\right|\right)\right)^{q}$$

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}h(n)\left(\sum_{i=-\infty}^{\infty}|a_i|^{q-1}\right)\sum_{i=-\infty}^{\infty}|a_i|\,E\max_{1\leq k\leq n}\left|\sum_{j=i+1}^{i+k}\left(Y_{nj}^{(1)}-EY_{nj}^{(1)}\right)\right|^q$$

(by HÖlder inequality)

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}h(n)\sum_{i=-\infty}^{\infty}|a_i|\,E\max_{1\leq k\leq n}\left|\sum_{j=i+1}^{i+k}\left(Y_{nj}^{(1)}-EY_{nj}^{(1)}\right)\right|^q \quad \text{(since } \sum_{i=-\infty}^{\infty}|a_i|<\infty)$$

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}h(n)\sum_{i=-\infty}^{\infty}|a_i|\left\{\sum_{j=i+1}^{i+k}\left(E\left(Y_{nj}^{(1)}-EY_{nj}^{(1)}\right)^2\right)^{q/2}+\sum_{j=i+1}^{i+k}E\left|Y_{nj}^{(1)}-EY_{nj}^{(1)}\right|^q\right\}$$

(by Rosenthal-type Maximal Inequality)

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}h(n)\sum_{i=-\infty}^{\infty}|a_i|\left\{\left(n\left(EY_1^2 I\{|Y_1|\leq n^{1/p}\}+n^{2/p}P\{|Y_1|>n^{1/p}\}\right)\right)^{q/2}\right\}$$
$$+n\left(E|Y_1|^q I\{|Y_1|\leq n^{1/p}\}+n^{q/p}P\{|Y_1|>n^{1/p}\}\right)$$

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}h(n)\left\{\left(n\left(EY_1^2 I\{|Y_1|\leq n^{1/p}\}+n^{2/p}P\{|Y_1|>n^{1/p}\}\right)\right)^{q/2}\right\}$$

$$+n\left(E|Y_1|^q I\{|Y_1|\leq n^{1/p}\}+n^{q/p}P\{|Y_1|>n^{1/p}\}\right) \quad \text{(since } \sum_{i=-\infty}^{\infty}|a_i|<\infty\text{)}$$

We consider two separate cases. If $rp<2$, let $q=2.$

$$I\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}h(n)\left(E|Y_1|^2 I\{|Y_1|\leq n^{1/p}\}+n^{2/p}h(n)P\{|Y_1|>n^{1/p}\}\right)$$

$$\leq C\sum_{n=1}^{\infty} n^{r-2-q/p}h(n)E|Y_1|^2 I\{|Y_1|\leq n^{1/p}\}+CE|Y_1|^{rp}h\left(|Y_1|^p\right) \quad \text{(by Lemma (i))}$$

$$+CE|Y_1|^{rp}h\left(|Y_1|^p<\infty\right) \quad \text{(by Lemma (ii) with } q=2.\text{)}.$$

If $rp<2$, let $q>\dfrac{2p(r-1)}{2-p}\geq 2$. Note that in this case

$$E|Y_1|^2 I\{|Y_1| \le n^{1/p}\} \le E|Y_1|^2 \le \left(E|Y_1|^{rp}\right)^{2/(rp)} < \infty$$

and by Markov inequality

$$n^{2/p} P\{|Y_1| > n^{1/p}\} \le E|Y_1|^2 < \infty .$$

Hence

$$I \le C\sum_{n=1}^{\infty} n^{r-2-q/p} h(n)\Big\{\Big(n\Big(E|Y_1|^2 I\{|Y_1| \le n^{1/p}\} + n^{2/p} h(n) P\{|Y_1| > n^{1/p}\}\Big)\Big)^{q/2}$$

$$+ n\Big(E|Y_1|^q I\{|Y_1| \le n^{1/p}\} + n^{q/p} P\{|Y_1| > n^{1/p}\}\Big)\Big\}$$

$$\le C\sum_{n=1}^{\infty} n^{r-2-q/p+q/2} h(n) + C\sum_{n=1}^{\infty} n^{r-1-q/p} h(n) E|Y_1|^p I\{|Y_1| > n^{1/p}\}$$

$$+ C\sum_{n=1}^{\infty} n^{n-1} h(n) P\{|Y_1| > n^{1/p}\}$$

$$\le C + CE|Y_1|^{rp} h\left(|Y_1|^p\right) < \infty \quad \text{(by Lemma (i) and (ii))}$$

Now we show the almost sure convergence. By the first part of Theorem, $EY_1 = 0$ and $E|Y_1|^p < \infty$ imply

$$\sum_{n=1}^{\infty} n^{-1} P\{\max_{1\le k\le n} |S_m| > \varepsilon n^{1/p}\} < \infty \text{ for all } \varepsilon > 0.$$

Hence

$$\infty > \sum_{n=1}^{\infty} n^{-1} P\{\max_{1\le k\le n} |S_m| > \varepsilon n^{1/p}\}$$

$$= \sum_{n=1}^{\infty} \sum_{n=2^{k-1}}^{2^k} n^{-1} P\{\max_{1\le m\le n} |S_m| > \varepsilon n^{1/p}\}$$

$$\ge 1/2 \sum_{k=1}^{\infty} P\{\max_{1\le m\le 2^k} |S_m| > \varepsilon 2^{k/p}\}.$$

By Borel-Cantelli lemma,

$$2^{-k/p} \max_{1 \le m \le 2^k} \left| S_m \to 0 \right| \text{ almost surely}$$

which implies that $S_n / n^{1/p} \to 0$ almost surely.

3. Concluding remarks

1. The careful analysis of the proof shows that Theorem remains true if we relax the assumption of ρ^--mixing to the condition that for a positive integer *s* and $0 \le t < (6p)^{-p/2}$, $\{Y_i, -\infty < i < \infty\}$ is a sequence of random variables with $\rho^-(s) \le t$.

2. It is easy to prove that Theorem remains true if the assumption of the identical distribution of the random variables $\{Y_i, -\infty < i < \infty\}$ is relaxed to the slightly weaker assumption that this sequence is stochastically dominated by a random variable Y with corresponding moment assumptions. Recall that a sequence of random variables $\{Y_i, -\infty < i < \infty\}$ is said to be stochastically dominated by a random variable Y if there is a constant $D > 0$ such that for all $t \ge 0$

$$\sup_{-\infty < i < \infty} \mathrm{P}\{|\mathrm{Y_i}| > \mathrm{t}\} \ \mathrm{DP}\{|\ \mathrm{Y}\ | > \mathrm{Dt}\}.$$

3. Since the notion of ρ^--mixing is weaker than both notions of negative association and $\rho*$-mixing, the statement of Theorem remains true if we consider the moving average process based on a sequence of negative associated or $\rho*$-mixing random variables. We would like to mention two publications by Baek et al. [1] and Liang et al. [8], where moving average processes based on a negative associated sequence of random variables are considered. Even for this particular case, our result is stronger since it deals with maximums of partial sums.

4. The case $pr = 1$ is not treated in Theorem. The authors believe that the result can be proved under the additional assumption that $\sum_{i=-\infty}^{\infty} |a_i|^{\theta} < \infty$ for some $0 < \theta < 1$, but this is still an open problem.

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