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Limiting behaviour of moving average processes under φ-mixing assumption

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ABSTRACT

Let $\{Y_i, -\infty < i < \infty\}$ be a doubly infinite sequence of identically distributed φ-mixing random variables, $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers. In this paper we prove the complete convergence and Marcinkiewicz–Zygmund strong law of large numbers for the partial sums of moving average processes $\{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\}$ based on the sequence $\{Y_i, -\infty < i < \infty\}$ of φ-mixing random variables, improving the result of [Zhang, L., 1996. Complete convergence of moving average processes under dependence assumptions. Statist. Probab. Lett. 30, 165–170].

1. Introduction and formulation of the main results

Let $\{Y_i, -\infty < i < +\infty\}$ be a doubly infinite sequence of identically distributed random variables and $\{a_i, -\infty < i < +\infty\}$ be an absolutely summable sequence of real numbers. Let

$$X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1$$

be the *moving average process based on the sequence* $\{Y_i, -\infty < i < +\infty\}$. As usual, we denote $S_n = \sum_{k=1}^{n} X_k, n \geq 1$, the sequence of partial sums.

Under the assumption that $\{Y_i, -\infty < i < +\infty\}$ is a sequence of independent identically distributed random variables, many limiting results have been obtained for the moving average process $\{X_n, n \geq 1\}$. For example, Ibragimov (1962) established the central limit theorem, Burton and Dehling (1990) obtained a large deviation principle, and Li et al. (1992) obtained the complete convergence result for $\{X_n, n \geq 1\}$.

Certainly, even if $\{Y_i, -\infty < i < +\infty\}$ is the sequence of independent identically distributed random variables, the moving average random variables $\{X_n, n \geq 1\}$ are dependent. This kind of dependence is called *weak dependence*. The partial sums of weakly dependent random variables $\{X_n, n \geq 1\}$ have similar limiting behaviour properties in comparison with the limiting properties of independent identically distributed random variables.

For example, we could present some of the previous results connected with complete convergence. The following was proved in Hsu and Robbins (1947).

Theorem A. *Suppose $\{X_n, n \geq 1\}$ is a sequence of independent identically distributed random variables. If $EX_1 = 0, E|X_1|^2 < \infty$, then $\sum_{n=1}^{\infty} P\{|S_n| \geq \varepsilon n\} < \infty$ for all $\varepsilon > 0$.*

The above result was extended by Li et al. (1992) for moving average processes.

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Theorem B. *Suppose $\{X_n, n \geq 1\}$ is the moving average process based on a sequence $\{Y_i, -\infty < i < \infty\}$ of independent identically distributed random variables with $EY_1 = 0$, $E|Y_1|^2 < \infty$. Then $\sum_{n=1}^{\infty} P\{|S_n| \geq \varepsilon n\} < \infty$ for all $\varepsilon > 0$.*

Very few results for a moving average process based on a dependent sequence are known. In this paper, we provide two results on the limiting behaviour of a moving average process based on a φ-mixing sequence.

Let $\{Y_i, -\infty < i < \infty\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and denote σ-algebras $\mathcal{F}_n^m = \sigma(Y_i, n \leq i \leq m)$, $-\infty \leq n \leq m \leq +\infty$.

Recall that a sequence of random variables $\{Y_i, -\infty < i < \infty\}$ is called *φ-mixing* if the mixing coefficient

$$\varphi(m) = \sup_{k \geq 1} \sup\{|P\{B|A\} - P\{B\}|, A \in \mathcal{F}_{-\infty}^k, P\{A\} \neq 0, B \in \mathcal{F}_{k+m}^{\infty}\} \to 0$$

as $m \to \infty$.

Recall that a function h is said to be *slowly varying at infinity* if it is real valued, positive and measurable on $[0, \infty)$, and if for each $\lambda > 0$

$$\lim_{x \to \infty} \frac{h(\lambda x)}{h(x)} = 1.$$

We refer to Seneta (1976) for other equivalent definitions and for a detailed and comprehensive study of properties of slowly varying functions.

In the following, we frequently use the following properties of slowly varying functions (cf. Seneta (1976)).

If h is a function slowly varying at infinity, then for any $0 \leq a \leq b \leq \infty$ and $s \neq -1$

$$\int_a^b x^s h(x)\, dx \leq C x^{s+1} h(x) \,|_a^b,$$

where C does not depend on a and b, and for any $\lambda > 0$

$$\max_{a \leq x \leq \lambda a} h(x) \leq C(\lambda) h(\lambda a).$$

Of course, these two inequalities take place only if the right hand sides make sense.

The following result of partial sums of φ-mixing random variables was proved in Shao (1988), Remarks 3.2 and 3.3.

Theorem C. *Let h be a function slowly varying at infinity, $1 \leq p < 2, r \geq 1$, and $\{X_i, i \geq 1\}$ be a sequence of identically distributed φ-mixing random variables with $EX_1 = 0$ and $E|X_1|^{rp} h(|X_1|^p) < \infty$.*

(i) *If $r > 1$ then*

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\left\{\max_{1 \leq k \leq n} |S_k| \geq \varepsilon n^{1/p}\right\} < \infty, \quad \text{for all } \varepsilon > 0.$$

and

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\left\{\sup_{k \geq n} \left|S_k / k^{1/p}\right| \geq \varepsilon\right\} < \infty, \quad \text{for all } \varepsilon > 0.$$

(ii) *If $r = 1$ and $\sum_{m=1}^{\infty} \varphi^{1/2}(2^m) < \infty$, then*

$$\sum_{n=1}^{\infty} \frac{h(n)}{n} P\left\{\max_{1 \leq k \leq n} |S_k| \geq \varepsilon n^{1/p}\right\} < \infty, \quad \text{for all } \varepsilon > 0.$$

For moving average processes, Zhang (1996) obtained the following result.

Theorem D. *Let h be a function slowly varying at infinity, $1 \leq p < 2$, and $r \geq 1$. Suppose that $\{X_n, n \geq 1\}$ is a moving average process based on a sequence $\{Y_i, -\infty < i < \infty\}$ of identically distributed φ-mixing random variables with $\sum_{m=1}^{\infty} \varphi^{1/2}(m) < \infty$. If $EY_1 = 0$ and $E|Y_1|^{rp} h(|Y_1|^p) < \infty$, then*

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\left\{|S_n| \geq \varepsilon n^{1/p}\right\} < \infty, \quad \text{for all } \varepsilon > 0.$$

Keeping in mind the above mentioned analogy between the "usual" limiting behaviour of random variables and limiting behaviour of the moving average process (cf. Theorems A and B), we note that a substantial gap between Theorems C and D is distinct. Firstly, when $r > 1$, Theorem C provides the result without any mixing rate, even when $r = 1$ Theorem C requires a weaker condition on mixing rate than Theorem D. Secondly, Theorem D does not discuss the complete convergence for the case of the maximums and supremums of the partial sums as it is done in Theorem C. Note that by the method of Zhang (1996) it is impossible to eliminate these differences. The main goal of the present investigation is to obtain the results similar to Theorem C, but for the moving average processes and using different methods from those in Zhang (1996).

Now we state the main results. Theorems 1 and 2 improve Theorem D and extend Theorem C on the case of moving average processes. The proofs will be detailed in the next section.

Theorem 1. *Let h be a function slowly varying at infinity, $1 \le p < 2$, and $r > 1$. Suppose that $\{X_n, n \ge 1\}$ is a moving average process based on a sequence $\{Y_i, -\infty < i < \infty\}$ of identically distributed φ-mixing random variables. If $EY_1 = 0$ and $E|Y_1|^{rp}h(|Y_1|^p) < \infty$, then*

(i) $\sum_{n=1}^{\infty} n^{r-2}h(n)P\left\{\max_{1\le k\le n}|S_k| \ge \varepsilon n^{1/p}\right\} < \infty$, *for all* $\varepsilon > 0$.
and
(ii) $\sum_{n=1}^{\infty} n^{r-2}h(n)P\left\{\sup_{k\ge n}\left|S_k/k^{1/p}\right| \ge \varepsilon\right\} < \infty$, *for all* $\varepsilon > 0$.

The second theorem treats the case $r = 1$.

Theorem 2. *Let h be a function slowly varying at infinity and $1 \le p < 2$. Assume that $\sum_{i=-\infty}^{\infty}|a_i|^\theta < \infty$, where θ belong to $(0, 1)$ if $p = 1$ and $\theta = 1$ if $1 < p < 2$. Suppose that $\{X_n, n \ge 1\}$ is a moving average process based on a sequence $\{Y_i, -\infty < i < \infty\}$ of identically distributed φ-mixing random variables with $\sum_{m=1}^{\infty}\varphi^{1/2}(2^m) < \infty$. If $EY_1 = 0$ and $E|Y_1|^p h(|Y_1|^p) < \infty$, then*

$$\sum_{n=1}^{\infty}\frac{h(n)}{n}P\left\{\max_{1\le k\le n}|S_k| \ge \varepsilon n^{1/p}\right\} < \infty, \quad \text{for all } \varepsilon > 0.$$

In particular, the assumptions $EY_1 = 0$ and $E|Y_1|^p < \infty$ imply the following Marcinkiewicz–Zygmund strong law of large numbers

$$S_n/n^{1/p} \to 0 \quad \text{almost surely as } n \to \infty.$$

2. Few technical lemmas

The following five lemmas will be useful. The first two lemmas can be found in Shao (1988) Lemma 3.1 and Corollary 2.1, hence we omit their proofs. For the first two lemmas we assume that $\{Y_n, n \ge 1\}$ is a φ-mixing sequence and $S_k(n) = \sum_{i=k+1}^{k+n} Y_i,\ n \ge 1, k \ge 0$.

Lemma 1. *Let $EY_i = 0$, $EY_i^2 < \infty$ for all $i \ge 1$. Then for all $n \ge 1$ and $k \ge 0$ we have*

$$ES_k^2(n) \le 8000n\exp\left\{6\sum_{i=1}^{[\log n]}\varphi^{1/2}(2^i)\right\}\max_{k+1\le i\le k+n} EY_i^2.$$

Lemma 2. *Suppose that there exists an array $\{C_{k,n}, k \ge 0, n \ge 1\}$ of positive numbers such that $\max_{1\le i\le n} ES_k^2(i) \le C_{k,n}$ for every $k \ge 0, n \ge 1$. Then for any $q \ge 2$, there exists $C = C(q, \varphi(\cdot))$ such that for any $k \ge 0, n \ge 1$*

$$E\max_{1\le i\le n}|S_k(i)|^q \le C\left(C_{k,n}^{q/2} + E\left(\max_{k<i\le k+n}|Y_i|^q\right)\right).$$

The next two lemmas seem to be known (cf., for example the proof of Theorem G in Chen et al. (2006)), but we include their short and simple proofs for the interested reader. Here we let h be a function slowly varying at infinity.

Lemma 3. *If $r > 1$ and $1 \le p < 2$, then for any $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} n^{r-2}h(n)P\left\{\sup_{k\ge n}\left|k^{-1/p}S_k\right| \ge \varepsilon\right\} \le C\sum_{n=1}^{\infty} n^{r-2}h(n)P\left\{\max_{1\le k\le n}|S_k| \ge (\varepsilon/2^{1/p})n^{1/p}\right\}.$$

Proof. We have the following estimations:

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{r-2}h(n)P\left\{\sup_{k\ge n}|S_k|/k^{1/p} > \varepsilon\right\} &= \sum_{m=1}^{\infty}\sum_{n=2^{m-1}}^{2^m-1} n^{r-2}h(n)P\left\{\sup_{k\ge n}|S_k|/k^{1/p} > \varepsilon\right\} \\
&\le C\sum_{m=1}^{\infty}P\left\{\sup_{k\ge 2^{m-1}}|S_k|/k^{1/p} > \varepsilon\right\}\sum_{n=2^{m-1}}^{2^m-1}2^{m(r-2)}h(2^m) \\
&\le C\sum_{m=1}^{\infty}2^{m(r-1)}h(2^m)P\left\{\sup_{k\ge 2^{m-1}}|S_k|/k^{1/p} > \varepsilon\right\} \\
&= C\sum_{m=1}^{\infty}2^{m(r-1)}h(2^m)P\left\{\sup_{l\ge m}\max_{2^{l-1}<k\le 2^l}|S_k|/k^{1/p} > \varepsilon\right\}
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{m=1}^{\infty}2^{m(r-1)}h(2^m)\sum_{l=m}^{\infty}P\left\{\max_{1\le k\le 2^l}|S_k|>\varepsilon 2^{(l-1)/p}\right\}\\
&= C\sum_{l=1}^{\infty}P\left\{\max_{1\le k\le 2^l}|S_k|>\varepsilon 2^{(l-1)/p}\right\}\sum_{m=1}^{l}2^{m(r-1)}h(2^m)\\
&\le C\sum_{l=1}^{\infty}2^{l(r-1)}h(2^l)P\left\{\max_{1\le k\le 2^l}|S_k|>\varepsilon 2^{(l-1)/p}\right\}\\
&\le C\sum_{l=1}^{\infty}\sum_{n=2^{l-1}}^{2^l-1}n^{r-2}h(n)P\left\{\max_{1\le k\le n}|S_k|>(\varepsilon/2^{1/p})n^{1/p}\right\}\\
&\le C\sum_{n=1}^{\infty}n^{r-2}h(n)P\left\{\max_{1\le k\le n}|S_k|>(\varepsilon/2^{1/p})n^{1/p}\right\}. \quad \square
\end{aligned}$$

Lemma 4. *Let Y be a random variable with $E|Y|^{rp}h(|Y|^p)<\infty$, where $r\ge 1$ and $p\ge 1$. If $q>rp$, then*

$$\sum_{n=1}^{\infty}n^{r-1-q/p}h(n)E|Y|^qI\{|Y|\le n^{1/p}\}\le CE|Y|^{rp}h(|Y|^p).$$

Proof. Since $r-q/p<0$, we have that

$$\begin{aligned}
\sum_{n=1}^{\infty}n^{r-1-q/p}h(n)E|Y|^qI\{|Y|\le n^{1/p}\} &= \sum_{n=1}^{\infty}n^{r-1-q/p}h(n)\sum_{m=1}^{n}E|Y|^qI\{m-1<|Y|^p\le m\}\\
&= \sum_{m=1}^{\infty}E|Y|^qI\{m-1<|Y|^p\le m\}\sum_{n=m}^{\infty}n^{r-1-q/p}h(n)\\
&\le C\sum_{m=1}^{\infty}m^{r-q/p}h(m)E|Y|^qI\{m-1<|Y|^p\le m\}\\
&\le C\sum_{m=1}^{\infty}Em^{r-q/p}h(m)|Y|^qI\{m-1<|Y|^p\le m\}\\
&\le C\sum_{m=1}^{\infty}E(|Y|^p)^{r-q/p}h(|Y|^p)|Y|^qI\{m-1<|Y|^p\le m\}\\
&\le C\sum_{m=1}^{\infty}E|Y|^{rp}h(|Y|^p)I\{m-1<|Y|^p\le m\}\\
&\le CE|Y|^{rp}h(|Y|^p). \quad \square
\end{aligned}$$

The last lemma presents a technical fact that is important in the proofs of Theorems 1 and 2.

Lemma 5. *Let h be a function slowly varying at infinity and $p\ge 1$. Suppose that $\{X_n, n\ge 1\}$ is a moving average process based on a sequence $\{Y_i, -\infty<i<\infty\}$ of mean zero identically distributed random variables such that $E|Y_1|^p<\infty$. For any $\varepsilon>0$ denote*

$$I =: \sum_{n=1}^{\infty}n^{r-2}h(n)P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_jI\{|Y_j|>n^{1/p}\}\right|\ge \varepsilon n^{1/p}/2\right\}$$

and

$$J =: \sum_{n=1}^{\infty}n^{r-2}h(n)P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+k}Y_{nj}\right|\ge \varepsilon n^{1/p}/4\right\},$$

where

$$Y_{nj}=Y_jI\{|Y_j|\le n^{1/p}\}-EY_jI\{|Y_j|\le n^{1/p}\}.$$

If $I < \infty$ *and* $J < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{r-2}h(n)P\left\{\max_{1\le k\le n}|S_k| \ge \varepsilon n^{1/p}\right\} \le I + J < \infty.$$

Proof. Note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n}\sum_{i=-\infty}^{\infty} a_i Y_{i+k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j$$

and since $\sum_{i=-\infty}^{\infty}|a_i| < \infty$,

$$\begin{aligned} n^{-1/p}\left|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j I\{|Y_j| \le n^{1/p}\}\right| &= n^{-1/p}\left|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j I\{|Y_j| > n^{1/p}\}\right| \quad (EY_j = 0) \\ &\le n^{-1/p}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n} E|Y_j|I\{|Y_j| > n^{1/p}\} \\ &\le n^{1-1/p}\left(\sum_{i=-\infty}^{\infty}|a_i|\right)E|Y_1|I\{|Y_1| > n^{1/p}\} \\ &\le CE(n^{1/p})^{p-1}|Y_1|I\{|Y_1| > n^{1/p}\} \\ &\le CE|Y_1|^p I\{|Y_1| > n^{1/p}\} \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Hence for n large enough we have

$$n^{1/p}E\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j I\{|Y_j| \le n^{1/p}\}\right| < \varepsilon/4.$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} n^{r-2}h(n)P\left\{\max_{1\le k\le n}|S_k| \ge \varepsilon n^{1/p}\right\} &\le C\sum_{n=1}^{\infty} n^{r-2}h(n)P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j I\{|Y_j| > n^{1/p}\}\right| \ge \varepsilon n^{1/p}/2\right\} \\ &\quad + C\sum_{n=1}^{\infty} n^{r-2}h(n)P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}\right| \ge \varepsilon n^{1/p}/4\right\} \\ &= I + J. \quad \square \end{aligned}$$

3. Proof of main results

With all the prerequisites accounted before, we could now prove the main results of the paper. We start with Theorem 1.

Proof. According to Lemma 3 it is enough to show that (i) holds. According to Lemma 5 it is enough to prove that $I < \infty$ and $J < \infty$.

For I, by Markov inequality we have

$$\begin{aligned} I &\le C\sum_{n=1}^{\infty} n^{r-2}h(n)n^{-1/p}E\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j I\{|Y_j| > n^{1/p}\}\right| \\ &\le C\sum_{n=1}^{\infty} n^{r-1-1/p}h(n)E|Y_1|I\{|Y_1| > n^{1/p}\} \\ &= C\sum_{n=1}^{\infty} n^{r-1-1/p}h(n)\sum_{m=n}^{\infty} E|Y_1|I\{m < |Y_1|^p \le m+1\} \\ &= C\sum_{m=1}^{\infty} E|Y_1|I\{m < |Y_1|^p \le m+1\}\sum_{n=1}^{m} n^{r-1-1/p}h(n) \\ &\le C\sum_{m=1}^{\infty} m^{r-1/p}h(m)E|Y_1|I\{m < |Y_1|^p \le m+1\} \\ &\le CE|Y_1|^{rp}h(|Y_1|^p) < \infty. \end{aligned}$$

For J, by Markov and Hölder inequalities, Lemmas 1 and 2, we have that for any $q \geq 2$

$$\begin{aligned}
J &\leq C\sum_{n=1}^{\infty} n^{r-2}h(n)n^{-q/p}E\max_{1\leq k\leq n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj}\right|^q \\
&\leq C\sum_{n=1}^{\infty} n^{r-2}h(n)n^{-q/p}E\left(\sum_{i=-\infty}^{\infty}\left(|a_i|^{1-1/q}\right)\left(|a_i|^{1/q}\max_{1\leq k\leq n}\left|\sum_{j=i+1}^{i+k} Y_{nj}\right|\right)\right)^q \\
&\leq C\sum_{n=1}^{\infty} n^{r-2-(q/p)}h(n)\left(\sum_{i=-\infty}^{\infty}|a_i|\right)^{q-1}\sum_{i=-\infty}^{\infty}|a_i|E\max_{1\leq k\leq n}\left|\sum_{j=i+1}^{i+k} Y_{nj}\right|^q \\
&\leq C\sum_{n=1}^{\infty} n^{r-2-(q/p)}h(n)\left(n\exp\left\{6\sum_{i=1}^{[\log n]}\varphi^{1/2}(2^i)\right\}\right)^{q/2}(E|Y_1|^2 I\{|Y_1|\leq n^{1/p}\})^{q/2} \\
&\quad + C\sum_{n=1}^{\infty} n^{r-1-(q/p)}h(n)E|Y_1|^q I\{|Y_1|\leq n^{1/p}\} \\
&=: J_1 + J_2.
\end{aligned}$$

Note that $\varphi(m) \to 0$ as $m \to \infty$, hence $\sum_{i=1}^{[\log n]}\varphi^{1/2}(2^i) = o(\log n)$. Furthermore, $\exp\left\{A\sum_{i=1}^{[\log n]}\varphi^{1/2}(2^i)\right\} = o(n^t)$ for any $A > 0$ and $t > 0$.

We consider two separate cases. If $rp < 2$, take $q = 2$. Note that in this case $r - (2r/p) < 0$. Take $t > 0$ small enough such that $r - (2r/p) + t < 0$. We have

$$\begin{aligned}
J_1 &= C\sum_{n=1}^{\infty} n^{r-2-(2/p)}h(n)\left(n\exp\left\{6\sum_{i=1}^{[\log n]}\varphi^{1/2}(2^i)\right\}\right)E|Y_1|^2 I\{|Y_1|\leq n^{1/p}\} \\
&\leq C\sum_{n=1}^{\infty} n^{r-(2/p)-1}h(n)\exp\left\{6\sum_{i=1}^{[\log n]}\varphi^{1/2}(2^i)\right\}E|Y_1|^2 I\{|Y_1|\leq n^{1/p}\} \\
&\leq C\sum_{n=1}^{\infty} n^{r-2/p+t-1}h(n)E|Y_1|^{rp}|Y_1|^{2-rp}I\{|Y_1|\leq n^{1/p}\} \\
&\leq C\sum_{n=1}^{\infty} n^{r-(2r/p)+t-1}h(n)E|Y_1|^{rp} < \infty.
\end{aligned}$$

If $rp \geq 2$, take $q > \frac{2p(r-1)}{2-p}$. We have that $r - (q/p) + (q/2) < 1$. Next, take $t > 0$ small enough such that $r - (q/p) + (q/2) + t < 1$. Note that in this case $E|Y_1|^2 < \infty$. We have

$$\begin{aligned}
J_1 &= C\sum_{n=1}^{\infty} n^{r-2-(q/p)}h(n)\left(n\exp\left\{6\sum_{i=1}^{[\log n]}\varphi^{1/2}(2^i)\right\}\right)^{q/2}(E|Y_1|^2 I\{|Y_1|\leq n^{1/p}\})^{q/2} \\
&\leq C\sum_{n=1}^{\infty} n^{r-(q/p)+(q/2)+t-2}h(n) < \infty.
\end{aligned}$$

By Lemma 4 we have that $J_2 < \infty$. □

Next, we prove Theorem 2.

Proof. By Lemma 5 we only need to show that $I < \infty$ and $J < \infty$ with $r = 1$.

For I, by Markov and C_r-inequalities (note that $\theta \leq 1$)

$$\begin{aligned}
I &\leq C\sum_{n=1}^{\infty} n^{-1}h(n)n^{-\theta/p}E\max_{1\leq k\leq n}\left|\sum_{i=-\infty}^{\infty} a_i\sum_{j=i+1}^{i+k} Y_j I\{|Y_j| > n^{1/p}\}\right|^\theta \\
&\leq C\sum_{n=1}^{\infty} n^{-\theta/p}h(n)E|Y_1|^\theta I\{|Y_1| > n^{1/p}\} \\
&= C\sum_{n=1}^{\infty} n^{-\theta/p}h(n)\sum_{m=n}^{\infty}E|Y_1|^\theta I\{m < |Y_1|^p \leq m+1\}
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{m=1}^{\infty} E|Y_1|^{\theta} I\{m < |Y_1|^p \le m+1\} \sum_{n=1}^{m} n^{-\theta/p} h(n) \\
&\le C\sum_{m=1}^{\infty} m^{1-\theta/p} h(m) E|Y_1|^{\theta} I\{m < |Y_1|^p \le m+1\} \\
&\le CE|Y_1|^p h(|Y_1|^p) < \infty.
\end{aligned}
$$

For J, by Markov and Hölder inequalities, and Lemma 1

$$
\begin{aligned}
J &\le C\sum_{n=1}^{\infty} n^{-1} h(n) n^{-2/p} E \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_{nj} \right|^2 \\
&\le C\sum_{n=1}^{\infty} n^{-1} h(n) n^{-2/p} E \left(\sum_{i=-\infty}^{\infty} |a_i|^{1/2} \left(|a_i|^{1/2} \max_{1\le k\le n} \left| \sum_{j=i+1}^{i+k} Y_{nj} \right| \right) \right)^2 \\
&\le C\sum_{n=1}^{\infty} n^{-1-2/p} h(n) \left(\sum_{i=-\infty}^{\infty} |a_i| \right) \sum_{i=-\infty}^{\infty} |a_i| E \max_{1\le k\le n} \left| \sum_{j=i+1}^{i+k} Y_{nj} \right|^2 \\
&\le C\sum_{n=1}^{\infty} n^{-1-2/p} h(n) \left(n \exp\left\{ 6 \sum_{i=1}^{[\log n]} \varphi^{1/2}(2^i) \right\} \right) E|Y_1|^2 I\{|Y_1| \le n^{1/p}\} \\
&\le C\sum_{n=1}^{\infty} n^{-2/p} h(n) E|Y_1|^2 I\{|Y_1| \le n^{1/p}\} < \infty.
\end{aligned}
$$

The last inequality holds by Lemma 4.

Now we will show almost sure convergence. By the first part of Theorem 2, $EY_1 = 0$ and $E|Y_1|^p < \infty$ imply

$$
\sum_{n=1}^{\infty} n^{-1} P\left\{ \max_{1\le k\le n} |S_k| \ge \varepsilon n^{1/p} \right\} < \infty, \quad \text{for all } \varepsilon > 0.
$$

Hence

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} P\left\{ \max_{1\le m\le n} |S_m| > \varepsilon n^{1/p} \right\} \\
&= \sum_{k=1}^{\infty} \sum_{n=2^{k-1}}^{2^k} n^{-1} P\left\{ \max_{1\le m\le n} |S_m| > \varepsilon n^{1/p} \right\} \\
&\ge 1/2 \sum_{k=1}^{\infty} P\left\{ \max_{1\le m\le 2^{k-1}} |S_m| > \varepsilon 2^{k/p} \right\}.
\end{aligned}
$$

By Borel–Cantelli lemma,

$$
2^{-k/p} \max_{1\le m\le 2^k} |S_m| \to 0 \quad \text{almost surely}
$$

which implies that $S_n/n^{1/p} \to 0$ almost surely. □

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References

Burton, R.M., Dehling, H., 1990. Large deviations for some weakly dependent random processes. Statist. Probab. Lett. 9, 397–401.
Chen, P., Hu, T.-C., Volodin, A., 2006. A note on the rate of complete convergence for maximums of partial sums for moving average processes in Randemacher type Banach spaces. Lobachevskii J. Math. 21, 45–55.
Hsu, P.L., Robbins, H., 1947. Complete convergence and the law of large numbers. Proc. Natl. Acad. Sci. USA 33, 25–31.
Ibragimov, I.A., 1962. Some limit theorem for stationary processes. Theory Probab. Appl. 7, 349–382.
Li, D., Rao, M.B., Wang, X.C., 1992. Complete convergence of moving average processes. Statist. Probab. Lett. 14, 111–114.
Seneta, E., 1976. Regularly Varying Function. In: Lecture Notes in Math., vol. 508. Springer, Berlin.
Shao, Q.M., 1988. A moment inequality and its application. Acta Math. Sinica 31, 736–747 (in Chinese).
Zhang, L., 1996. Complete convergence of moving average processes under dependence assumptions. Statist. Probab. Lett. 30, 165–170.