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COMPLETE MOMENT CONVERGENCE FOR L^P-MIXINGALES*

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Abstract In this paper, the complete moment convergence for L^p-mixingales are studied. Sufficient conditions are given for the complete moment convergence for the maximal partial sums of B-valued L^p-mixingales by utilizing the Rosenthal maximal type inequality for B-valued martingale difference sequence, which extend and improve the related known works in the literature.

Key words complete moment convergence; q-smooth Banach space; L^p-mixingales

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1 Introduction

Let B be a real separable Banach space with norm $\|\cdot\|$, $\{\Omega, \mathcal{F}, P\}$ be a probability space. An $\mathcal{F}$-measurable function from Ω into B is called a B-valued random variable. The expected value of a B-valued random variable X is defined to be the Bochner integral (when $E\|X\| < \infty$) and is denoted by EX. B is said to be q-smooth ($1 \le q \le 2$) if there exists a constant $C_q > 0$ such that for every B-valued L^q-integrable martingale difference sequence $\{D_n, n \ge 1\}$,

$$E\left\|\sum_{i=1}^{n} D_i\right\|^q \le C_q \sum_{i=1}^{n} E\|D_i\|^q, n \ge 1.$$

Obviously, if B is the d-dimensional Euclidean space ($d \ge 1$), then B is 2-smooth.

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Throughout this paper, let $\{X_{n,i}, n, i \geq 1\}$ be an array of B-valued L^p-integrable random variables $(p > 0)$, $\{\mathcal{F}_{n,i}, -\infty < i < \infty, n \geq 1\}$ be a family of sub σ-algebras of $\mathcal{F}$ such that for each $n \geq 1$, $\{\mathcal{F}_{n,i}, -\infty < i < \infty\}$ is increasing in i. Then $\{X_{n,i}, \mathcal{F}_{n,i}\}$ is called an L^p-mixingale array if there exist nonnegative constants $\{C_{n,i}, n, i \geq 1\}$ and $\{\psi(j), j \geq 0\}$ such that $\psi(j) \downarrow 0$ as $j \to \infty$ and for all $i \geq 1$ and $j \geq 0$, we have

$$\|E(X_{n,i}|\mathcal{F}_{n,i-j})\|_p \leq C_{n,i}\psi(j) \quad \text{and} \quad \|X_{n,i} - E(X_{n,i}|\mathcal{F}_{n,i+j})\|_p \leq C_{n,i}\psi(j+1),$$

where $\|X\|_p = (E\|X\|^p)^{1/p}$.

Mcleish [29] first introduced the concept of L^2-mixingale sequences and proved some convergence theorems and the strong laws of large numbers for such sequences. Afterwards, McLeish [30, 31] proved invariance principles for mixingales. Yin [42] generalized McLeish's concept of mixingale to operator-valued mixingale, and proved the operator-valued mixingale convergence theorems. Mixingales cover a quite wide scope, Hall and Heyde [17] pointed out that martingales, linear processes and uniformly mixing processes all belong to the scope of mixingales. It is more important for mixingales that they have various applications in the probability limit theory, strong (or weak) consistency of estimators and kernel regression function estimation. For these reasons many authors had more and more interest in the study of mixingales, including Andrews [1], Chen and White [5], Davidson [7], Davidson and De Jong [8], De Jong [9–12], Fazekas and Klesov [13], Gan [14, 15], Hansen [18, 19], Hong et al. [20], Hu [22], Liang and Ren [27], Meng and Lin [28], Yang et al. [41] and so on. We recommend the paper of Hu [23] for more information. Hu [23] gave some more general sufficient conditions for the complete convergence for B-valued L^p-mixingales including the following result.

Theorem 1.1 Let $1 < q \leq 2, 1/q < \alpha < 1, 1 \leq p \leq 2$, B be a q-smooth Banach space, $\{X_n, \mathcal{F}_n\}$ be a B-valued L^p-mixingale, and X be a real random variable satisfying $P(\|X_n\| > x) \leq P(|X| > x)$ for all $x > 0$ and every $n \geq 1$. Let $h(x) > 0$ be a slowly varying function at infinity. Suppose that $E|X|^{1/\alpha}h(|X|^{1/\alpha}) < \infty$ and

$$\sum_{n=1}^{\infty} n^{-\alpha p-1}h(n)\,(\psi(1))^p \left(\sum_{i=1}^{n} C_i\right)^p < \infty$$

hold. Then

$$\sum_{n=1}^{\infty} n^{-1}h(n)P(\|S_n\| \geq \varepsilon n^{\alpha}) < \infty, \forall\ \varepsilon > 0,$$

where $S_n = \sum\limits_{i=1}^{j} X_i, n \geq 1$.

A sequence of B-valued random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant $\theta \in B$ if

$$\sum_{n=1}^{\infty} P(\|U_n - \theta\| \geq \varepsilon) < \infty, \forall \varepsilon > 0.$$

The concept of complete convergence was introduced by Hsu and Robbins [21]. Moreover, they proved that the sequence of arithmetic means of real independent identically distribution random variables converges completely to the expected value if the variance of the summands is finite. This result was generalized and extended by many authors. One can refer to Bai and Su [2], Baum and Katz [3], Hu [22, 23], Kuczmaszewska [24], Li [25], Liang and Ren [27], Qiu and Chen ([32, 34]), Wang and Hu [40], and so forth.

Chow [6] first investigated the complete moment convergence, which is more exact than the complete convergence. He obtained the following result.

Theorem 1.2 Let $1/2 < \alpha \le 1, \alpha p \ge 1$, and Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. real valued random variables with $EX = 0$. If $E\{|X|^p + |X|\log(1 + |X|)\} < \infty$, then

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} E\left\{\left|\sum_{k=1}^{n} X_k\right| - \varepsilon n^{\alpha}\right\}_+ < \infty, \forall \varepsilon > 0,$$

where and in the following $x_+ = \max\{0, x\}$, $x_+^q = (x_+)^q$.

Chow's result was generalized and extended by many authors. See the works by Chen and Wang [4], Liang et al.[26], Qiu and Chen [32–34], Qiu et al.[35], Sung [37], Wang and Su [38], Wang and Zhao [39], Wang and Hu [40], etc.

The aim of this paper is to extend and improve Theorem 1.2 to B-valued L^p-mixingales ($p \ge 1$), since there aren't papers in the current literature reporting on complete moment convergence for mixingales.

In a general way, to prove the complete moment convergence, we should first establish the corresponding complete convergence. In this paper, we find a new condition which can imply the complete convergence and the complete moment convergence directly, see Lemma 2.5. So the method of proving the main results is different.

Recall that a function $h(x)$ is said to be slowly varying at infinity if it is real valued, positive and measurable on $[0, \infty)$, and if for each $\lambda > 0$,

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = 1.$$

We refer to Seneta [36] for other equivalent definitions and for a detailed and comprehensive study of properties of slowly varying functions.

Let X be a real random variable, $\{X_{n,i}\} \prec X$ means that for all $x > 0$ and some constant $K > 0$,

$$\frac{1}{n}\sum_{i=1}^{n} P(\|X_{n,i}\| > x) \le KP(|X| > x), \forall\ n \ge 1$$

holds.

Throughout this paper, C will represent a positive constant not depending on n which may change from one place to another, and $I(A)$ represents the indicator function of the set A.

2 Preparations

In order to prove our main results, we need the following lemmas.

Lemma 2.1 (see [16]) Let $1 \le q \le 2, B$ be a q-smooth Banach space. For any $t > 0$, there exists a positive constant C_t depending only on t such that for every B-valued martingale difference sequence $\{X_n, \mathcal{F}_n, n \ge 1\}$,

(i) $E\max\limits_{1\le k\le n}\left\|\sum\limits_{i=1}^{k} X_i\right\|^t \le C_t \sum\limits_{i=1}^{n} E\|X_i\|^t$, $\forall t \in [1, q]$, $\forall n \ge 1$;

(ii) $E\max\limits_{1\le k\le n}\left\|\sum\limits_{i=1}^{k} X_i\right\|^t \le C_t\Big\{\sum\limits_{i=1}^{n} E\|X_i\|^t + E\Big(\sum\limits_{i=1}^{n} E\left(\|X_i\|^q|\mathcal{F}_{i-1}\right)\Big)^{t/q}\Big\}$, $\forall t \in [q, \infty)$, $\forall n \ge 1$.

Lemma 2.2 (see [24]) Let β be a positive constant. Suppose that $\{X_n, n \geq 1\}$ is a sequence of B-valued random variables satisfying $\{X_n\} \prec X$.

(i) If $E|X|^\beta < \infty$, then $\frac{1}{n}\sum_{j=1}^{n} E\|X_j\|^\beta \leq CE|X|^\beta$,

(ii) $\frac{1}{n}\sum_{j=1}^{n} E\|X_j\|^\beta I(\|X_j\| \leq x) \leq C\left\{E|X|^\beta I(|X| \leq x) + x^\beta P(|X| > x)\right\}, \forall\ x > 0$,

(iii) $\frac{1}{n}\sum_{j=1}^{n} E\|X_j\|^\beta I(\|X_j\| > x) \leq CE|X|^\beta I(|X| > x), \forall\ x > 0$.

Lemma 2.3 (see [36]) If $h(x)$ is a slowly varying function at infinity, then for any $s > 0$,

$$C_1 n^{-s} h(n) \leq \sum_{i=n}^{\infty} i^{-1-s} h(i) \leq C_2 n^{-s} h(n)$$

and

$$C_3 n^s h(n) \leq \sum_{i=1}^{n} i^{-1+s} h(i) \leq C_4 n^s h(n),$$

where $C_1, C_2, C_3, C_4 > 0$ depend only on s.

The following lemma can be easily proved by Lemma 2.3. Here we omit the details of the proof.

Lemma 2.4 Let $\alpha > 0, \tau > 0$. Let X be a real random variable, $h(x)$ be a slowly varying function at infinity.

(i) Suppose that $E|X|^\tau h(|X|^{1/\alpha}) < \infty$, then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-1} h(n) \int_{(\varepsilon n^\alpha)^v}^{\infty} x^{-t/v} E|X|^t I(|X| > x^{1/v}) \mathrm{d}x < \infty, \forall v \in (0,\tau), t \in [0,\tau)$$

and

$$\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-1} h(n) \int_{(\varepsilon n^\alpha)^v}^{\infty} x^{-t/v} E|X|^t I(|X| \leq x^{1/v}) \mathrm{d}x < \infty, \forall v \in (0,\tau), t \in (\tau,\infty).$$

(ii) Suppose that $E|X|\log|X| < \infty$, then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} \int_{\varepsilon n^\alpha}^{\infty} x^{-t} E|X|^t I(|X| > x) \mathrm{d}x < \infty, \forall t \in [0,1)$$

and

$$\sum_{n=1}^{\infty} n^{-1} \int_{\varepsilon n^\alpha}^{\infty} x^{-t} E|X|^t I(|X| \leq x) \mathrm{d}x < \infty, \forall t \in (1,\infty).$$

Lemma 2.5 Let $v > 0, \{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of positive real numbers. Let $\{X_{n,i}\}$ be an array of B-valued random variables. Suppose that

$$\sum_{n=1}^{\infty} b_n a_n^{-v} \int_{(\varepsilon a_n)^v}^{\infty} P\Big(\max_{1\leq j\leq n} \|S_{n,j}\| > x^{1/v}\Big) \mathrm{d}x < \infty$$

holds for all $\varepsilon > 0$. Then

$$\sum_{n=1}^{\infty} b_n a_n^{-v} E\Big\{\max_{1\leq j\leq n} \|S_{n,j}\| - \varepsilon a_n\Big\}_+^v < \infty, \forall\ \varepsilon > 0,$$

where $S_{n,j} = \sum_{i=1}^{j} X_{n,i}$.

Proof For any fixed $\varepsilon > 0$, note that

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} b_n a_n^{-v} \int_{(2^{-1}\varepsilon a_n)^v}^{\infty} P\left(\max_{1\le j\le n} \|S_{n,j}\| > x^{1/v}\right) \mathrm{d}x \\
&\ge \sum_{n=1}^{\infty} b_n a_n^{-v} \int_{(2^{-1}\varepsilon a_n)^v}^{(\varepsilon a_n)^v} P\left(\max_{1\le j\le n} \|S_{n,j}\| > x^{1/v}\right) \mathrm{d}x \\
&\ge \sum_{n=1}^{\infty} b_n a_n^{-v} \int_{(2^{-1}\varepsilon a_n)^v}^{(\varepsilon a_n)^v} P\left(\max_{1\le j\le n} \|S_{n,j}\| > \varepsilon a_n\right) \mathrm{d}x \\
&= (1-2^{-v})\varepsilon^v \sum_{n=1}^{\infty} b_n P\left(\max_{1\le j\le n} \|S_{n,j}\| > \varepsilon a_n\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{n=1}^{\infty} b_n a_n^{-v} E\left\{\max_{1\le j\le n} \|S_{n,j}\| - \varepsilon a_n\right\}_+^v \\
&= \sum_{n=1}^{\infty} b_n a_n^{-v} \int_0^{\infty} P\left(\max_{1\le j\le n} \|S_{n,j}\| - \varepsilon a_n > x^{1/v}\right) \mathrm{d}x \\
&= \sum_{n=1}^{\infty} b_n a_n^{-v} \int_0^{(\varepsilon a_n)^v} P\left(\max_{1\le j\le n} \|S_{n,j}\| - \varepsilon a_n > x^{1/v}\right) \mathrm{d}x \\
&\quad + \sum_{n=1}^{\infty} b_n a_n^{-v} \int_{(\varepsilon a_n)^v}^{\infty} P\left(\max_{1\le j\le n} \|S_{n,j}\| - \varepsilon a_n > x^{1/v}\right) \mathrm{d}x \\
&\le \varepsilon^v \sum_{n=1}^{\infty} b_n P\left(\max_{1\le j\le n} \|S_{n,j}\| > \varepsilon a_n\right) \\
&\quad + \sum_{n=1}^{\infty} b_n a_n^{-v} \int_{(\varepsilon a_n)^v}^{\infty} P\left(\max_{1\le j\le n} \|S_{n,j}\| > x^{1/v}\right) \mathrm{d}x \\
&< \infty.
\end{aligned}$$

□

3 Main Results and Proofs

Theorem 3.1 Let $1 \le q \le 2, \min\{p,\tau\} \ge 1, \alpha q > 1, \alpha\tau \ge 1, 0 < v < \min\{\tau, p\}$. Let B be a q-smooth Banach space, and $\{X_{n,i}, \mathcal{F}_{n,i}\}$ be a B-valued L^p-mixingale array satisfying $\{X_{n,i}\} \prec X$. Let $h(x) > 0$ be a slowly varying function at infinity. When $\tau \ge q$, we further assume that $E\big(\sup_{n\ge1,i\ge1} E(\|X_{n,i}\|^q|\mathcal{F}_{n,i-1})\big)^{m/q} < \infty$ for some $m > q(\alpha\tau-1)/(\alpha q-1)$. Suppose that $E|X|^\tau h(|X|^{1/\alpha}) < \infty$ and

$$\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha p-2} h(n) \left(\psi(1)\right)^p \left(\sum_{i=1}^{n} C_{n,i}\right)^p < \infty \tag{3.1}$$

hold. Then

$$\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2} h(n) E\left\{\max_{1\le j\le n} \|S_{n,j}\| - \varepsilon n^\alpha\right\}_+^v < \infty, \forall \varepsilon > 0, \tag{3.2}$$

$$\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)E\left\{\max_{1\le j\le n}\|S_{n,n}^{(j)}\|-\varepsilon n^{\alpha}\right\}_{+}^{v}<\infty,\forall\varepsilon>0,\tag{3.3}$$

$$\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)E\left\{\max_{1\le j\le n}\|X_{n,j}\|-\varepsilon n^{\alpha}\right\}_{+}^{v}<\infty,\forall\varepsilon>0,\tag{3.4}$$

where $S_{n,j}=\sum_{i=1}^{j}X_{n,i}, S_{n,n}^{(j)}=S_{n,n}-X_{n,j}, n\ge 1, j=1,2,\cdots,n$.

Proof First, we prove (3.2). We will apply Lemma 2.5 to the array $\{X_{n,i}\}$ and $a_n=n^{\alpha}, b_n=n^{\alpha\tau-2}h(n)$. For $x>0$, set $Y_{n,i}^{(x)}=X_{n,i}I(\|X_{n,i}\|\le x^{1/v}), Z_{n,i}^{(x)}=X_{n,i}-Y_{n,i}^{(x)}, U_{n,i}^{(x)}=E(Y_{n,i}^{(x)}|\mathcal{F}_{n,i})-E(Y_{n,i}^{(x)}|\mathcal{F}_{n,i-1}), V_{n,i}^{(x)}=E(Z_{n,i}^{(x)}|\mathcal{F}_{n,i})-E(Z_{n,i}^{(x)}|\mathcal{F}_{n,i-1})$,

$$X_{n,i}=X_{n,i}-E(X_{n,i}|\mathcal{F}_{n,i})+U_{n,i}^{(x)}+V_{n,i}^{(x)}+E\left(X_{n,i}|\mathcal{F}_{n,i-1}\right).$$

Therefore, for any fixed $\varepsilon>0$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty}P\left(\max_{1\le j\le n}\|S_{n,j}\|>x^{1/v}\right)\mathrm{d}x\\
\le&\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty}P\left(\max_{1\le j\le n}\left\|\sum_{i=1}^{j}\left(X_{n,i}-E(X_{n,i}|\mathcal{F}_{n,i})\right)\right\|\ge x^{1/v}/4\right)\mathrm{d}x\\
&+\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty}P\left(\max_{1\le j\le n}\left\|\sum_{i=1}^{j}U_{n,i}^{(x)}\right\|\ge x^{1/v}/4\right)\mathrm{d}x\\
&+\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty}P\left(\max_{1\le j\le n}\left\|\sum_{i=1}^{j}V_{n,i}^{(x)}\right\|\ge x^{1/v}/4\right)\mathrm{d}x\\
&+\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty}P\left(\max_{1\le j\le n}\left\|\sum_{i=1}^{j}E\left(X_{n,i}|\mathcal{F}_{n,i-1}\right)\right\|\ge x^{1/v}/4\right)\mathrm{d}x\\
:=&J_1+J_2+J_3+J_4.
\end{aligned}\tag{3.5}$$

So, to prove (3.2), it suffices to show that $J_i<\infty$ for $i=1,2,3,4$ by (3.5) and Lemma 2.5.

For J_1, by the Markov inequality, the Minkowski inequality, the L^p-mixingale property, (3.1) and $0<v<p$,

$$\begin{aligned}
J_1&\le C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)E\left(\max_{1\le j\le n}\left\|\sum_{i=1}^{j}\left(X_{n,i}-E(X_{n,i}|\mathcal{F}_{n,i})\right)\right\|\right)^{p}\int_{(\varepsilon n^{\alpha})^{v}}^{\infty}x^{-p/v}\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha p-2}h(n)E\left(\sum_{i=1}^{n}\|X_{n,i}-E(X_{n,i}|\mathcal{F}_{n,i})\|\right)^{p}\\
&\le C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha p-2}h(n)\left(\sum_{i=1}^{n}\|X_{n,i}-E(X_{n,i}|\mathcal{F}_{n,i})\|_{p}\right)^{p}\\
&\le C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha p-2}h(n)(\psi(1))^{p}\left(\sum_{i=1}^{n}C_{n,i}\right)^{p}\\
&<\infty.
\end{aligned}\tag{3.6}$$

Similarly, we can prove $J_4<\infty$.

For J_2, note that $\{U_{n,i}^{(x)}, \mathcal{F}_{n,i}, 1 \le i \le n\}$ is a martingale difference sequence for fixed $n \ge 2$ and any $x > 0$, we consider the following two cases.

Case 1 $\tau \ge q$. We have $\alpha\tau - 2 - \alpha m + m/q < -1$ and $m > \tau$ by $m > q(\alpha\tau - 1)/(\alpha q - 1)$. By the Markov inequality and Lemma 2.1(ii),

$$\begin{aligned}
J_2 &\le C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty} x^{-m/v}E\left(\max_{1\le j\le n}\left\|\sum_{i=1}^{j}U_{n,i}^{(x)}\right\|\right)^{m}\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty} x^{-m/v}\sum_{i=1}^{n}E\|U_{n,i}^{(x)}\|^{m}\mathrm{d}x\\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty} x^{-m/v}E\left(\sum_{i=1}^{n}E\left(\|U_{n,i}^{(x)}\|^{q}|\mathcal{F}_{n,i-1}\right)\right)^{m/q}\mathrm{d}x\\
&:= J_{21} + J_{22}.
\end{aligned} \tag{3.7}$$

For J_{21}, by the C_r-inequality, Jensen inequality, Lemma 2.2(ii), the moment condition, $0 < v < \tau < m$ and Lemma 2.4(i),

$$\begin{aligned}
J_{21} &\le C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty} x^{-m/v}\sum_{i=1}^{n}E\left\{E(\|Y_{n,i}^{(x)}\|^{m}|\mathcal{F}_{n,i}) + E(\|Y_{n,i}^{(x)}\|^{m}|\mathcal{F}_{n,i-1})\right\}\mathrm{d}x\\
&= 2C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty} x^{-m/v}\sum_{i=1}^{n}E\|Y_{n,i}^{(x)}\|^{m}\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-1}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty}\left(x^{-m/v}E|X|^{m}I(|X|\le x^{1/v}) + P(|X| > x^{1/v})\right)\mathrm{d}x\\
&< \infty.
\end{aligned} \tag{3.8}$$

For J_{22}, by the C_r-inequality, Jensen inequality and Lemma 2.3,

$$\begin{aligned}
J_{22} &\le C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2+m/q}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty} x^{-m/v}E\left(\sup_{n\ge1,i\ge1}E\left(\|X_{n,i}\|^{q}|\mathcal{F}_{n,i-1}\right)\right)^{m/q}\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha m-2+m/q}h(n)\\
&< \infty.
\end{aligned} \tag{3.9}$$

Case 2 $1 \le \tau < q$. By Lemma 2.1(i), $0 < v < \tau < q$ and Lemma 2.4(i),

$$\begin{aligned}
J_2 &\le C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty} x^{-q/v}E\left(\max_{1\le j\le n}\left\|\sum_{i=1}^{j}U_{n,i}^{(x)}\right\|\right)^{q}\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty} x^{-q/v}\sum_{i=1}^{n}E\|U_{n,i}^{(x)}\|^{q}\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-1}h(n)\int_{(\varepsilon n^{\alpha})^{v}}^{\infty}\left(x^{-q/v}E|X|^{q}I(|X|\le x^{1/v}) + P(|X| > x^{1/v})\right)\mathrm{d}x\\
&< \infty.
\end{aligned} \tag{3.10}$$

For J_3, by the moment condition and Lemma 2.4(i),

$$\begin{aligned} J_3 \le & \sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2} h(n) \int_{(\varepsilon n^\alpha)^v}^{\infty} P\left(\max_{1\le j\le n}\left\|\sum_{i=1}^{j} E\left(Z_{n,i}^{(x)}|\mathcal{F}_{n,i}\right)\right\| > x^{1/v}/8\right)\mathrm{d}x \\ & + \sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2} h(n) \int_{(\varepsilon n^\alpha)^v}^{\infty} P\left(\max_{1\le j\le n}\left\|\sum_{i=1}^{j} E\left(Z_{n,i}^{(x)}|\mathcal{F}_{n,i-1}\right)\right\| > x^{1/v}/8\right)\mathrm{d}x \\ \le & C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2} h(n) \int_{(\varepsilon n^\alpha)^v}^{\infty} P\left(\bigcup_{i=1}^{n}(\|X_{n,i}\| > x^{1/v})\right)\mathrm{d}x \\ \le & C\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-1} h(n) \int_{(\varepsilon n^\alpha)^v}^{\infty} P(|X| > x^{1/v})\mathrm{d}x \\ < & \infty. \end{aligned} \tag{3.11}$$

From the statements above, we complete the proof of (3.2).

(3.2)⇒(3.3) Note that $\|S_{n,n}^{(j)}\| = \|S_{n,n} - X_{n,j}\| \le \|S_{n,n}\| + \|X_{n,j}\| = \|S_{n,n}\| + \|S_{n,j} - S_{n,j-1}\| \le \|S_{n,n}\| + \|S_{n,j}\| + \|S_{n,j-1}\| \le 3\max_{1\le j\le n}\|S_{n,j}\|, \forall\ 1\le j\le n$, where $S_{n,0}=0$, hence

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2} h(n) E\left\{\max_{1\le j\le n}\|S_{n,n}^{(j)}\| - \varepsilon n^\alpha\right\}_+^v \\ \le & \sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2} h(n) E\left\{3\max_{1\le j\le n}\|S_{n,j}\| - \varepsilon n^\alpha\right\}_+^v \\ = & 3^v \sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2} h(n) E\left\{\max_{1\le j\le n}\|S_{n,j}\| - \varepsilon n^\alpha/3\right\}_+^v . \end{aligned} \tag{3.12}$$

Therefore, (3.3) follows from (3.2).

(3.3)⇒(3.4) Since $\frac{1}{2}\|S_{n,n}\| \le \frac{n-1}{n}\|S_{n,n}\| = \|\frac{1}{n}\sum_{j=1}^{n} S_{n,n}^{(j)}\| \le \max_{1\le j\le n}\|S_{n,n}^{(j)}\|$, and $\|X_{n,j}\| = \|S_{n,n} - S_{n,n}^{(j)}\| \le \|S_{n,n}\| + \|S_{n,n}^{(j)}\| \le 3\max_{1\le j\le n}\|S_{n,n}^{(j)}\|, \forall\ n\ge 2$, similar to the proof of (3.12), (3.4) follows from (3.3). □

For $\tau = 1$ and $h(x)\equiv 1$, we have the following Theorem 3.2.

Theorem 3.2 Let $1 < q \le 2, \alpha \ge 1, p > 1$. Let B be a q-smooth Banach space, and $\{X_{n,i},\mathcal{F}_{n,i}\}$ be a B-valued L^p-mixingale array satisfying $\{X_{n,i}\}\prec X$. Suppose that $E|X|\log|X| < \infty$ and

$$\sum_{n=1}^{\infty} n^{\alpha-\alpha p-2}\left(\psi(1)\right)^p \left(\sum_{i=1}^{n} C_{n,i}\right)^p < \infty \tag{3.13}$$

hold. Then

$$\sum_{n=1}^{\infty} n^{-2} E\left\{\max_{1\le j\le n}\|S_{n,j}\| - \varepsilon n^\alpha\right\}_+ < \infty, \forall\varepsilon > 0, \tag{3.14}$$

$$\sum_{n=1}^{\infty} n^{-2} E\left\{\max_{1\le j\le n}\|S_{n,n}^{(j)}\| - \varepsilon n^\alpha\right\}_+ < \infty, \forall\varepsilon > 0, \tag{3.15}$$

$$\sum_{n=1}^{\infty} n^{-2} E\left\{\max_{1\le j\le n}\|X_{n,j}\| - \varepsilon n^\alpha\right\}_+ < \infty, \forall\varepsilon > 0. \tag{3.16}$$

Proof From the proof of Theorem 3.1, we only need to prove (3.14). From the proof (3.2), in order to prove (3.14), we only need to prove $J_2 < \infty$ and $J_3 < \infty$ for $\tau = v = 1, h(x) \equiv 1$. For J_2, similar to the proof of (3.10), by the Markov inequality, Lemma 2.1(i), Lemma 2.2(ii), $q > 1$ and Lemma 2.4(ii),

$$\begin{aligned} J_2 &\le C\sum_{n=1}^{\infty} n^{-2}\int_{\varepsilon n^{\alpha}}^{\infty} x^{-q} E\left(\max_{1\le j\le n}\left\|\sum_{i=1}^{j} U_{n,i}^{(x)}\right\|\right)^q \mathrm{d}x \\ &\le C\sum_{n=1}^{\infty} n^{-2}\int_{\varepsilon n^{\alpha}}^{\infty} x^{-q}\sum_{i=1}^{n} E\|U_{n,i}^{(x)}\|^q \mathrm{d}x \\ &\le C\sum_{n=1}^{\infty} n^{-1}\int_{\varepsilon n^{\alpha}}^{\infty} \left\{x^{-q}E|X|^q I(|X|\le x) + P(|X|>x)\right\}\mathrm{d}x \\ &< \infty. \end{aligned} \tag{3.17}$$

For J_3, similar to the proof of (3.11), by Lemma 2.4(ii),

$$\begin{aligned} J_3 &\le C\sum_{n=1}^{\infty} n^{-2}\int_{\varepsilon n^{\alpha}}^{\infty} P\left(\bigcup_{i=1}^{n}(\|X_{n,i}\| > x)\right)\mathrm{d}x \\ &\le C\sum_{n=1}^{\infty} n^{-1}\int_{\varepsilon n^{\alpha}}^{\infty} P(|X|>x)\mathrm{d}x \\ &< \infty. \end{aligned} \tag{3.18}$$

Therefore, (3.14) holds. □

In the sequence case, i.e., if $X_{n,i} = X_i, \mathcal{F}_{n,i} = \mathcal{F}_i, C_{n,i} = C_i$ for every $n \ge 1$. We have the following results.

Theorem 3.3 Let $1 \le q \le 2, \min\{\tau, p\} \ge 1, \alpha\cdot\min\{\tau, q\} > 1, 0 < v < \min\{\tau, p\}$. Let B be a q-smooth Banach space, and $\{X_n, \mathcal{F}_n\}$ be a B-valued L^p-mixingale sequence satisfying $\{X_n\} \prec X$. Let $h(x) > 0$ be a slowly varying function at infinity. When $\tau \ge q$, we further assume that $E\big(\sup_{n\ge1} E(\|X_n\|^q|\mathcal{F}_{n-1})\big)^{m/q} < \infty$ for some $m > q(\alpha\tau-1)/(\alpha q-1)$. Suppose that $E|X|^{\tau}h(|X|^{1/\alpha}) < \infty$ and

$$\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha p-2}h(n)\,(\psi(1))^p\left(\sum_{i=1}^{n} C_i\right)^p < \infty \tag{3.19}$$

hold. Then

$$\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)E\left\{\max_{1\le j\le n}\|S_j\| - \varepsilon n^{\alpha}\right\}_+^v < \infty, \forall\, \varepsilon > 0, \tag{3.20}$$

$$\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)E\left\{\max_{1\le j\le n}\|S_n^{(j)}\| - \varepsilon n^{\alpha}\right\}_+^v < \infty, \forall\, \varepsilon > 0, \tag{3.21}$$

$$\sum_{n=1}^{\infty} n^{\alpha\tau-\alpha v-2}h(n)E\left\{\max_{1\le j\le n}\|X_j\| - \varepsilon n^{\alpha}\right\}_+^v < \infty, \forall\, \varepsilon > 0, \tag{3.22}$$

$$\sum_{n=1}^{\infty} n^{\alpha\tau-2}h(n)E\left\{\sup_{j\ge n} j^{-\alpha}\|S_j\| - \varepsilon\right\}_+^v < \infty, \forall\, \varepsilon > 0, \tag{3.23}$$

$$\sum_{n=1}^{\infty} n^{\alpha\tau-2}h(n)E\left\{\sup_{j\geq n} j^{-\alpha}\|X_j\|-\varepsilon\right\}_+^v<\infty, \forall\ \varepsilon>0, \tag{3.24}$$

where $S_n=\sum\limits_{i=1}^{n}X_i, S_n^{(j)}=S_n-X_j, n\geq 1, j=1,2,\cdots,n.$

Proof From Theorem 3.1, (3.20)–(3.22) hold. So, we only need to prove (3.23) and (3.24). First, we prove (3.23). By Lemma 2.3, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha\tau-2}h(n)E\left\{\sup_{j\geq n} j^{-\alpha}\|S_j\|-\varepsilon\right\}_+^v\\
&=\sum_{n=1}^{\infty} n^{\alpha\tau-2}h(n)\int_0^\infty P\left(\sup_{j\geq n} j^{-\alpha}\|S_j\|-\varepsilon>x^{1/v}\right)\mathrm{d}x\\
&=\sum_{i=1}^{\infty}\sum_{2^{i-1}\leq n<2^i} n^{\alpha\tau-2}h(n)\int_0^\infty P\left(\sup_{j\geq n} j^{-\alpha}\|S_j\|-\varepsilon>x^{1/v}\right)\mathrm{d}x\\
&\leq C\sum_{i=1}^{\infty}2^{i(\alpha\tau-1)}h(2^i)\int_0^\infty P\left(\sup_{j\geq 2^{i-1}} j^{-\alpha}\|S_j\|-\varepsilon>x^{1/v}\right)\mathrm{d}x\\
&\leq C\sum_{i=1}^{\infty}2^{i(\alpha\tau-1)}h(2^i)\int_0^\infty \sum_{k=i}^{\infty}P\left(\max_{2^{k-1}\leq j<2^k} j^{-\alpha}\|S_j\|-\varepsilon>x^{1/v}\right)\mathrm{d}x\\
&=C\sum_{k=1}^{\infty}\int_0^\infty P\left(\max_{2^{k-1}\leq j<2^k} j^{-\alpha}\|S_j\|-\varepsilon>x^{1/v}\right)\mathrm{d}x\cdot\sum_{i=1}^{k}2^{i(\alpha\tau-1)}h(2^i)\\
&\leq C\sum_{k=1}^{\infty}2^{k(\alpha\tau-1)}h(2^k)\int_0^\infty P\left(\max_{1\leq j<2^k}\|S_j\|>(\varepsilon+x^{1/v})2^{(k-1)\alpha}\right)\mathrm{d}x\\
&\leq C\sum_{k=1}^{\infty}2^{k(\alpha\tau-\alpha v-1)}h(2^k)\int_0^\infty P\left(\max_{1\leq j\leq 2^k}\|S_j\|>\varepsilon 2^{(k-1)\alpha}+t^{1/v}\right)\mathrm{d}t\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha\tau-\alpha v-2}h(n)\int_0^\infty P\left(\max_{1\leq j\leq n}\|S_j\|>2^{-2\alpha}n^\alpha\varepsilon+t^{1/v}\right)\mathrm{d}t\\
&=C\sum_{n=1}^{\infty}n^{\alpha\tau-\alpha v-2}h(n)E\left\{\max_{1\leq j\leq n}\|S_j\|-\varepsilon_0 n^\alpha\right\}_+^v \qquad (\varepsilon_0=2^{-2\alpha}\varepsilon).
\end{aligned} \tag{3.25}$$

Therefore, (3.23) follows by (3.20).

(3.23)⇒(3.24) Note $j^{-\alpha}\|X_j\|=j^{-\alpha}\|S_j-S_{j-1}\|\leq j^{-\alpha}(\|S_j\|+\|S_{j-1}\|)\leq j^{-\alpha}\|S_j\|+(j-1)^{-\alpha}\|S_{j-1}\|$ for every $j\geq 2$. Thus, (3.24) follows by (3.23). □

By Theorem 3.2, similar to the proof of Theorem 3.3, we have the following theorem.

Theorem 3.4 Let $1<q\leq 2, \alpha\geq 1, p>1$. Let B be a q-smooth Banach space, and $\{X_n,\mathcal{F}_n\}$ be a B-valued L^p-mixingale sequence satisfying $\{X_n\}\prec X$. Suppose that $E|X|\log|X|<\infty$ and

$$\sum_{n=1}^{\infty} n^{\alpha-\alpha p-2}\left(\psi(1)\right)^p\left(\sum_{i=1}^{n}C_i\right)^p<\infty$$

hold. Then

$$\sum_{n=1}^{\infty} n^{-2}E\left\{\max_{1\leq j\leq n}\|S_j\|-\varepsilon n^\alpha\right\}_+<\infty, \forall\varepsilon>0,$$

$$\sum_{n=1}^{\infty} n^{-2} E\left\{\max_{1\le j\le n} \|S_n^{(j)}\| - \varepsilon n^{\alpha}\right\}_+ < \infty, \forall \varepsilon > 0,$$
$$\sum_{n=1}^{\infty} n^{-2} E\left\{\max_{1\le j\le n} \|X_j\| - \varepsilon n^{\alpha}\right\}_+ < \infty, \forall \varepsilon > 0,$$
$$\sum_{n=1}^{\infty} n^{\alpha-2} E\left\{\sup_{j\ge n} j^{-\alpha}\|S_j\| - \varepsilon\right\}_+ < \infty, \forall \varepsilon > 0,$$
$$\sum_{n=1}^{\infty} n^{\alpha-2} E\left\{\sup_{j\ge n} j^{-\alpha}\|X_j\| - \varepsilon\right\}_+ < \infty, \forall \varepsilon > 0.$$

Remark 3.5 If we take $h(x) \equiv 1$ in Theorem 3.1, note that the complete moment convergence implies the complete convergence, so we not only get the Baum- Katz-type theorem (see Baum-Katz-type [3]) for a B-valued L^p-mixingale, but we also consider the case of $\alpha\tau = 1$.

Remark 3.6 Compared with Theorem 1.1, our results in Theorem 3.1 not only consider the maximal partial sum, but also expand the scope of α and τ. Our results enrich Theorem 1.1.

Remark 3.7 Compared with Theorem 1.2, our results not only consider the maximal partial sum, but also extend and enrich Theorem 1.2.

Remark 3.8 Our results extend the corresponding results of Wang and Hu [40] for a martingale difference sequence.

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