Metrika (2017) 80:605–625
DOI 10.1007/s00184-017-0618-z

Weak and strong laws of large numbers for arrays of rowwise END random variables and their applications

Aiting Shen[1] · Andrei Volodin[2]

Received: 1 March 2016 / Published online: 22 May 2017

Abstract In the paper, the Marcinkiewicz–Zygmund type moment inequality for extended negatively dependent (END, in short) random variables is established. Under some suitable conditions of uniform integrability, the L_r convergence, weak law of large numbers and strong law of large numbers for usual normed sums and weighted sums of arrays of rowwise END random variables are investigated by using the Marcinkiewicz–Zygmund type moment inequality. In addition, some applications of the L_r convergence, weak and strong laws of large numbers to nonparametric regression models based on END errors are provided. The results obtained in the paper generalize or improve some corresponding ones for negatively associated random variables and negatively orthant dependent random variables.

Keywords Extended negatively dependent random variables · L_r convergence · Marcinkiewicz–Zygmund type moment inequality · Law of large numbers · Nonparametric regression models

Mathematics Subject Classification 60F05 · 60F15 · 60F25 · 62G05

Supported by the National Natural Science Foundation of China (11501004, 11671012), the Natural Science Foundation of Anhui Province (1508085J06) and the Provincial Natural Science Research Project of Anhui Colleges (KJ2015A018).

✉ Aiting Shen
empress201010@126.com

1 School of Mathematical Sciences, Anhui University, Hefei 230601, People's Republic of China

2 Department of Mathematics and Statistics, University of Regina, Regina, SK S4S 0A2, Canada

1 Introduction

Let $1 \leq r \leq 2$ and $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $EX_n = 0$ and $E|X_n|^r < \infty$ for all $n \geq 1$. Bahr and Esseen (1965) showed that for any $n \geq 1$,

$$E\left|\sum_{i=1}^{n} X_i\right|^r \leq C_r \sum_{i=1}^{n} E\left|X_i\right|^r, \tag{1.1}$$

where C_r is a positive constant depending only on r.

The formula (1.1) is called the r-th Bahr–Esseen type moment inequality or Marcinkiewicz–Zygmund type moment inequality.

As we known that the Marcinkiewicz–Zygmund type moment inequality plays an important role in probability limit theory and mathematical statistics, especially in establishing strong convergence, weak convergence and large sample properties of statistics in many stochastic models. There are many sequences of random variables satisfying the Marcinkiewicz–Zygmund type moment inequality under some suitable conditions, such as martingale difference sequence (see Chatterji 1969), $\tilde{\rho}$-mixing sequence (see Bryc and Smolenski 1993 or Wu 2006), negatively associated sequence (NA, in short, see Shao 2000), negatively orthant dependent sequence (NOD, in short, see Asadian et al. 2006), negatively superadditive dependent sequence (NSD, in short, see Hu 2000 or Wang et al. 2014), asymptotically almost negatively associated sequence with the mixing coefficients satisfying certain conditions (AANA, in short, see Yuan and An 2009), and so on. However, there is no literature discussing the Marcinkiewicz–Zygmund type moment inequality for extended negatively dependent sequence (END, in short) which includes independent sequence, NA sequence, NSD sequence and NOD sequence as special cases. The main purpose of the paper is to establish the Marcinkiewicz–Zygmund type moment inequality for END random variables and give some applications to L_r convergence, weak and strong law of large numbers under some suitable conditions. In addition, we will present some applications of the L_r convergence, weak and strong law of large numbers to nonparametric regression models based on END errors.

Now, let us recall the the definition of extended negatively dependent random variables which was introduced by Liu (2009) as follows.

Definition 1.1 A finite collection of random variables $X_1, X_2, \ldots, X_n$ is said to be extended negatively dependent (END, in short) if there exists a constant $M > 0$ such that both

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq M \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq M \prod_{i=1}^{n} P(X_i \leq x_i)$$

hold for all real numbers $x_1, x_2, \ldots, x_n$. An infinite sequence $\{X_n, n \geq 1\}$ is said to be END if every finite subcollection is END.

An array of random variables $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ is called rowwise END random variables if for every $n \geq 1$, $\{X_{ni}, 1 \leq i \leq n\}$ are END random variables.

If $M = 1$ in Definition 1.1, then the END structure reduces to the well known notion of NOD random variables, which was introduced by Lehmann (1966) (cf. also Joag-Dev and Proschan 1983). The END structure can reflect not only a negatively structure but also a positive one to some extent. Liu (2009) pointed out that the END random variables can be taken as negatively or positively dependent and provided some interesting examples to support this idea. Joag-Dev and Proschan (1983) also pointed out that NA random variables must be NOD and NOD is not necessarily NA, thus NA random variables are END. In addition, Christofides and Vaggelatou (2004) indicated that NA implies NSD and Hu (2000) pointed out that NSD is NOD. Hence, the class of END random variables includes independent sequence, NA sequence, NSD sequence and NOD sequence as special cases.

Since the concept of END structure was introduced by Liu (2009), many authors studied the probability limit properties for END random variables and provided some interesting applications. See for example, Liu (2010) studied the sufficient and necessary conditions of moderate deviations for END random variables with heavy tails; Chen et al. (2010) established the Kolmogorov strong law of large numbers for END random variables and gave applications to risk theory and renewal theory; Shen (2011) presented Rosenthal type moment inequality for END random variables and gave some applications; Wang and Wang (2013) investigated a more general precise large deviation result for random sums of END real-valued random variables in the presence of consistent variation; Qiu et al. (2013), Wang et al. (2013), Wang et al. (2013), Wang et al. (2014), Wu et al. (2014) and Hu et al. (2015) provided some results on complete convergence for sequences of END random variables or arrays of rowwise END random variables; Cheng and Li (2014) established the asymptotics for the tail probability of random sums with a heavy-tailed random number and END summands; Wang et al. (2015) studied the complete consistency for the estimator of nonparametric regression models based on END errors, and so forth.

Remark 1.1 As is mentioned in Liu (2009), the END structure can reflect not only a negative dependence structure but also a positive one (inequalities from the definition of NOD random variables hold both in reverse direction), to some extend. We refer the interested readers to Example 4.1 in Liu (2009) where END random variables can be taken as negatively or positively dependent. Here, we provide two examples possessing the END structure.

The first one comes from Example 4.2 in Liu (2009). For any $n \geq 1$, let $X_1, X_2, \ldots, X_n$ be dependent according to a copula function $C(u_1, u_2, \ldots, u_n)$ with absolutely continuous distribution functions $F_1, F_2, \ldots, F_n$. Assume that the joint copula density

$$C_{1,2,\ldots,n}(u_1, u_2, \ldots, u_n) = \frac{\partial^n}{\partial u_1 \partial u_2 \ldots \partial u_n} C(u_1, u_2, \ldots, u_n)$$

exists and is uniformly bounded in the whole domain. Then random variables $\{X_n, n \geq 1\}$ are END. As noted in Example 4.2 in Liu (2009), for example, copulas in the Frank family of the form

$$C_\alpha(u_1, u_2, \ldots, u_n) = \frac{1}{\alpha} \ln \left(1 + \frac{(e^{\alpha u_1} - 1) \ldots (e^{\alpha u_n} - 1)}{(e^\alpha - 1)^n}\right), \ \alpha < 0$$

belong to this category.

The another one comes from Chen et al. (2010). Recall that an n-dimensional Farlie–Gumbel–Morgenstern (FGM, in short) distribution has the following form

$$F_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n) = \left(\prod_{k=1}^{n} F_k(x_k)\right) \left(1 + \sum_{1 \leq i < j \leq n} a_{ij} \bar{F}_i(x_i) \bar{F}_j(x_j)\right),$$

where $F_k = 1 - \bar{F}_k$ for $k = 1, 2, \ldots, n$ are corresponding marginal distributions and a_{ij} are real numbers choose such that $F_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n)$ is a proper n-dimensional distribution. Chen et al. (2010) pointed out that every n-dimensional FGM distribution describes a specific END structure.

The following concept of stochastic domination will be used in this work.

Definition 1.2 An array $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_{ni}| > x) \leq C P(|X| > x)$$

for all $x \geq 0$, $i \geq 1$ and $n \geq 1$.

By using the concept of stochastic domination, we can get the following important property for stochastic domination.

Property 1.1 *Suppose that the array $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ is stochastically dominated by a random variable X. Then for all $\alpha > 0$, there exists a positive constant C such that $E|X_{ni}|^\alpha \leq C E|X|^\alpha$ for all $1 \leq i \leq n$ and $n \geq 1$.*

This structure of the paper is organized as follows: some important properties of END random variables are provided in Sect. 2, including the Marcinkiewicz–Zygmund type moment inequality and Rosenthal type moment inequality. These properties will be used to prove the main results of the paper. In Sect. 3, some results on L_r convergence, weak and strong law of large numbers for arrays of rowwise END random variables are established. Finally, some applications of the L_r convergence, weak and strong law of large numbers to nonparametric regression models based on END errors are provided in Sect. 4.

Throughout the paper, let C denote a positive constant not depending on n, which may be different in various places. Let $I(A)$ be the indicator function of the set A. Denote $\log x = \ln \max(x, e)$, $x^+ = xI(x > 0)$ and $x^- = -xI(x < 0)$.

2 Properties of END random variables

In this section, we will present some important properties of END random variables including the Marcinkiewicz–Zygmund type moment inequality and Rosenthal type moment inequality. These properties play important roles to prove the main results of the paper.

The first one is a basic property of END random variables, which can be found in Liu (2010) for instance.

Lemma 2.1 *Let random variables $X_1, X_2, \ldots, X_n$ be END, $f_1, f_2, \ldots, f_n$ be all nondecreasing (or all nonincreasing) functions, then random variables $f_1(X_1)$, $f_2(X_2), \ldots, f_n(X_n)$ are also END.*

The next one is the Rosenthal type moment inequality for END random variables, which was established by Shen (2011). This inequality with exponent 2 can be used to prove the Marcinkiewicz–Zygmund type moment inequality.

Lemma 2.2 *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for some $p \geq 2$ and any $n \geq 1$. Then there exist positive constants C_p depending only on p such that for any $n \geq 1$,*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq C_p \left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}. \tag{2.1}$$

With the Rosenthal type moment inequality accounted for, one can get the Marcinkiewicz–Zygmund type moment inequality for END random variables as follows. The proof is similar to that of Lemma 2.1 in Chen et al. (2014). For convenience of the reader, we will present the proof of Lemma 2.3 in Appendix.

Lemma 2.3 *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables with $EX_n = 0$ and $E|X_n|^r < \infty$ for some $1 \leq r \leq 2$ and any $n \geq 1$. Then there exist positive constants c_r depending only on r such that for any $n \geq 1$,*

$$E\left|\sum_{i=1}^{n} X_i\right|^r \leq c_r \sum_{i=1}^{n} E|X_i|^r. \tag{2.2}$$

Using Lemma 2.3, we can get the following corollary by the same argument as Theorem 2.3.1 in Stout (1974).

Corollary 2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables with $EX_n = 0$ and $E|X_n|^r < \infty$ for some $1 \leq r \leq 2$ and any $n \geq 1$. Then there exist positive constant c_r depending only on r such that for any $n \geq 1$,*

$$E\left(\max_{1 \leq j \leq n}\left|\sum_{i=1}^{j} X_i\right|^r\right) \leq c_r \log^r n \sum_{i=1}^{n} E|X_i|^r. \tag{2.3}$$

Remark 2.1 Assume that (2.2) holds for any $n \geq 1$ and $\sum_{i=1}^{\infty} X_i$ converges almost surely. Then we have by Fatou's lemma that

$$E\left|\sum_{i=1}^{\infty} X_i\right|^r \leq c_r \sum_{i=1}^{\infty} E|X_i|^r. \tag{2.4}$$

Remark 2.2 Let $\{a_n, n \geq 1\}$ be a sequence of real numbers. Under the conditions of Lemma 2.3, we have for $n \geq 1$ that

$$E\left|\sum_{i=1}^{n} a_i X_i\right|^r \leq 2^{r-1} c_r \sum_{i=1}^{n} E|a_i X_i|^r. \tag{2.5}$$

Assume further that $\sum_{i=1}^{\infty} a_i X_i$ converges almost surely, we have by Fatou's lemma that

$$E\left|\sum_{i=1}^{\infty} a_i X_i\right|^r \leq 2^{r-1} c_r \sum_{i=1}^{\infty} E|a_i X_i|^r. \tag{2.6}$$

We only need to note that $a_{ni} = a_{ni}^+ - a_{ni}^-$, and for fixed $n \geq 1$, $\{a_i^+ X_i, 1 \leq i \leq n\}$ and $\{a_i^- X_i, 1 \leq i \leq n\}$ are both END random variables by Lemma 2.1.

3 Main results and their proofs

In Sect. 2, the Marcinkiewicz–Zygmund type moment inequality for END random variables was established. In this section, we will give some applications of the Marcinkiewicz–Zygmund type moment inequality to L_r convergence, weak and strong laws of large numbers for arrays of rowwise END random variables under some uniformly integrable conditions.

In the following, let $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$, let $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ be two sequences of integers (not necessary positive or finite) such that $v_n > u_n$ for all $n \geq 1$ and $v_n - u_n \to \infty$ as $n \to \infty$. Let $\{k_n, n \geq 1\}$ be a sequence of positive numbers such that $k_n \to \infty$ as $n \to \infty$ and $\{h(n), n \geq 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$.

3.1 L_r convergence and weak law of large numbers

The notion of h-integrability with exponent r was introduced by Sung et al. (2008), which deals with usual normed sums of random variables as follows.

Definition 3.1 Let $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of random variables and $r > 0$. The array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is said to be h-integrable with exponent r if

$$\sup_{n\geq 1}\frac{1}{k_n}\sum_{i=u_n}^{v_n}E|X_{ni}|^r<\infty \text{ and } \lim_{n\to\infty}\frac{1}{k_n}\sum_{i=u_n}^{v_n}E|X_{ni}|^r I(|X_{ni}|^r>h(n))=0.$$

Under the conditions of h-integrability with exponent r, Sung et al. (2008) further studied the L_r convergence and weak law of large numbers for arrays of rowwise NA random variables.

Inspired by the concept of h-integrability with exponent r, Wang and Hu (2014) introduced a new and weaker concept of uniform integrability as follows.

Definition 3.2 Let $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of random variables and $r > 0$. The array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is said to be residually h-integrable (R-h-integrable, in short) with exponent r if

$$\sup_{n\geq 1}\frac{1}{k_n}\sum_{i=u_n}^{v_n}E|X_{ni}|^r<\infty \text{ and } \lim_{n\to\infty}\frac{1}{k_n}\sum_{i=u_n}^{v_n}E\left(|X_{ni}|-h^{1/r}(n)\right)^r I(|X_{ni}|^r>h(n))=0.$$

Under the condition of R-h-integrability with exponent r, Wang and Hu (2014) established some weak laws of large numbers for arrays of dependent random variables. Noting that

$$\left(|X_{ni}|-h^{1/r}(n)\right)^r I(|X_{ni}|^r>h(n))\leq |X_{ni}|^r I(|X_{ni}|^r>h(n)),$$

hence, the concept of R-h-integrability with exponent r is weaker than h-integrability with exponent r.

For more details about the L_r convergence and weak law of large numbers for normed sums or weighted sums of random variables based on h-integrability or R-h-integrability, one can refer to Yuan and Tao (2008), Ordóñez et al. (2012), Shen et al. (2013), Sung (2013), and so on.

Inspired by Wang and Hu (2014) and Sung (2013), we get the following results on L_r convergence and weak law of large numbers for arrays of rowwise END random variables. the first one deals with the L_r convergence and weak law of large numbers for normed sums of arrays of rowwise END random variables.

Theorem 3.1 *Suppose that $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is an array of rowwise END R-h-integrable with exponent $1 \leq r < 2$ random variables. Let $k_n \to \infty$, $h(n) \uparrow \infty$, and $h(n)/k_n \to 0$ as $n \to \infty$. Then*

$$\frac{1}{k_n^{1/r}}\sum_{i=u_n}^{v_n}(X_{ni}-EX_{ni})\to 0$$

in L_r and, hence, in probability as $n \to \infty$.

Proof For fixed $n \geq 1$, denote for $u_n \leq i \leq v_n$ that

$$\begin{aligned} Y_{ni} &= -h^{1/r}(n) I\left(X_{ni} < -h^{1/r}(n)\right) + X_{ni} I\left(|X_{ni}| \leq h^{1/r}(n)\right) \\ &\quad + h^{1/r}(n) I\left(X_{ni} > h^{1/r}(n)\right), \\ Z_{ni} &= X_{ni} - Y_{ni} = \left(X_{ni} + h^{1/r}(n)\right) I\left(X_{ni} < -h^{1/r}(n)\right) \\ &\quad + \left(X_{ni} - h^{1/r}(n)\right) I\left(X_{ni} > h^{1/r}(n)\right), \\ S_n &= \frac{1}{k_n^{1/r}} \sum_{i=u_n}^{v_n} (Y_{ni} - EY_{ni}), \ T_n = \frac{1}{k_n^{1/r}} \sum_{i=u_n}^{v_n} (Z_{ni} - EZ_{ni}). \end{aligned}$$

Noting that

$$\frac{1}{k_n^{1/r}} \sum_{i=u_n}^{v_n} (X_{ni} - EX_{ni}) = S_n + T_n, \quad n \geq 1,$$

we have by C_r-inequality that

$$E\left|\frac{1}{k_n^{1/r}} \sum_{i=u_n}^{v_n} (X_{ni} - EX_{ni})\right|^r \leq CE|S_n|^r + CE|T_n|^r.$$

To prove $\frac{1}{k_n^{1/r}} \sum_{i=u_n}^{v_n} (X_{ni} - EX_{ni}) \to 0$ in L_r, we only need to show $E|S_n|^r \to 0$ and $E|T_n|^r \to 0$ as $n \to \infty$, where $1 \leq r < 2$.

Firstly, we will show that $E|S_n|^r \to 0$ as $n \to \infty$. Note that $1 \leq r < 2$, it suffices to show $ES_n^2 \to 0$ as $n \to \infty$.

For fixed $n \geq 1$, it is easily checked that $\{Y_{ni} - EY_{ni}, u_n \leq i \leq v_n\}$ are END random variables by Lemma 2.1. Noting that $1 \leq r < 2$ and $|Y_{ni}| = \min\{|X_{ni}|, h^{1/r}(n)\}$, we have by Lemma 2.3 or Remark 2.1 that

$$\begin{aligned} ES_n^2 &= E\left|\frac{1}{k_n^{1/r}} \sum_{i=u_n}^{v_n} (Y_{ni} - EY_{ni})\right|^2 \\ &\leq \frac{C}{k_n^{2/r}} \sum_{i=u_n}^{v_n} E(Y_{ni} - EY_{ni})^2 \leq \frac{C}{k_n^{2/r}} \sum_{i=u_n}^{v_n} EY_{ni}^2 \\ &\leq \frac{C}{k_n^{2/r}} \cdot h^{(2-r)/r}(n) \cdot \sum_{i=u_n}^{v_n} E|Y_{ni}|^r \\ &\leq C\left[\frac{h(n)}{k_n}\right]^{(2-r)/r} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r \\ &\to 0 \text{ as } n \to \infty, \end{aligned}$$

which implies that $ES_n^2 \to 0$ as $n \to \infty$ and thus, $E|S_n|^r \to 0$ as $n \to \infty$.

Next, we will show that $E|T_n|^r \to 0$ as $n \to \infty$. For fixed $n \geq 1$, we can see that $\{Z_{ni} - EZ_{ni}, u_n \leq i \leq v_n\}$ are still END random variables by Lemma 2.1 again. Noting that

$$|Z_{ni}| = \left(|X_{ni}| - h^{1/r}(n)\right) I\left(|X_{ni}| > h^{1/r}(n)\right),$$

we have by Lemma 2.3 or Remark 2.1 again that

$$\begin{aligned} E|T_n|^r &= E\left|\frac{1}{k_n^{1/r}}\sum_{i=u_n}^{v_n}(Z_{ni} - EZ_{ni})\right|^r \\ &\leq \frac{C}{k_n}\sum_{i=u_n}^{v_n} E\left|Z_{ni} - EZ_{ni}\right|^r \leq \frac{C}{k_n}\sum_{i=u_n}^{v_n} E\left|Z_{ni}\right|^r \\ &\leq \frac{C}{k_n}\sum_{i=u_n}^{v_n} E\left(|X_{ni}| - h^{1/r}(n)\right)^r I\left(|X_{ni}|^r > h(n)\right) \\ &\to 0 \text{ as } n \to \infty, \end{aligned}$$

which implies that $E|T_n|^r \to 0$ as $n \to \infty$. This completes the proof of the theorem. □

Remark 3.1 Note that the concept of R-h-integrability with exponent r is weaker than h-integrability with exponent r and END is weaker than NA. Hence, the result of Theorem 3.1 generalizes and improves the corresponding one of Sung et al. (2008) for NA random variables to the case of END random variables. In addition, the result of Theorem 3.1 generalizes the corresponding one of Wang and Hu (2014) for NOD random variables to the case of END random variables.

The next one deals with the L_r convergence and weak law of large numbers for weighted sums of END random variables. The proof is similar to that of Theorem 2.1 in Sung (2013). So the details are omitted. We should point out that the key technique used here is still the Marcinkiewicz–Zygmund type moment inequality for END random variables.

Theorem 3.2 *Let* $1 \leq r < 2$, $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ *be an array of rowwise END random variables and* $\{a_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ *be an array of constants. Assume that the following conditions hold:*

(i) $\sup_{n\geq 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r < \infty$;
(ii) for any $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r I(|X_{ni}|^r > \epsilon) = 0.$$

Then

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - EX_{ni}) \to 0$$

in L_r and, hence, in probability as $n \to \infty$.

With Theorem 3.2 accounted for, we can get the following corollary. The proof is similar to that of Corollary 2.1 in Sung (2013), so the details are omitted.

Corollary 3.1 *Let $\{a_{ni}, u_n \le i \le v_n, n \ge 1\}$ be an array of constants satisfying $k_n \doteq 1/\sup_{u_n \le i \le v_n} |a_{ni}|^r \to \infty$, $0 < h(n) \uparrow \infty$ and $h(n)/k_n \to 0$ as $n \to \infty$. Let $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ be an array of rowwise END h-integrable with exponent $1 \le r < 2$ random variables. Then*

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - EX_{ni}) \to 0$$

in L_r and, hence, in probability as $n \to \infty$.

If we take $a_{ni} = k_n^{-1/r}$ for $u_n \le i \le v_n$ and $n \ge 1$ in Corollary 3.1, then we can get the following corollary.

Corollary 3.2 *Let $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ be an array of rowwise END h-integrable with exponent $1 \le r < 2$ random variables, $k_n \to \infty$, $0 < h(n) \uparrow \infty$ and $h(n)/k_n \to 0$ as $n \to \infty$. Then*

$$\frac{\sum_{i=u_n}^{v_n} (X_{ni} - EX_{ni})}{k_n^{1/r}} \to 0$$

in L_r and, hence, in probability as $n \to \infty$.

Remark 3.2 Note that the condition "$k_n \doteq 1/\sup_{u_n \le i \le v_n} |a_{ni}|^r \to \infty$, $0 < h(n) \uparrow \infty$ and $h(n)/k_n \to 0$ as $n \to \infty$" in Corollary 3.1 in the paper is weaker than "$k_n \doteq 1/\sup_{u_n \le i \le v_n} |a_{ni}| \to \infty$, $0 < h(n) \uparrow \infty$ and $h(n)/k_n \to 0$ as $n \to \infty$" in Corollary 3.6 of Wang and Hu (2014). Hence, our results of Theorem 3.2 and Corollary 3.1 generalize and improve the corresponding one of Corollary 3.6 in Wang and Hu (2014) for NOD random variables to the case of END random variables.

3.2 Strong law of large numbers

In order to establish the strong version of Theorem 3.1, we should introduce the concept of strongly residual h-integrability with exponent r, which deals with usual normed sums of random variables as follows.

Definition 3.3 Let $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of random variables and $r > 0$. The array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is said to be strongly residually h-integrable (SR-h-integrable, for short) with exponent r if

$$\sup_{n\geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E\left(|X_{ni}| - h^{1/r}(n)\right)^r I\left(|X_{ni}|^r > h(n)\right) < \infty.$$

Remark 3.3 We point out that the concept of SR–h-integrability with exponent r is stronger than the concept of R–h-integrability with exponent r. SR–h-integrability with exponent r implies R–h-integrability with exponent r.

Our main result on the strong law of large numbers for usual normed sums of arrays of rowwise END random variables is as follows.

Theorem 3.3 *Suppose that $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ is an array of rowwise END SR-h-integrable with exponent $1 \leq r < 2$ random variables. Let $k_n \to \infty$, $h(n) \uparrow \infty$, and $\sum_{n=1}^{\infty} \left(\frac{h(n)}{k_n}\right)^{\frac{2-r}{r}} < \infty$. Then $\frac{1}{k_n^{1/r}} \sum_{i=u_n}^{v_n} (X_{ni} - EX_{ni}) \to 0$ a.s. as $n \to \infty$.*

Proof We use the same notations as those in Theorem 3.1. In order to prove $\frac{1}{k_n^{1/r}} \sum_{i=u_n}^{v_n} (X_{ni} - EX_{ni}) \to 0$ $a.s.$ as $n \to \infty$, we only need to show

$$S_n \doteq \frac{1}{k_n^{1/r}} \sum_{i=u_n}^{v_n} (Y_{ni} - EY_{ni}) \to 0 \ \ a.s. \ \text{ as } \ n \to \infty, \tag{3.1}$$

and

$$T_n \doteq \frac{1}{k_n^{1/r}} \sum_{i=u_n}^{v_n} (Z_{ni} - EZ_{ni}) \to 0 \ \ a.s. \ \text{ as } \ n \to \infty. \tag{3.2}$$

For (3.1), noting that $1 \leq r < 2$ and $|Y_{ni}| = \min\{|X_{ni}|, h^{1/r}(n)\}$, we have by Markov's inequality, Lemma 2.3 or Remark 2.1 that for any $\varepsilon > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} P(|S_n| > \varepsilon) &\leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{k_n^{2/r}} E\left|\sum_{i=u_n}^{v_n} (Y_{ni} - EY_{ni})\right|^2 \\ &\leq C \sum_{n=1}^{\infty} \frac{1}{k_n^{2/r}} \sum_{i=u_n}^{v_n} EY_{ni}^2\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty}\frac{h^{(2-r)/r}(n)}{k_n^{2/r}}\sum_{i=u_n}^{v_n}E\left|Y_{ni}\right|^r\\
&\leq C\sum_{n=1}^{\infty}\left(\frac{h(n)}{k_n}\right)^{\frac{2-r}{r}}\cdot\left(\sup_{n\geq 1}\frac{1}{k_n}\sum_{i=u_n}^{v_n}E\left|X_{ni}\right|^r\right)\\
&<\infty,
\end{aligned}
$$

which together with Borel–Cantelli lemma implies (3.1).

For (3.2), noting that $|Z_{ni}| = (|X_{ni}| - h^{1/r}(n))I(|X_{ni}|^r > h(n))$, we have by Markov's inequality, Lemma 2.3 or Remark 2.1 again that for any $\varepsilon > 0$,

$$
\begin{aligned}
\sum_{n=1}^{\infty}P\left(|T_n|>\varepsilon\right) &\leq \frac{1}{\varepsilon^r}\sum_{n=1}^{\infty}\frac{1}{k_n}E\left|\sum_{i=u_n}^{v_n}(Z_{ni}-EZ_{ni})\right|^r\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{k_n}\sum_{i=u_n}^{v_n}E\left|Z_{ni}\right|^r\\
&= C\sum_{n=1}^{\infty}\frac{1}{k_n}\sum_{i=u_n}^{v_n}E\left(|X_{ni}|-h^{1/r}(n)\right)^r I\left(|X_{ni}|^r>h(n)\right)\\
&<\infty,
\end{aligned}
$$

which together with Borel–Cantelli lemma yields (3.2). This completes the proof of the theorem. □

Using Theorem 3.3, we can get the following strong law of large numbers for weighted sums of arrays of rowwise END random variables.

Corollary 3.3 *Let* $1 \leq r < 2$, $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ *be an array of rowwise END random variables and* $\{a_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ *be an array of constants. Let* $h(n) \uparrow \infty$, *and*

(i) $\sup_{n\geq 1}\sum_{i=u_n}^{v_n}|a_{ni}|^r E|X_{ni}|^r < \infty$;

(ii) $\sum_{n=1}^{\infty}\sum_{i=u_n}^{v_n}|a_{ni}|^r E|X_{ni}|^r I(|X_{ni}|^r > h(n)) < \infty$;

(iii) $\sum_{n=1}^{\infty}\left(h(n)\sup_{u_n\leq i\leq v_n}|a_{ni}|^r\right)^{\frac{2-r}{r}} < \infty$.

Then

$$
\sum_{i=u_n}^{v_n}a_{ni}(X_{ni}-EX_{ni})\to 0\ a.s.\ \ as\ \ n\to\infty. \tag{3.3}
$$

Proof Denote $k_n = 1/\sup_{u_n \le i \le v_n} |a_{ni}|^r$. It follows by condition (iii) that $h(n)/k_n \to 0$ as $n \to \infty$, and thus $k_n \to \infty$ as $n \to \infty$.

Without loss of generality, we assume that $a_{ni} \ge 0$ for all $u_n \le i \le v_n$ and $n \ge 1$. Otherwise, we will use a_{ni}^+ and a_{ni}^- instead of a_{ni} respectively and note that $a_{ni} = a_{ni}^+ - a_{ni}^-$. Hence, it follows by Lemma 2.1 that $\{k_n^{1/r} a_{ni} X_{ni}, u_n \le i \le v_n, n \ge 1\}$ is still an array of rowwise END random variables.

Taking $k_n^{1/r} a_{ni} X_{ni}$ instead of X_{ni} in Theorem 3.3, we have by condition (i) that

$$\sup_{n\ge 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E\left|k_n^{1/r} a_{ni} X_{ni}\right|^r = \sup_{n\ge 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r < \infty. \tag{3.4}$$

Noting that $\left|k_n^{1/r} a_{ni}\right| \le 1$ for all $u_n \le i \le v_n$ and $n \ge 1$, we have by condition (ii) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E\left(\left|k_n^{1/r} a_{ni} X_{ni}\right| - h^{1/r}(n)\right)^r I\left(|k_n^{1/r} a_{ni} X_{ni}|^r > h(n)\right)\\
&\le \sum_{n=1}^{\infty} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E\left|k_n^{1/r} a_{ni} X_{ni}\right|^r I\left(|k_n^{1/r} a_{ni} X_{ni}|^r > h(n)\right)\\
&\le \sum_{n=1}^{\infty} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r I\left(|X_{ni}|^r > h(n)\right)\\
&< \infty.
\end{aligned} \tag{3.5}$$

Hence, the desired result (3.3) follows by (3.4), (3.5) and Theorem 3.3 immediately. The proof is completed. □

Remark 3.4 According to the proofs of Theorem 3.3 and Corollary 3.3, one can get the complete convergence for arrays of rowwise END random variables, which is much stronger than almost sure convergence. Under the conditions of Theorem 3.3, we have that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\left|\frac{1}{k_n^{1/r}} \sum_{i=u_n}^{v_n} (X_{ni} - EX_{ni})\right| > \varepsilon\right) < \infty; \tag{3.6}$$

under the conditions of Corollary 3.3, we have that for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - EX_{ni})\right| > \varepsilon\right) < \infty. \tag{3.7}$$

4 Applications

In Sect. 3, we established the L_r convergence, weak and strong laws of large numbers for arrays of rowwise END random variables under some uniformly integrable conditions. In this section, we will present some applications of the L_r convergence, weak and strong laws of large numbers to nonparametric regression models based on END errors.

Consider the following nonparametric regression model:

$$Y_{nk} = g(x_{nk}) + \varepsilon_{nk}, \quad k = 1, 2, \dots, n, \ n \geq 1, \tag{4.1}$$

where x_{nk} are known fixed design points from A, and $A \subset \mathbb{R}^m$ is a given compact set for some $m \geq 1$, $g(\cdot)$ is an unknown regression function defined on A, and the ε_{nk} are random errors. As an estimator of $g(\cdot)$, we consider the weighted regression estimator as follows:

$$g_n(x) = \sum_{k=1}^{n} W_{nk}(x) Y_{nk}, \quad x \in A \subset \mathbb{R}^m, \tag{4.2}$$

where $W_{nk}(x) = W_{nk}(x; x_{n1}, x_{n2}, \dots, x_{nn})$, $k = 1, 2, \dots, n$ are the weight functions.

The above weighted regression estimator for nonparametric regression model was first adapted by Georgiev (1985). Since then, many authors devoted to studying the asymptotic properties of $g_n(x)$ and providing many interesting results. We refer the readers to Roussas (1989), Fan (1990), Roussas et al. (1992), Tran et al. (1996), Liang and Jing (2005), Wang et al. (2014), Wang and Si (2015), Chen et al. (2016) for instance. The purpose of this section is to further investigate the strong consistency and mean consistency for the estimator $g_n(x)$ in the nonparametric regression model based on END errors by using the results obtained in Sect. 3.

In this section, let $c(g)$ denote the set of continuity points of the function g on A. The symbol $\|x\|$ denotes the Euclidean norm. For any fixed design point $x \in A$, the following assumptions on weight functions $W_{nk}(x)$ will be used:

(H_1) $\sum_{k=1}^{n} W_{nk}(x) \to 1$ as $n \to \infty$;

(H_2) $\sum_{k=1}^{n} |W_{nk}(x)| \leq C < \infty$ for all n;

(H_3) $\sum_{k=1}^{n} |W_{nk}(x)| \cdot |g(x_{nk}) - g(x)| I(\|x_{nk} - x\| > a) \to 0$ as $n \to \infty$ for all $a > 0$.

We point out that the design assumptions $(H_1) - (H_3)$ are regular conditions for nonparametric regression models and are very general. For more details, one can refer to Liang and Jing (2005) and Wang et al. (2014) for instance. Based on the assumptions above, we present the following results on strong consistency and mean consistency of the nonparametric regression estimator $g_n(x)$.

The first one is the strong consistency of the nonparametric regression estimator $g_n(x)$

Theorem 4.1 *Let* $\{\varepsilon_{nk}, 1 \leq k \leq n, n \geq 1\}$ *be an array of rowwise END random variables with mean zero which is stochastically dominated by a random variable* X *with* $E|X|^r < \infty$ *for some* $1 < r < 2$. *Suppose that the conditions* $(H_1) - (H_3)$ *hold, and*

$$\max_{1 \leq k \leq n} |W_{nk}(x)| = O(n^{-u}) \text{ for some } u > \max\left\{\frac{1}{2-r}, \frac{1}{r-1}\right\}. \tag{4.3}$$

Then for all $x \in c(g)$,

$$g_n(x) \to g(x) \ a.s. \tag{4.4}$$

Proof For $a > 0$ and $x \in c(g)$, we obtain from (4.1) and (4.2) that

$$\begin{aligned} |Eg_n(x) - g(x)| \leq & \sum_{k=1}^{n} |W_{nk}(x)| \cdot |g(x_{nk}) - g(x)| I(\|x_{nk} - x\| \leq a) \\ & + \sum_{k=1}^{n} |W_{nk}(x)| \cdot |g(x_{nk}) - g(x)| I(\|x_{nk} - x\| > a) \\ & + |g(x)| \cdot \left| \sum_{k=1}^{n} W_{nk}(x) - 1 \right|. \end{aligned} \tag{4.5}$$

It follows from $x \in c(g)$ that for all $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for all $x^{'}$ which satisfy $\|x^{'} - x\| < \delta$, we have $|g(x^{'}) - g(x)| < \varepsilon$. Hence we take $0 < a < \delta$ in (4.5) and obtain that

$$\begin{aligned} |Eg_n(x) - g(x)| \leq & \sum_{k=1}^{n} \varepsilon |W_{nk}(x)| \\ & + \sum_{k=1}^{n} |W_{nk}(x)| \cdot |g(x_{nk}) - g(x)| I(\|x_{nk} - x\| > a) \\ & + |g(x)| \cdot \left| \sum_{k=1}^{n} W_{nk}(x) - 1 \right|. \end{aligned}$$

Then by assumptions (H_1)-(H_3) and the arbitrariness of $\varepsilon > 0$, we have that for all $x \in c(g)$,

$$\lim_{n \to \infty} Eg_n(x) = g(x). \tag{4.6}$$

Hence, to prove (4.4), it suffices to prove

$$g_n(x) - Eg_n(x) = \sum_{k=1}^{n} W_{nk}(x)\varepsilon_{nk} \to 0 \ \ a.s. \tag{4.7}$$

We will apply Corollary 3.3 with $X_{nk} = \varepsilon_{nk}$, $a_{nk} = W_{nk}(x)$, $u_n = 1$, $v_n = n$ and $h(n) = n^a$, where $0 < a < r\left(u - \frac{1}{2-r}\right)$. By $E|X|^r < \infty$, conditions (H_2), (4.3) and Property 1.1, we have that

$$\begin{aligned}\sup_{n\ge1} \sum_{i=u_n}^{v_n} |a_{ni}|^r \, E|X_{ni}|^r &\le C \sup_{n\ge1} \left(\max_{1\le k\le n} |W_{nk}(x)|\right)^{r-1} \sum_{k=1}^{n} |W_{nk}(x)| \, E|X|^r \\ &\le C \sup_{n\ge1} n^{-u(r-1)} \le C < \infty,\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{i=u_n}^{v_n} |a_{ni}|^r \, E|X_{ni}|^r I\left(|X_{ni}|^r > h(n)\right) &\le C \sum_{n=1}^{\infty} \sum_{k=1}^{n} |W_{nk}(x)|^r \, E|X|^r \\ &\le C \sum_{n=1}^{\infty} n^{-u(r-1)} < \infty,\end{aligned}$$

$$\sum_{n=1}^{\infty} \left(h(n) \sup_{u_n\le i\le v_n} |a_{ni}|^r\right)^{\frac{2-r}{r}} \le C \sum_{n=1}^{\infty} n^{(a-ur)(2-r)/r} < \infty.$$

Thus, the conditions (i)–(iii) in Corollary 3.3 are satisfied. Noting that $E\varepsilon_{nk} = 0$, we can immediately get the desired result (4.7) by Corollary 3.3. This completes the proof of the theorem. □

The next one is the mean consistency and weak consistency of the nonparametric regression estimator $g_n(x)$.

Theorem 4.2 *Let $\{\varepsilon_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of rowwise END random variables with mean zero which is stochastically dominated by a random variable X with $E|X|^r < \infty$ for some $1 < r < 2$. Suppose that the conditions $(H_1) - (H_3)$ hold, and*

$$\max_{1\le k\le n} |W_{nk}(x)| = O(n^{-u}) \ \ \textit{for some} \ \ u > 0. \tag{4.8}$$

Then for all $x \in c(g)$,

$$g_n(x) \to g(x) \ \textit{in} \ L_r, \tag{4.9}$$

and thus,

$$g_n(x) \to g(x) \text{ in probability}. \tag{4.10}$$

Proof Similar to the proof of Theorem 4.1, we can see that (4.6) still holds. Note that

$$E\,|g_n(x) - g(x)|^r \le 2^{r-1} E\,|g_n(x) - Eg_n(x)|^r + 2^{r-1}\,|Eg_n(x) - g(x)|^r\,.$$

Hence, to prove (4.9), it suffices to prove

$$E\,|g_n(x) - Eg_n(x)|^r = E\left|\sum_{k=1}^{n} W_{nk}(x)\varepsilon_{nk}\right|^r \to 0 \text{ as } n \to \infty. \tag{4.11}$$

We will apply Theorem 3.2 with $X_{nk} = \varepsilon_{nk}$, $a_{nk} = W_{nk}(x)$, $u_n = 1$ and $v_n = n$. By $E|X|^r < \infty$, conditions (H_2), (4.8) and Property 1.1, we have that

$$\begin{aligned}\sup_{n\ge1}\sum_{i=u_n}^{v_n}|a_{ni}|^r E|X_{ni}|^r &\le C\sup_{n\ge1}\left(\max_{1\le k\le n}|W_{nk}(x)|\right)^{r-1}\sum_{k=1}^{n}|W_{nk}(x)|\,E|X|^r\\ &\le C\sup_{n\ge1} n^{-u(r-1)} \le C < \infty,\end{aligned}$$

and for any $\epsilon > 0$,

$$\sum_{i=u_n}^{v_n}|a_{ni}|^r E|X_{ni}|^r I(|X_{ni}|^r > \epsilon) \le Cn^{-u(r-1)} \to 0 \text{ as } n \to \infty.$$

Thus, the conditions (i) and (ii) in Theorem 3.2 are satisfied. Noting that $E\varepsilon_{nk} = 0$, we can immediately get the desired result (4.11) by Theorem 3.2. This completes the proof of the theorem. □

Acknowledgements The authors are most grateful to anonymous referees for careful reading of the manuscript and valuable suggestions which helped in improving an earlier version of this paper.

Appendix

Proof of Lemma 2.3. If $r = 1$ or $r = 2$, then we can see that (2.2) holds trivially by C_r-inequality and Lemma 2.2 with $p = 2$, respectively. So in the following, we only need to consider the case $1 < r < 2$.

For fixed $n \ge 1$, denote $M_n = \sum_{i=1}^{n} E|X_i|^r$. Without loss of generality, we assume that $M_n > 0$. For any $\varepsilon > 1$, it is easily seen that

$$E\left|\sum_{i=1}^{n} X_i\right|^r \le (1+\varepsilon)M_n + \int_{(1+\varepsilon)M_n}^{\infty} P\left(\left|\sum_{i=1}^{n} X_i\right| > t^{1/r}\right)dt. \tag{4.12}$$

For fixed $n \geq 1$ and $t \geq (1+\varepsilon)M_n$, denote for $1 \leq i \leq n$ that

$$Y_i = -t^{1/r} I(X_i < -t^{1/r}) + X_i I(|X_i| \leq t^{1/r}) + t^{1/r} I(X_i > t^{1/r}).$$

It follows by (4.12) that

$$\begin{aligned} E\left|\sum_{i=1}^n X_i\right|^r &\leq (1+\varepsilon)M_n + \int_{(1+\varepsilon)M_n}^{\infty} \sum_{i=1}^n P\left(|X_i| > t^{1/r}\right) dt \\ &\quad + \int_{(1+\varepsilon)M_n}^{\infty} P\left(\left|\sum_{i=1}^n Y_i\right| > t^{1/r}\right) dt \\ &\leq (1+\varepsilon)M_n + \int_{(1+\varepsilon)M_n}^{\infty} \sum_{i=1}^n P\left(|X_i| > t^{1/r}\right) dt \\ &\quad + \int_{(1+\varepsilon)M_n}^{\infty} P\left(\left|\sum_{i=1}^n (Y_i - EY_i)\right| > t^{1/r} - \left|\sum_{i=1}^n EY_i\right|\right) dt \\ &\doteq (1+\varepsilon)M_n + I_1 + I_2. \end{aligned} \tag{4.13}$$

For I_1, we have

$$I_1 \leq \sum_{i=1}^n \int_0^{\infty} P\left(|X_i| > t^{1/r}\right) dt = \sum_{i=1}^n E|X_i|^r = M_n. \tag{4.14}$$

Note that

$$\begin{aligned} \sup_{t \geq (1+\varepsilon)M_n} t^{-1/r} \left|\sum_{i=1}^n EY_i\right| &\leq 2 \sup_{t \geq (1+\varepsilon)M_n} t^{-1/r} \cdot t^{1/r-1} \sum_{i=1}^n E|X_i|^r I(|X_i| > t^{1/r}) \\ &\leq 2(1+\varepsilon)^{-1}. \end{aligned} \tag{4.15}$$

Hence, by (4.15), Markov's inequality and Lemma 2.2 with $p = 2$, we can get that

$$\begin{aligned} I_2 &\leq \int_{(1+\varepsilon)M_n}^{\infty} P\left(\left|\sum_{i=1}^n (Y_i - EY_i)\right| > \left[1 - 2(1+\varepsilon)^{-1}\right] t^{1/r}\right) dt \\ &\leq \left[1 - 2(1+\varepsilon)^{-1}\right]^{-2} \int_{(1+\varepsilon)M_n}^{\infty} t^{-2/r} E\left|\sum_{i=1}^n (Y_i - EY_i)\right|^2 dt \\ &\leq C_2 \left[1 - 2(1+\varepsilon)^{-1}\right]^{-2} \sum_{i=1}^n \int_{(1+\varepsilon)M_n}^{\infty} t^{-2/r} EX_i^2 I(|X_i| \leq t^{1/r}) dt \\ &\quad + C_2 \left[1 - 2(1+\varepsilon)^{-1}\right]^{-2} \sum_{i=1}^n \int_{(1+\varepsilon)M_n}^{\infty} P(|X_i| > t^{1/r}) dt \end{aligned}$$

$$\doteq I_{21} + I_{22}. \tag{4.16}$$

Here, C_2 is defined by Lemma 2.2. For I_{22}, we have

$$\begin{aligned} I_{22} &\le C_2 \left[1 - 2(1+\varepsilon)^{-1}\right]^{-2} \sum_{i=1}^{n} \int_0^\infty P(|X_i| > t^{1/r}) dt \\ &= C_2 \left[1 - 2(1+\varepsilon)^{-1}\right]^{-2} M_n. \end{aligned} \tag{4.17}$$

For I_{21}, we can see that

$$\begin{aligned} &\int_{(1+\varepsilon)M_n}^\infty t^{-2/r} E X_i^2 I(|X_i| \le t^{1/r}) dt \le \int_{(1+\varepsilon)M_n}^\infty t^{-2/r} dt \int_0^{(1+\varepsilon)^{2/r} M_n^{2/r}} \\ &\quad P(|X_i| > y^{1/2}) dy + \int_{(1+\varepsilon)M_n}^\infty t^{-2/r} dt \int_{(1+\varepsilon)^{2/r} M_n^{2/r}}^{t^{2/r}} P(|X_i| > y^{1/2}) dy \\ &\doteq J_1 + J_2. \end{aligned} \tag{4.18}$$

For J_1, it follows by Markov's inequality that

$$\begin{aligned} J_1 &\le \frac{r}{2-r} (1+\varepsilon)^{1-2/r} M_n^{1-2/r} \int_0^{(1+\varepsilon)^{2/r} M_n^{2/r}} E|X_i|^r y^{-r/2} dy \\ &= \frac{2r}{(2-r)^2} E|X_i|^r. \end{aligned} \tag{4.19}$$

For J_2, we have

$$\begin{aligned} J_2 &= \int_{(1+\varepsilon)^{2/r} M_n^{2/r}}^\infty P(|X_i| > y^{1/2}) dy \int_{y^{r/2}}^\infty t^{-2/r} dt \\ &= \frac{r}{2-r} \int_{(1+\varepsilon)^{2/r} M_n^{2/r}}^\infty y^{r/2-1} P(|X_i| > y^{1/2}) dy \\ &\le \frac{r}{2-r} \int_0^\infty y^{r/2-1} P(|X_i| > y^{1/2}) dy = \frac{2}{2-r} E|X_i|^r. \end{aligned} \tag{4.20}$$

Hence, by (4.16)-(4.20), we can get that

$$\begin{aligned} I_2 &\le C_2 \left[1 - 2(1+\varepsilon)^{-1}\right]^{-2} M_n + C_2 \left[1 - 2(1+\varepsilon)^{-1}\right]^{-2} \left[\frac{2r}{(2-r)^2} + \frac{2}{2-r}\right] M_n \\ &= C_2 \left[1 - 2(1+\varepsilon)^{-1}\right]^{-2} \left[1 + \left(\frac{2}{2-r}\right)^2\right] M_n. \end{aligned} \tag{4.21}$$

By (4.13), (4.14) and (4.21), we have

$$E\left|\sum_{i=1}^{n} X_i\right|^r \leq \left\{2+\varepsilon+C_2\left[1-2(1+\varepsilon)^{-1}\right]^{-2}\left[1+\left(\frac{2}{2-r}\right)^2\right]\right\} M_n \\ \doteq f(\varepsilon)M_n. \tag{4.22}$$

It is easily checked that $f(\varepsilon)$ is positive and continuous on $(1, \infty)$, and

$$\lim_{\varepsilon\to 1^+} f(\varepsilon) = \lim_{\varepsilon\to\infty} f(\varepsilon) = \infty.$$

Hence, $f(\varepsilon)$ has the minimum on $(1, \infty)$. Set $c_r = \inf_{1<\varepsilon<\infty} f(\varepsilon)$. It is obvious that $c_r > 3$ does not depend on n, and thus (2.2) holds. This completes the proof of the lemma. □

References

Asadian N, Fakoor V, Bozorgnia A (2006) Rosenthal's type inequalities for negatively orthant dependent random variables. J Iran Stat Soc 5:69–75

Bahr BV, Esseen CG (1965) Inequalities for the r-th absolute moment of a sum of random variables, $1 \leq r \leq 2$. Ann Math Stat 36:299–303

Bryc W, Smolenski W (1993) Moment conditions for almost sure convergence of weakly correlated random variables. Proc Am Math Soc 119:629–635

Chatterji SD (1969) An L_p-convergence theorem. Ann Math Stat 40:1068–1070

Chen Y, Chen A, Ng KW (2010) The strong law of large numbers for extend negatively dependent random variables. J Appl Probab 47:908–922

Chen PY, Bai P, Sung SH (2014) The von Bahr-Esseen moment inequality for pairwise independent random variables and applications. J Math Anal Appl 419:1290–1302

Chen ZY, Wang HB, Wang XJ (2016) The consistency for the estimator of nonparametric regression model based on martingale difference errors. Stat Papers 57(2):451–469

Cheng FY, Li N (2014) Asymptotics for the tail probability of random sums with a heavy-tailed random number and extended negatively dependent summands. Chin Ann Math Ser B 35(1):69–78

Christofides TC, Vaggelatou E (2004) A connection between supermodular ordering and positive/negative association. J Multivar Anal 88:138–151

Fan Y (1990) Consistent nonparametric multiple regression for dependent heterogeneous processes: the fixed design case. J Multivar Anal 33:72–88

Georgiev AA (1985) Local properties of function fitting estimates with applications to system identification. In: Grossmann W et al. (eds) Mathematical statistics and applications, vol B. Proceedings 4th Pannonian symposium on mathematical statistics, 4–10, September 1983. Bad Tatzmannsdorf, Austria, Reidel, Dordrecht, pp 141–151

Hu TZ (2000) Negatively superadditive dependence of random variables with applications. Chin J Appl Probab Stat 16:133–144

Hu TC, Wang KL, Rosalsky A (2015) Complete convergence theorems for extended negatively dependent random variables. Sankhyā A Indian J Stat 77(1):1–29

Joag-Dev K, Proschan F (1983) Negative association of random variables with applications. Ann Stat 11:286–295

Lehmann E (1966) Some concepts of dependence. Ann Math Stat 37:1137–1153

Liang HY, Jing BY (2005) Asymptotic properties for estimates of nonparametric regression models based on negatively associated sequences. J Multivar Anal 95:227–245

Liu L (2009) Precise large deviations for dependent random variables with heavy tails. Stat Probab Lett 79:1290–1298

Liu L (2010) Necessary and sufficient conditions for moderate deviations of dependent random variables with heavy tails. Sci China Ser A Math 53(6):1421–1434

Ordóñez Cabrera M, Rosalsky A, Volodin A (2012) Some theorems on conditional mean convergence and conditional almost sure convergence for randomly weighted sums of dependent random variables. Test 21:369–385

Qiu DH, Chen PY, Antonini RG, Volodin A (2013) On the complete convergence for arrays of rowwise extended negatively dependent random variables. J Korean Math Soc 50(2):379–392

Roussas GG (1989) Consistent regression estimation with fixed design points under dependence conditions. Stat Probab Lett 8:41–50

Roussas GG, Tran LT, Ioannides DA (1992) Fixed design regression for time series: asymptotic normality. J Multivar Anal 40:262–291

Shao QM (2000) A comparison theorem on moment inequalities between negatively associated and independent random variables. J Theor Probab 13(2):343–356

Shen AT (2011) Probability inequalities for END sequence and their applications. J Inequal Appl 2011:98

Shen AT, Wu RC, Chen Y, Zhou Y (2013) Conditional convergence for randomly weighted sums of random variables based on conditional residual h-integrability. J Inequal Appl 2013:11

Stout WF (1974) Almost sure convergence. Academic Press, New York

Sung SH, Lisawadi S, Volodin A (2008) Weak laws of large numbers for arrays under a condition of uniform integrability. J Korean Math Soc 45(1):289–300

Sung SH (2013) Convergence in r-mean of weighted sums of NQD random variables. Appl Math Lett 26:18–24

Tran LT, Roussas GG, Yakowitz S, Van BT (1996) Fixed design regression for linear time series. Ann Stat 24:975–991

Wang XJ, Hu TC, Volodin A, Hu SH (2013) Complete convergence for weighted sums and arrays of rowwise extended negatively dependent random variables. Commun Stat Theory Methods 42:2391–2401

Wang XJ, Wang SJ, Hu SH, Ling JM, Wei YF (2013) On complete convergence of weighted sums for arrays of rowwise extended negatively dependent random variables. Stoch Int J Probab Stoch Process 85(6):1060–1072

Wang XJ, Li XQ, Hu SH, Wang XH (2014) On complete convergence for an extended negatively dependent sequence. Commun Stat Theory Methods 43(14):2923–2937

Wang XJ, Xu C, Hu TC, Volodin A, Hu SH (2014) On complete convergence for widely orthant-dependent random variables and its applications in nonparametric regression models. Test 23:607–629

Wang XJ, Deng X, Zheng LL, Hu SH (2014) Complete convergence for arrays of rowwise negatively superadditive-dependent random variables and its applications. Stat J Theor Appl Stat 48(4):834–850

Wang XJ, Zheng LL, Xu C, Hu SH (2015) Complete consistency for the estimator of nonparametric regression models based on extended negatively dependent errors. Stat J Theor Appl Stat 49(2):396–407

Wang XH, Hu SH (2014) Weak laws of large numbers for arrays of dependent random variables. Stoch Int J Probab Stoch Process 86(5):759–775

Wang XJ, Si ZY (2015) Complete consistency of the estimator of nonparametric regression model under ND sequence. Stat Papers 56:585–596

Wang SJ, Wang XJ (2013) Precise large deviations for random sums of END real-valued random variables with consistent variation. J Math Anal Appl 402:660–667

Wu QY (2006) Probability limit theory of mixing sequences. Science Press of China, Beijing

Wu YF, Cabrea MO, Volodin A (2014) Complete convergence and complete moment convergence for arrays of rowwise end random variables. Glasnik Matematički 49(69):449–468

Yuan DM, An J (2009) Rosenthal type inequalities for asymptotically almost negatively associated random variables and applications. Sci China Ser A Math 52(9):1887–1904

Yuan DM, Tao B (2008) Mean convergence theorems for weighted sums of arrays of residually h-integrable random variables concerning the weights under dependence assumptions. Acta Applicandae Mathematicae 103:221–234