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CONVERGENCE OF RANDOMLY WEIGHTED SUMS OF BANACH-SPACE-VALUED RANDOM ELEMENTS UNDER SOME CONDITIONS OF UNIFORM INTEGRABILITY*

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1. Introduction

There exists extensive literature on the weak or strong convergence of weighted partial sums $\sum_{j=1}^{n} a_{nj} X_j$, where $\{X_n, n \geq 1\}$ is a sequence of random variables and $\{a_{nj}, 1 \leq j \leq n, n \geq 1\}$ is an array of (nonrandom) constants. In this scope, Rosalsky and Sreehari (see [10]) provide a complete list of references from 1965 to 1995. We only refer now to a few of them: the ones that are at the heart of the matter in this note.

Wang and Rao (see [12]) extend the classical Rohatgi's weak law of large numbers (WLLN) (see [9]) for weighted sums of random variables to the case of uniform integrable random variables.

Chandra, in [1], obtains several variations and extensions of Khintchine's WLLN by introducing the condition of uniform integrability in the Cesaro sense, which is weaker than the uniform integrability.

Gut, in [2], obtains a WLLN for an array $\{X_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ of random variables such that $\{|X_{nj}|^p\}$, $0 < p < 2$, is uniformly integrable in the Cesàro sense.

Ordóñez Cabrera (see [6]), in studying the convergence of weighted sums $S_n = \sum_j a_{nj} X_{nj}$, introduces and characterizes the condition of $\{a_{nj}\}$-uniform integrability of $\{X_{nj}\}$, which is weaker than the condition of uniform integrability and leads to the Cesàro uniform integrability as a particular case (see Definition 1).

This condition can be adapted to an array $\{V_{nj}\}$ of random elements in a real separable Banach space, and one obtains a result of convergence in L_r, $0 < r < 1$, for a sequence of weighted sums $S_n = \sum_j a_{nj} X_{nj}$. In order to obtain the convergence in L_1, Ordóñez Cabrera (see [7]) introduces the notion of $\{a_{nj}\}$-compactly uniform pth-order integrability of $\{V_{nj}\}$, $p > 0$.

Beginning in the seventies, the random nature of many problems arising in the applied sciences was noted. This led to mathematical models that dealt with the limiting behavior of weighted sums of random elements in normed linear spaces, where the weights are random variables. The particular structure of these models can be subsumed within the general structure of the (weak or strong) convergence of randomly weighted sums $\sum_{j=1}^{n} A_{nj} V_{nj}$ (partial sums) or $\sum_{j=1}^{\infty} A_{nj} V_{nj}$, where $\{A_{nj}\}$ and $\{V_{nj}\}$ are respective arrays of random variables and random elements in a separable normed linear space (or in a Banach space), defined on the same probability space $(\Omega, \mathcal{A}, \mathbf{P})$.

Simultaneously, this structure can be subsumed within a more general structure by considering randomly weighted sums $\sum_{j=u_n}^{v_n} A_{nj} V_{nj}$, where $\{u_n \geq -\infty, n \geq 1\}$ and $\{v_n \leq +\infty, n \geq 1\}$ are two sequences of integers, $v_n > u_n$ for all $n \geq 1$. If $u_n = 1$, $v_n = n$, $n \geq 1$, we have randomly weighted partial sums.

In this note, we introduce the notion of $\{A_{nj}\}$-conditional uniform integrability relative to a sequence $\{\mathcal{B}_n\}$ of σ-algebras for an array $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ of Banach-space-valued random elements. In the particular case $\mathcal{B}_n = \{\varnothing, \Omega\}$, for every $n \geq 1$ we have the characterization of $\{A_{nj}\}$-uniform integrability.

Under the condition

$$\sup_{n \geq 1} \sum_{j=u_n}^{v_n} |A_{nj}| \leq C \quad \text{a.e.},$$

where C is a constant, we obtain a characterization of these notions. These results extend, in a natural way, the characterizations of $\{a_{nj}\}$-uniform integrability (for nonrandom weights) in [6].

By supposing the hypothesis of $\{A_{nj}\}$-conditional uniform integrability relative to a sequence $\{\mathcal{B}_n\}$ of σ-algebras, we obtain later a result of convergence in L_1 and a result of almost everywhere (a.e.) convergence of the sequence of conditional expectations of randomly weighted sums of random elements taking values in a separable Banach space.

That previous notion is extended in a natural way, giving rise to the notion of $\{A_{nj}\}$-conditional compactly uniform integrability relative to $\{\mathcal{B}_n\}$. This notion is characterized in terms of $\{A_{nj}\}$-conditional uniform integrability relative

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to $\{\mathcal{B}_n\}$ and $\{A_{nj}\}$-conditional tightness relative to $\{\mathcal{B}_n\}$.

The concept of compactly uniform integrability has often been used to obtain limit laws for sums of random elements taking values in a Banach space $\mathcal{X}$, which has a Schauder basis. The crucial point for the fruitfulness of the results obtained is the fact that the identity operator in $\mathcal{X}$ can be approximated by the partial-sum operators, corresponding to the Schauder basis, uniformly on compact sets.

In the last section of this note, we extend the habitual field of applications of the concepts of compactly uniform integrability to the scope of separable Banach spaces with the bounded approximation property (B.A.P.), which is a condition weaker than the existence of a Schauder basis, and we show that under the hypothesis of conditional compactly uniform integrability, the study of a conditional law of large numbers in a separable Banach space with B.A.P. can be reduced to the study of a similar limit law for finite-dimensional random elements.

2. Definitions

Let $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ be two sequences of integers (not necessarily positive or finite) such that $v_n > u_n$ for all $n \geq 1$ and $v_n - u_n \to \infty$ as $n \to \infty$. Let $\{a_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of real constants. Consider an array of random variables $\{A_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ and an array $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ of random elements in a real separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$, defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Let $\{\mathcal{B}_n, n \geq 1\}$ be a sequence of sub σ-algebras of $\mathcal{A}$. For each $n \geq 1$, denote by $\mathbf{E}^{\mathcal{B}_n}(Y)$ the conditional expectation of the random variable Y relative to $\mathcal{B}_n$, and by $\mathbf{P}^{\mathcal{B}_n}(A)$ denote the conditional probability of the event $A \in \mathcal{A}$ relative to $\mathcal{B}_n$.

The concepts that we will define later have their origin in the following two notions, introduced in [6] and [7], respectively.

(1) An array of random variables $\{X_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{a_{nj}\}$-uniform integrable if

$$\lim_{a\to\infty} \sup_{n\geq 1} \sum_{j=u_n}^{v_n} |a_{nj}| \mathbf{E}(|X_{nj}| I_{[|X_{nj}|>a]}) = 0.$$

(2) Let $p > 0$. The array $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{a_{nj}\}$-compactly uniformly pth-order integrable if, for each $\varepsilon > 0$, there exists a compact subset $K = K(\varepsilon)$ of $\mathcal{X}$ such that

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |a_{nj}| \mathbf{E}(\|V_{nj}\|^p I_{[V_{nj}\notin K]}) < \varepsilon.$$

We now introduce several concepts of uniform integrability for the array of random elements, concerning the array of random variables.

(3) We say that $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{A_{nj}\}$-conditionally uniformly integrable relative to $\{\mathcal{B}_n\}$ if, for each $\varepsilon > 0$, there exists $a_0 = a_0(\varepsilon) > 0$ such that

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |A_{nj}| \mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\| I_{[\|V_{nj}\|>a_0]}) < \varepsilon \quad \text{a.e.}$$

If $\mathcal{B}_n = \{\varnothing, \Omega\}$ for all $n \geq 1$, we have the analogous unconditioned concept.

(4) We say that $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{A_{nj}\}$-uniformly integrable if, for each $\varepsilon > 0$, there exists $a_0 = a_0(\varepsilon) > 0$ such that

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |A_{nj}| \mathbf{E}(\|V_{nj}\| I_{[\|V_{nj}\|>a_0]}) < \varepsilon \quad \text{a.e.}$$

Remark 1. If $A_{nj} = a_{nj}$ (nonrandom) a.e., for all $u_n \leq j \leq v_n$, $n \geq 1$, and $\mathcal{B}_n = \{\varnothing, \Omega\}$, for all $n \geq 1$, then Definitions (1), (3), and (4) coincide for the array of random variables $\{\|V_{nj}\|, u_n \leq j \leq v_n, n \geq 1\}$.

(5) We say that $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{a_{nj}\}$-tight if, for each $\varepsilon > 0$, there exists a compact subset $K = K(\varepsilon)$ of $\mathcal{X}$ such that

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |a_{nj}| \mathbf{P}[V_{nj} \notin K] < \varepsilon.$$

Remark 2. If

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |a_{nj}| \leq C$$

for some constant $C > 0$, this condition is weaker than the classical condition of tightness of $\{V_{nj}, u_n \le j \le v_n, n \ge 1\}$.

(6) We say that $\{V_{nj}, u_n \le j \le v_n, n \ge 1\}$ is conditionally tight relative to $\{\mathcal{B}_n\}$ if, for each $\varepsilon > 0$, there exists a compact subset $K = K(\varepsilon)$ of $\mathcal{X}$ such that

$$\sup_{n\ge1}\sup_{u_n\le j\le v_n} \mathbf{P}^{\mathcal{B}_n}[V_{nj} \notin K] < \varepsilon \quad \text{a.e.}$$

Remark 3. Definition (6) implies the tightness of $\{V_{nj}, u_n \le j \le v_n, n \ge 1\}$, since

$$\mathbf{P}[V_{nj} \notin K] = \mathbf{E}(I_{[V_{nj}\notin K]} = \mathbf{E}(\mathbf{E}^{\mathcal{B}_n}(I_{[V_{nj}\notin K]})) = \mathbf{E}(\mathbf{P}^{\mathcal{B}_n}[V_{nj} \notin K]).$$

(7) We say that $\{V_{nj}, u_n \le j \le v_n, n \ge 1\}$ is $\{A_{nj}\}$-conditionally tight relative to $\{\mathcal{B}_n\}$ if, for each $\varepsilon > 0$, there exists a compact subset $K = K(\varepsilon)$ of $\mathcal{X}$ such that

$$\sup_{n\ge1}\sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{P}^{\mathcal{B}_n}[V_{nj} \notin K] < \varepsilon \quad \text{a.e.}$$

Remark 4. If

$$\sup_{n\ge1}\sum_{j=u_n}^{v_n} |A_{nj}| \le C \quad \text{a.e.}$$

for some constant $C > 0$, then (6) implies (7).

(8) We say that $\{V_{nj}, u_n \le j \le v_n, n \ge 1\}$ is $\{A_{nj}\}$-conditionally compactly uniformly integrable relative to $\{\mathcal{B}_n\}$ if, for each $\varepsilon > 0$, there exists a compact subset $K = K(\varepsilon)$ of $\mathcal{X}$ such that

$$\sup_{n\ge1}\sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\| I_{[V_{nj}\notin K]}) < \varepsilon \quad \text{a.e.}$$

(9) Let $p > 0$. We say that $\{V_{nj}, u_n \le j \le v_n, n \ge 1\}$ is $\{A_{nj}\}$-conditionally compactly uniformly pth-order integrable relative to $\{\mathcal{B}_n\}$ if (8) remains with $\|V_{nj}\|^p$ instead of $\|V_{nj}\|$, i.e., if, for each $\varepsilon > 0$, there exists a compact subset $K = K(\varepsilon)$ of $\mathcal{X}$ such that

$$\sup_{n\ge1}\sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\|^p I_{[V_{nj}\notin K]}) < \varepsilon \quad \text{a.e.}$$

3. Characterizations

In this section, we will obtain characterizations of the various concepts of uniform integrability that have been introduced in the previous section.

Theorem 3.1. *Let $\{A_{nj}, u_n \le j \le v_n, n \ge 1\}$ be an array of random variables and let $\{V_{nj}, u_n \le j \le v_n, n \ge 1\}$ be an array of random elements in a separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$, defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, with*

$$\sup_{n\ge1}\sum_{j=u_n}^{v_n} |A_{nj}| \le C \quad \text{a.e.,}$$

for some constant $C < \infty$. Let $\{\mathcal{B}_n, n \ge 1\}$ be a sequence of sub σ-algebras of $\mathcal{A}$. Then $\{V_{nj}, u_n \le j \le v_n, n \ge 1\}$ is $\{A_{nj}\}$-conditionally uniformly integrable relative to $\{\mathcal{B}_n\}$ if and only if

(a) $\displaystyle\sup_{n\ge1}\sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{E}^{\mathcal{B}_n}\|V_{nj}\| = M < \infty$ a.e.;

(b) *for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, whenever $\{B_{nj}, u_n \le j \le v_n, n \ge 1\}$ is an array of events satisfying*

$$\sup_{n\ge1}\sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{P}^{\mathcal{B}_n}(B_{nj}) < \delta \quad \text{a.e.,}$$

then

$$\sup_{n\geq 1}\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\|I_{B_{nj}})<\varepsilon \quad \text{a.e.}$$

Proof. Let $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of random elements that is $\{A_{nj}\}$-conditionally uniformly integrable relative to $\mathcal{B}_n$. Then, given $\varepsilon > 0$, there exists $a > 0$ such that

$$\sup_{n\geq 1}\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\|I_{[|V_{nj}|>a]})<\frac{\varepsilon}{2} \quad \text{a.e.}$$

Then we have a.e.

$$\mathbf{E}^{\mathcal{B}_n}\|V_{nj}\|=\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\|I_{[||V_{nj}||\leq a]}+\|V_{nj}\|I_{[\|V_{nj}\|>a]})\leq a+\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\|I_{[\|V_{nj}\|>a]}).$$

Therefore, for every $n \geq 1$,

$$\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{E}^{\mathcal{B}_n}\|V_{nj}\|\leq a\sum_{j=u_n}^{v_n}|A_{nj}|+\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\|I_{[\|V_{nj}\|>a]}) \quad \text{a.e.,}$$

and so,

$$\sup_{n\geq 1}\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{E}^{\mathcal{B}_n}\|V_{nj}\|=M<\infty \quad \text{a.e.}$$

Now let $\varepsilon > 0$ and let $\{B_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of events with

$$\sup_{n\geq 1}\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{P}^{\mathcal{B}_n}(B_{nj})<\frac{\varepsilon}{2a}=\delta \quad \text{a.e.}$$

Then, for every $n \geq 1$,

$$|A_{nj}|\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\|I_{B_{nj}})=\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{E}^{\mathcal{B}_n}\|V_{nj}\|(I_{B_{nj}\cap[\|V_{nj}\|\leq a]}+I_{B_{nj}\cap[\|V_{nj}\|>a]})$$
$$\leq a\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{P}^{\mathcal{B}_n}(B_{nj})+\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\|I_{[\|V_{nj}\|>a]})<a\frac{\varepsilon}{2a}+\frac{\varepsilon}{2}=\varepsilon \quad \text{a.e.}$$

Conversely, for each $a > 0$ and every $n \geq 1$,

$$\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{P}^{\mathcal{B}_n}([\|V_{nj}\|>a])=\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{E}^{\mathcal{B}_n}I_{[\|V_{nj}\|>a]}\leq\frac{1}{a}\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{E}^{\mathcal{B}_n}\|V_{nj}\|\leq\frac{M}{a} \quad \text{a.e.,}$$

since $aI_{[\|V_{nj}\|>a]} \leq \|V_{nj}\|$ a.e.

Given $\varepsilon > 0$, for each $a \geq a_0 = 2M/\delta$ and every $n \geq 1$, we have

$$\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{P}^{\mathcal{B}_n}([\|V_{nj}\|>a])\leq\frac{M}{a_0}=\frac{\delta}{2}<\delta \quad \text{a.e.}$$

Therefore, the array of events $\{B_{nj}\} = \{[\|V_{nj}\| > a]\}$, for each $a > a_0$, verifies condition (b). So,

$$\sup_{n\geq 1}\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\|I_{[\|V_{nj}\|>a]})<\varepsilon \quad \text{a.e.,}$$

for each $a > a_0$, i.e., $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{A_{nj}\}$-conditionally uniformly integrable relative to $\mathcal{B}_n$.

By considering the sequence of σ-algebras $\mathcal{B}_n = \{\varnothing, \Omega\}$, for every $n \geq 1$, we obtain the characterization of $\{A_{nj}\}$-uniform integrability.

Corollary 3.2. *Let $\{A_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of random variables and let $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of random elements in a separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$, defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, with*

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |A_{nj}| \leq C \quad \text{a.e.},$$

for some constant $C < \infty$. Then $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{A_{nj}\}$-uniformly integrable if and only if

(a) $\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{E}\|V_{nj}\| = M < \infty \quad \text{a.e.};$

(b) *for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, whenever $\{B_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is an array of events satisfying*

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{P}(B_{nj}) < \delta \quad \text{a.e.},$$

then

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{E}(\|V_{nj}\| I_{B_{nj}}) < \varepsilon \quad \text{a.e.}$$

Note that if, in particular, $A_{nj} = a_{nj}$ (nonrandom) for all $u_n \leq j \leq v_n$, $n \geq 1$, Corollary 3.2 gives the characterization of $\{a_{nj}\}$-uniform integrability of an array $\{X_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ of random variables in [6].

Theorem 3.3. *Let $\{A_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of random variables and let $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of random elements in a separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$, defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, with*

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |A_{nj}| \leq C \quad \text{a.e.},$$

for some constant $C < \infty$. Let $\{\mathcal{B}_n, n \geq 1\}$ be a sequence of sub σ-algebras of $\mathcal{A}$. Then $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{A_{nj}\}$-conditionally compactly uniformly integrable relative to $\{\mathcal{B}_n\}$ if and only if $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{A_{nj}\}$-conditionally uniformly integrable relative to $\{\mathcal{B}_n\}$ and $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{A_{nj}\}$-conditionally tight relative to $\{\mathcal{B}_n\}$.

Proof. Let $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be $\{A_{nj}\}$-conditionally uniformly integrable relative to $\{\mathcal{B}_n\}$. By Theorem 3.1, given $\varepsilon > 0$, there exists $\delta > 0$ such that, whenever $\{B_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is an array of events satisfying

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{P}^{\mathcal{B}_n}(B_{nj}) < \delta \quad \text{a.e.},$$

then

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\| I_{B_{nj}}) < \varepsilon \quad \text{a.e.}$$

By the hypothesis of $\{A_{nj}\}$-conditional tightness, there exists a compact subset K of $\mathcal{X}$ such that

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{P}^{\mathcal{B}_n}[V_{nj} \notin K] < \delta \quad \text{a.e.},$$

and, therefore,

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\| I_{[V_{nj}\notin K]}) < \varepsilon \quad \text{a.e.}$$

So, $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{A_{nj}\}$-conditionally compactly uniformly integrable relative to $\mathcal{B}_n$.

Conversely, suppose that $\{V_{nj}, u_n \le j \le v_n, n \ge 1\}$ is $\{A_{nj}\}$-conditionally compactly uniformly integrable relative to $\{\mathcal{B}_n\}$. Given $\varepsilon > 0$, for each $i \ge 1$, there exists a compact $K_i \subset \mathcal{X}$ such that

$$\sup_{n\ge 1} \sum_{j=u_n}^{v_n} |A_{nj}| \mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\| I_{[V_{nj}\notin K_i]}) < \frac{\varepsilon}{i2^i} \quad \text{a.e.}$$

Denote by $B(0, 1/i)$ the open ball in $\mathcal{X}$ with center 0 and radius $1/i$. Then

$$\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\| I_{[V_{nj}\notin K_i]}) \ge \mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\| I_{[V_{nj}\in K_i^c\cap B^c(0,1/i)]}) \ge \frac{1}{i}\mathbf{E}^{\mathcal{B}_n} I_{[V_{nj}\in K_i^c\cap B^c(0,1/i)]} = \frac{1}{i}\mathbf{P}^{\mathcal{B}_n}[V_{nj} \in K_i^c \cap B^c(0,1/i)] \quad \text{a.e.},$$

which implies that

$$\sup_{n\ge 1} \sum_{j=u_n}^{v_n} |A_{nj}| \mathbf{P}^{\mathcal{B}_n}[V_{nj} \in K_i^c \cap B^c(0,1/i)] \le \sup_{n\ge 1} i \sum_{j=u_n}^{v_n} |A_{nj}| \mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\| I_{[V_{nj}\notin K_i]}) < \frac{\varepsilon}{2^i} \quad \text{a.e.}$$

Put

$$K = \bigcap_{i=1}^{\infty}\left(K_i \cup B\left(0, \frac{1}{i}\right)\right).$$

The closure of K in $\mathcal{X}$, $\overline{K}$, is a compact set (see Lemma 2.2 in [13]). Then

$$\mathbf{P}^{\mathcal{B}_n}[V_{nj} \in \overline{K}^c] \le \mathbf{P}^{\mathcal{B}_n}[V_{nj} \in K^c] = \mathbf{P}^{\mathcal{B}_n}[V_{nj} \in \bigcup_{i=1}^{\infty}(K_i^c \cap B^c(0,1/i))] \le \sum_{i=1}^{\infty} \mathbf{P}^{\mathcal{B}_n}[V_{nj} \in K_i^c \cap B^c(0,1/i)] \quad \text{a.e.}$$

Therefore, for each $n \ge 1$,

$$\begin{aligned}\sum_{j=u_n}^{v_n} |A_{nj}| \mathbf{P}^{\mathcal{B}_n}[V_{nj} \notin \overline{K}] &\le \sum_{j=u_n}^{v_n} |A_{nj}| \left(\sum_{i=1}^{\infty} \mathbf{P}^{\mathcal{B}_n}[V_{nj} \in K_i^c \cap B^c(0,1/i)]\right) \\ &= \sum_{i=1}^{\infty}\sum_{j=u_n}^{v_n} |A_{nj}| \mathbf{P}^{\mathcal{B}_n}[V_{nj} \in K_i^c \cap B^c(0,1/i)] < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon \quad \text{a.e.}\end{aligned}$$

So, $\{V_{nj}, u_n \le j \le v_n, n \ge 1\}$ is $\{A_{nj}\}$-conditionally tight relative to $\{\mathcal{B}_n\}$.

On the other hand, given $\varepsilon > 0$, there exists a compact subset K of $\mathcal{X}$ such that

$$\sup_{n\ge 1} \sum_{j=u_n}^{v_n} |A_{nj}| \mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\| I_{[V_{nj}\notin K]}) < \varepsilon \quad \text{a.e.}$$

The compactness of K implies that there exists $r > 0$ such that $K \subset \overline{B}(0, r)$ and so, $[\|V_{nj}\| > r] \subset [V_{nj} \notin K]$, for every $n \ge 1$, $u_n \le j \le v_n$.

Therefore, we have

$$\sup_{n\ge 1} \sum_{j=u_n}^{v_n} |A_{nj}| \mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\| I_{[\|V_{nj}\|>r]}) \le \sup_{n\ge 1} \sum_{j=u_n}^{v_n} |A_{nj}| \mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\| I_{[V_{nj}\notin K]}) < \varepsilon \quad \text{a.e.}$$

Thus, $\{V_{nj}, u_n \le j \le v_n, n \ge 1\}$ is $\{A_{nj}\}$-conditionally uniformly integrable relative to $\{\mathcal{B}_n\}$.

4. Some Conditional Limit Laws for Randomly Weighted Sums

In this section, we will use the notions introduced before in order to obtain some conditional limit laws (i.e., limit laws for certain conditional expectations) for randomly weighted sums of random elements taking values in a separable Banach space.

Theorem 4.1. *Let $\{A_{nj}, u_n \le j \le v_n, n \ge 1\}$ be an array of random variables and let $\{V_{nj}, u_n \le j \le v_n, n \ge 1\}$ be an array of random elements in a separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$, defined on a probability space*

$(\Omega, \mathcal{A}, \mathbf{P})$, *with* $\sum_{j=u_n}^{v_n} |A_{nj}| = o(1)$ *a.e. Let* $\{\mathcal{B}_n, n \geq 1\}$ *be a sequence of sub* σ*-algebras of* $\mathcal{A}$*, and suppose that the* A_{nj}*,* $u_n \leq j \leq v_n$*, are* $\mathcal{B}_n$*-measurable, for each* $n \geq 1$*. Suppose that* $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ *is* $\{A_{nj}\}$*-conditionally uniformly integrable relative to* $\{\mathcal{B}_n\}$*. Then*

$$\mathbf{E}^{\mathcal{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|V_{nj}\| \longrightarrow 0 \quad \text{a.e. as} \quad n \to \infty$$

and, consequently,

$$\mathbf{E}^{\mathcal{B}_n} \left\| \sum_{j=u_n}^{v_n} A_{nj} V_{nj} \right\| \longrightarrow 0 \quad \text{a.e.}$$

Proof. Given $\varepsilon > 0$, there exists $a > 0$ such that

$$\sup_{n \geq 1} \left(\sum_{j=u_n}^{v_n} |A_{nj}| \mathbf{E}^{\mathcal{B}_n} (\|V_{nj}\| I_{[\|V_{nj}\| > a]}) \right) < \frac{\varepsilon}{2} \quad \text{a.e.}$$

Define $V'_{nj} = V_{nj} I_{[\|V_{nj}\| \leq a]}$ and $V''_{nj} = V_{nj} I_{[\|V_{nj}\| > a]}$. Then

$$\begin{aligned} \mathbf{E}^{\mathcal{B}_n} \left(\sum_{j=u_n}^{v_n} |A_{nj}| \|V_{nj}\| \right) &\leq \mathbf{E}^{\mathcal{B}_n} \left(\sum_{j=u_n}^{v_n} |A_{nj}| \|V'_{nj}\| \right) + \mathbf{E}^{\mathcal{B}_n} \left(\sum_{j=u_n}^{v_n} |A_{nj}| \|V''_{nj}\| \right) \\ &\leq a \left(\sum_{j=u_n}^{v_n} |A_{nj}| \right) + \left(\sum_{j=u_n}^{v_n} |A_{nj}| \mathbf{E}^{\mathcal{B}_n} \|V''_{nj}\| \right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{a.e.,} \end{aligned}$$

for all n greater than a certain n_0.

In the following theorem, we obtain the convergence in L_1 (and, consequently, the weak convergence) of random weighted sums under a condition of convergence on expectations $\mathbf{E}|A_{nj}|$.

Theorem 4.2. *Let* $\{A_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ *be an array of random variables and let* $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ *be an array of random elements in a separable Banach space* $\mathcal{X}$ *with norm* $\|\cdot\|$*, defined on a probability space* $(\Omega, \mathcal{A}, \mathbf{P})$*, with*

$$\lim_{n \to \infty} \sum_{j=u_n}^{v_n} \mathbf{E}|A_{nj}| = 0.$$

Let $\{\mathcal{B}_n, n \geq 1\}$ *be a sequence of sub* σ*-algebras of* $\mathcal{A}$*, and suppose that* A_{nj}*,* $u_n \leq j \leq v_n$*, are* $\mathcal{B}_n$*-measurable, for each* $n \geq 1$*. Suppose that* $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ *is* $\{A_{nj}\}$*-conditionally uniformly integrable relative to* $\{\mathcal{B}_n\}$*. Then*

$$\sum_{j=u_n}^{v_n} |A_{nj}| \|V_{nj}\| \longrightarrow 0 \quad \text{in } L_1 \text{ as} \quad n \to \infty$$

and, consequently,

$$\left\| \sum_{j=u_n}^{v_n} A_{nj} V_{nj} \right\| \longrightarrow 0 \quad \text{in } L_1.$$

Proof. The beginning is the same as in Theorem 4.1, and we obtain the boundedness

$$\mathbf{E}^{\mathcal{B}_n} \left(\sum_{j=u_n}^{v_n} |A_{nj}| \|V_{nj}\| \right) \leq a \left(\sum_{j=u_n}^{v_n} |A_{nj}| \right) + \left(\sum_{j=u_n}^{v_n} |A_{nj}| \mathbf{E}^{\mathcal{B}_n} \|V''_{nj}\| \right) \quad \text{a.e.}$$

There exists $n_0 \geq 1$ such that the expectation of the first sum is less than $\varepsilon/2$ for all $n \geq n_0$, and the expectation of the second sum is less than $\varepsilon/2$ by the choice of a. Then, given $\varepsilon > 0$, there exists $n_0 \geq 1$ such that, for all $n \geq n_0$,

$$\mathbf{E} \left(\mathbf{E}^{\mathcal{B}_n} \left(\sum_{j=u_n}^{v_n} |A_{nj}| \|V_{nj}\| \right) \right) = \mathbf{E} \left(\sum_{j=u_n}^{v_n} |A_{nj}| \|V_{nj}\| \right) < \varepsilon.$$

Next, we prove that the condition of $\{A_{nj}\}$-conditional compactly uniform integrability relative to a sequence of σ-algebras $\{\mathcal{B}_n\}$ allows us to reduce the study of conditional convergence of a randomly weighted sum of random elements in some infinite-dimensional Banach spaces to a similar problem posed for random elements in finite-dimensional spaces.

Henceforth, the term "operator" means a continuous linear operator.

Recall that a separable Banach space $\mathcal{X}$ has the bounded approximation property B.A.P. if the identity operator I on $\mathcal{X}$ is a pointwise limit of a sequence of finite-rank operators in the strong operator topology, that is, if there exists a sequence $\{T_n, n \geq 1\}$ of finite-rank operators on $\mathcal{X}$ such that

$$\lim_{n\to\infty}\left\|x - \sum_{k=1}^{n} T_k(x)\right\| = 0, \quad x \in \mathcal{X}.$$

An application of the Banach–Steinhaus principle implies that there exists a constant $M < \infty$ such that

$$\left\|\sum_{k=1}^{n} T_k\right\| \leq M, \quad \text{for all} \quad n \geq 1.$$

We will denote

$$U_n = \sum_{k=1}^{n} T_k.$$

We will need the following lemma from [8].

Lemma 4.3. *Let $\mathcal{X}$ be a separable Banach space with the B.A.P. Let K be a compact subset of $\mathcal{X}$. Then, for each $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \geq 1$ such that $\|x - U_n(x)\| < \varepsilon$ for all $x \in K$ and $n \geq n_0$.*

We now prove the following theorem.

Theorem 4.4. *Let $\{A_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of random variables and let $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of random elements in a separable Banach space $(\mathcal{X}, \|\cdot\|)$ with B.A.P., defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, with*

$$\sup_{n\geq 1}\sum_{j=u_n}^{v_n} |A_{nj}| \leq C \quad \text{a.e.},$$

for some constant $C < \infty$. Let $\{\mathcal{B}_n, n \geq 1\}$ be a sequence of sub σ-algebras of $\mathcal{A}$, and suppose that the A_{nj}, $u_n \leq j \leq v_n$, are $\mathcal{B}_n$-measurable for each $n \geq 1$. Suppose that $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{A_{nj}\}$-conditionally compactly uniformly integrable relative to $\{\mathcal{B}_n\}$. Then

$$\mathbf{E}^{\mathcal{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\right\| \longrightarrow 0 \quad \text{a.e. as} \quad n \to \infty,$$

if and only if there exists $t_1 \geq 1$ such that

$$\mathbf{E}^{\mathcal{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}U_t(V_{nj})\right\| \longrightarrow 0 \quad \text{a.e. for each } t \geq t_1 \text{ as} \quad n \to \infty.$$

Proof. Suppose that

$$\mathbf{E}^{\mathcal{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}U_t(V_{nj})\right\| \longrightarrow 0 \quad \text{a.e.},$$

for each $t \geq t_1$. Given $\varepsilon > 0$, there exists a compact $K \subset \mathcal{X}$ such that

$$\sup_{n\geq 1}\sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{E}^{\mathcal{B}_n}(\|V_{nj}\|I_{[V_{nj}\notin K]}) < \frac{\varepsilon}{4(M+1)} \quad \text{a.e.}$$

Define, for each $n \geq 1$, $u_n \leq j \leq v_n$,

$$W_{nj} = V_{nj}I_{[V_{nj}\in K]}, \qquad Y_{nj} = V_{nj}I_{[V_{nj}\notin K]}.$$

The compactness of K implies (see Lemma 4.3) that there exists $t_0 \geq 1$ such that, for every $t \geq t_0$ and every $n \geq 1$, $u_n \leq j \leq v_n$,

$$\|W_{nj} - U_t(W_{nj})\| < \frac{\varepsilon}{4C},$$

which implies

$$\mathbf{E}^{\mathcal{B}_n}\|W_{nj} - U_t(W_{nj})\| < \frac{\varepsilon}{4C} \quad \text{a.e.}$$

Moreover, for every $n \geq 1$,

$$\sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{E}^{\mathcal{B}_n}\|Y_{nj} - U_t(Y_{nj})\| \leq (M+1)\sum_{j=u_n}^{v_n} |A_{nj}|\mathbf{E}^{\mathcal{B}_n}\|Y_{nj}\| < \frac{\varepsilon}{4} \quad \text{a.e.}$$

Therefore, for every $t \geq t_0$,

$$\sup_{n\geq 1}\mathbf{E}^{\mathcal{B}_n}\sum_{j=u_n}^{v_n}|A_{nj}|\|V_{nj}U_t(V_{nj})\| \leq \sup_{n\geq 1}\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{E}^{\mathcal{B}_n}\|W_{nj} - U_t(W_{nj})\| + \sup_{n\geq 1}\sum_{j=u_n}^{v_n}|A_{nj}|\mathbf{E}^{\mathcal{B}_n}\|Y_{nj} - U_t(Y_{nj})\| < \frac{\varepsilon}{2} \quad \text{a.e.}$$

We now take $t \geq \max\{t_0, t_1\}$. By the hypothesis, there exists $n_0 \geq 1$ such that, for all $n \geq n_0$,

$$\mathbf{E}^{\mathcal{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}U_t(V_{nj})\right\| < \frac{\varepsilon}{2} \quad \text{a.e.}$$

Therefore, for all $n \geq n_0$,

$$\mathbf{E}^{\mathcal{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\right\| = \mathbf{E}^{\mathcal{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}(V_{nj} - U_t(V_{nj}) + U_t(V_{nj}))\right\|$$
$$\leq \mathbf{E}^{\mathcal{B}_n}\sum_{j=u_n}^{v_n}|A_{nj}|\|V_{nj} - U_t(V_{nj})\| + \mathbf{E}^{\mathcal{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}U_t(V_{nj})\right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{a.e.}$$

Now suppose that

$$\mathbf{E}^{\mathcal{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\right\| \longrightarrow 0 \quad \text{a.e. as} \quad n \to \infty.$$

Then, for every $t \geq 1$,

$$\left\|\sum_{j=u_n}^{v_n} A_{nj}U_t(V_{nj})\right\| = \left\|U_t\left(\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\right)\right\| \leq M\left\|\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\right\|,$$

which implies that

$$\mathbf{E}^{\mathcal{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}U_t(V_{nj})\right\| \leq M\mathbf{E}^{\mathcal{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\right\| \quad \text{a.e.,}$$

and so,

$$\mathbf{E}^{\mathcal{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}U_t(V_{nj})\right\| \longrightarrow 0 \quad \text{a.e. as} \quad n \to \infty.$$

Remark 5. A separable Banach space $\mathcal{X}$ has the B.A.P. if and only if $\mathcal{X}$ is isomorphic to a complemented subspace of a space with a basis (see [5]). Szarek, in [11], proved that there exists a separable Banach space that has the B.A.P. but fails to have a basis, i.e., the B.A.P. is a condition weaker than the existence of a Schauder basis.

Thus, Theorem 4.4 remains true if $\mathcal{X}$ is a Banach space with a Schauder basis and, for each $t \geq 1$, U_t denotes the tth partial-sum operator corresponding to the basis.

Remark 6. A basic hypothesis in all of the previous theorems in this section is the condition "A_{nj}, $u_n \leq j \leq v_n$, are $\mathcal{B}_n$-measurable for each $n \geq 1$." A particular case of great interest, in which this condition is satisfied, is when $\mathcal{B}_n = \sigma(A_{nj}, u_n \leq j \leq v_n)$, i.e., when $\mathcal{B}_n$ is the σ-algebra generated by $\{A_{nj}, u_n \leq j \leq v_n\}$, for each $n \geq 1$.

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