ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2019, Vol. 40, No. 12, pp. 2034–2042. © Pleiades Publishing, Ltd., 2019.

On the Concept of A-statistical Uniform Integrability and the Law of Large Numbers

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(Submitted by A. M. Elizarov)

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Received June 22, 2019; revised July 26, 2019; accepted August 10, 2019

Abstract—In this paper, we introduce the concept of A-statistical uniform integrability of sequences of random variables which is not only more general than the concept of uniform integrability, but is also weaker than the concept of uniform integrability. We also give some characterizations of A-statistical uniform integrability and prove a law of large numbers.

DOI: 10.1134/S1995080219120035

Keywords and phrases: *Sequence of random variables, uniform integrability, statistical convergence*

1. INTRODUCTION

Uniform integrability is an important concept of both probability theory and functional analysis that brings a compactness type concept for families of random variables. The concept of uniform integrability plays an especially important role in the area of limit theorems of probability theory. For instance, convergence in probability, combined with the additional condition of uniform integrability implies mean convergence; this implication fails without uniform integrability. The versions of uniform integrability such as the Cesàro uniform integrability [1], or uniform integrability with respect to a real array $\{a_{nk}\}$ [2] are also very useful in the limit theorems for sums of random variables.

The main motivation of the summability theory is to make a non-convergent sequence or series converge in a general sense. Therefore, the summability theory has many applications in probability limit theorems, approximation theory with positive linear operators, and differential equations, whenever the ordinary limit does not exist (see [3–13]).

Based on a study of Zygmund [14], the concept of statistical convergence, which is weaker than the ordinary convergence, was first examined by Steinhaus [15] and Fast [16] and was later reintroduced by Schoenberg [17]. Since then, many authors have studied this interesting concept and, have investigated some properties of it. For example, the statistical convergence of a sequence has a strong relationship between the ordinary convergence of its sub-sequences (see, [18, 19]). Some further results on statistical convergence can be found in [20–24]. The idea of statistical convergence was generalized to both A-statistical convergence and ideal convergence. Many authors have studied these new types of convergence as well as their properties and applications (see, [25–30]).

Statistical convergence is related to the concept of density (see, [31–35]). The main motivation behind statistical and A-statistical convergence is to deal with the ordinary convergence of a sub-sequence over density 1 subset of indices. This idea was also extended to the first countable topological

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spaces (see, [36]). Considering A-density 1 or, equivalently, A-density 0 sets, many concepts of classical analysis on sequences was generalized. For instance, the concept of statistical limit points of a sequence was first introduced by Fridy [37]. Further results on this and related concepts can be found in [38–40]. Fridy and Orhan [41] introduced the concepts of statistical limit superior and limit inferior, Demirci [42] introduced A-statistical limit superior and limit inferior concepts, whereas Küçükarslan and Altınok [43] introduced the concepts of A-statistical supremum and infimum.

In this paper, we introduce the concept of A-statistical uniform integrability of sequences of random variables, where A is a non-negative regular summability matrix. We use the concept of A-statistical supremum to introduce the concept of A-statistical uniform integrability which is not only more general than the concept of uniform integrability, but is also weaker than the concept of uniform integrability.

2. PRELIMINARIES

Let $x = \{x_k\}$ be a real sequence and let $A = \{a_{nk}\}$ be a summability matrix. If the sequence $\{(Ax)_n\}$ is convergent to a real number α then we say that the sequence x is A-summable to the real number α, where the series$(Ax)_n = \sum_k a_{nk}x_k$ is convergent for any $n \in \mathbb{N}$ and $\mathbb{N}$ is the set of positive integers. A summability matrix A is said to be regular if $\lim\limits_{n\to\infty}(Ax)_n = L$ whenever $\lim\limits_{k\to\infty} x_k = L$ (see, [44]). Throughout this paper we assume that $A = \{a_{nk}\}$ is a non-negative regular summability matrix.

Let $K \subset \mathbb{N}$. Then the number$\delta_A(K) := \lim_{n\to\infty} \sum_{k\in K} a_{nk}$ is said to be the A-density of K whenever the limit exists (see, [31–34]). Regularity of the summability matrix A ensures that $0 \le \delta_A(K) \le 1$ whenever $\delta_A(K)$ exists. If we consider that $A = C$, the Cesàro matrix, then $\delta(K) := \delta_C(K)$ is called the density of K (see, [35]), where $C = (c_{nk})$ is the summability matrix defined by

$$c_{nk} = \begin{cases} 1/n, & \text{if } k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

A real sequence $x = \{x_k\}$ is said to be *A-statistically convergent* (see, [25, 26]) to a real number α if for any $\varepsilon > 0$

$$\delta_A(\{k \in \mathbb{N} : |x_k - \alpha| \ge \varepsilon\}) = 0.$$

In this case, we write $st_A - \lim\limits_{k\to\infty} x_k = \alpha$. If we consider the Cesàro matrix, then C-statistical convergence is called *statistical convergence* [15, 16, 18]. In general, A-statistical convergence is regular (i.e., it preserves ordinary limit) and there exists some sequences that are A-statistically convergent but not ordinary convergent. Recall that if a sequence $x = \{x_k\}$ is statistically convergent to a real number α then there exists a sub-sequence $\{x_{k_j}\}$ such that $\lim\limits_{j\to\infty} x_{k_j} = \alpha$ and $\delta_A(\{k_j : j \in \mathbb{N}\}) = 1$ (see, [18, 19]).

A real number M is said to be a *A-statistical upper bound* of a sequence $\{x_k\}$ if

$$\delta_A(\{k \in \mathbb{N} : x_k > M\}) = 0.$$

In this case $\{x_k\}$ is said to be *A-statistically upper bounded*. The infimum of the set of all A-statistical upper bounds of a A-statistically upper bounded sequence is said to be the *A-statistical supremum* of $\{x_k\}$ and is denoted by $\sup_{st_A}\limits_{k\in\mathbb{N}} x_k$ [43]. If the sequence $x = \{x_k\}$ is not A-statistically upper bounded, then we define $\sup_{st_A}\limits_{k\in\mathbb{N}} x_k = \infty$. We use the notation $\sup_{st}\limits_{k\in\mathbb{N}} x_k$, whenever $A = C$.

A similar definition for *A-statistical infimum*, $\inf_{st_A}\limits_{k\in\mathbb{N}} x_k$, was given in [43]. It was also shown in [43] that

$$\inf_{k\in\mathbb{N}} x_k \le \inf_{st_A}\limits_{k\in\mathbb{N}} x_k \le \sup_{st_A}\limits_{k\in\mathbb{N}} x_k \le \sup_{k\in\mathbb{N}} x_k \tag{1}$$

for any real sequence $\{x_k\}$.

The following example shows that there exists some sequences such that the inequalities in (1) can be strict:

Example 1. Consider the sequence $x = \{x_k\}$ given by

$$x_k = \begin{cases} 2, & \text{if } k \text{ is even and perfect square,} \\ 1, & \text{if } k \text{ is even and not perfect square,} \\ 0, & \text{if } k \text{ is odd and not perfect square,} \\ -1, & \text{if } k \text{ is odd and perfect square,} \end{cases}$$

and observe that $\inf_{st}\limits_{k\in\mathbb{N}} x_k = 0$ and $\sup_{st}\limits_{k\in\mathbb{N}} x_k = 1$ whereas $\inf\limits_{k\in\mathbb{N}} x_k = -1$ and $\sup\limits_{k\in\mathbb{N}} x_k = 2$.

We use the following remark in our proofs:

Remark 1. If $\sup_{st_A}\limits_{k\in\mathbb{N}} x_k = M < \infty$ then, by the definition for any $\varepsilon > 0$, there exists $b < M + \varepsilon$ such that $\delta_A(\{k \in \mathbb{N} : x_k > b\}) = 0$. Therefore, we get$\delta_A(\{k \in \mathbb{N} : x_k > M + \varepsilon\}) = 0$.

3. STATISTICAL UNIFORM INTEGRABILITY

Throughout this paper, all random variables are defined on a fixed but otherwise arbitrary probability space $(\Omega, \mathcal{F}, P)$. The expected value of a random variable X is denoted by $\mathbb{E}X$ and we use the notation I for the indicator function.

A sequence of random variables $\{X_k\}$ is said to be *uniformly integrable* (see, [45]) if

$$\lim_{c\to\infty} \sup_{k\in\mathbb{N}} \mathbb{E}\,|X_k|\, I_{\{|X_k|>c\}} = 0.$$

Now, we define a new type of uniform integrability that is called A-statistical uniform integrability:

Definition 1. *A sequence of random variables $\{X_k\}$ is said to be A-statistically uniformly integrable if*

$$\lim_{c\to\infty} \sup_{st_A}\limits_{k\in\mathbb{N}} \mathbb{E}\,|X_k|\, I_{\{|X_k|>c\}} = 0.$$

If we consider the identity matrix for A, then A-statistical uniform integrability reduces to uniform integrability and it is trivial by (1) that if a sequence of random variables $\{X_k\}$ is uniformly integrable then it is A-statistically uniformly integrable. The following example shows that converse of this statement is not true in general:

Example 2. Let $K \subset \mathbb{N}$ be a set of A-density zero. Consider the sequence of random variables $\{X_k\}$ defined by

$$X_k = \begin{cases} \pm k, & \text{with probability } 1/2 \text{ if } k \in K\ , \\ 0, & \text{otherwise.} \end{cases}$$

As $\sup\limits_{k\in\mathbb{N}} \mathbb{E}\,|X_k| = \infty$, $\{X_k\}$ is not uniformly integrable (see, Theorem 4.5.3 of [45]). On the other hand, we have for any $c > 0$ that

$$\mathbb{E}\,|X_k|\, I_{\{|X_k|>c\}} \le \mathbb{E}\,|X_k|\, I_{\{|X_k|>0\}} = \left\{ \begin{cases} k, & \text{if } k \in K, \\ 0, & \text{otherwise} \end{cases} \right.$$

which yields that $\sup_{st_A}\limits_{k\in\mathbb{N}} \mathbb{E}\,|X_k|\, I_{\{|X_k|>c\}} = 0$. Hence we have $\{X_k\}$ is A-statistically uniformly integrable.

Note that if we consider the Cesàro matrix then the set of perfect square integers has density zero.

The following theorem is a characterization of A-statistical uniform integrability:

Theorem 1. *A sequence of random variables $\{X_k\}$ is A-statistically uniformly integrable if and only if*

(i) $\sup_{st_A}\limits_{k\in\mathbb{N}} \mathbb{E}\,|X_k| < \infty$;

(ii) *For every $\varepsilon > 0$, there exist $\nu(\varepsilon) > 0$ such that* $\sup_{k\in\mathbb{N}}{}_{st_A}\mathbb{E}\,|X_k|\,I_F \le \varepsilon$ *for any measurable subset F of Ω with $P(F) \le \nu(\varepsilon)$.*

Proof. Let $\{X_k\}$ be a A-statistically uniformly integrable sequence of random variables. Then for any $\varepsilon > 0$ there exists $a > 0$ such that

$$\delta_A\left(\left\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_{\{|X_k|>a\}} > \varepsilon/2\right\}\right) = 0. \tag{2}$$

On the other hand, we have

$$\{k \in \mathbb{N} : \mathbb{E}\,|X_k| > a + \varepsilon/2\} \subset \left\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_{\{|X_k|\le a\}} > a\right\}$$
$$\cup \left\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_{\{|X_k|>a\}} > \varepsilon/2\right\} = \left\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_{\{|X_k|>a\}} > \varepsilon/2\right\},$$

because $\left\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_{\{|X_k|\le a\}} > a\right\} = \varnothing$. Thus, we obtain by (2) that

$$0 \le \delta_A\left(\{k \in \mathbb{N} : \mathbb{E}\,|X_k| > a + \varepsilon/2\}\right) \le \delta_A\left(\left\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_{\{|X_k|>a\}} > \varepsilon/2\right\}\right) = 0.$$

Therefore, $\delta_A\left(\{k \in \mathbb{N} : \mathbb{E}\,|X_k| > a + \varepsilon/2\}\right) = 0$ which yields that $a + \varepsilon/2$ is a A-statistical upper bound of $\{\mathbb{E}\,|X_k|\}$. Hence, (i) holds.

Now, let choose $\nu(\varepsilon) = \varepsilon/2a$. Let F be a measurable subset of Ω such that $P(F) \le \nu(\varepsilon)$. Then, we have

$$\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_F > \varepsilon\}$$
$$\subset \left\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_{F\cap\{|X_k|\le a\}} > \varepsilon/2\right\} \cup \left\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_{F\cap\{|X_k|>a\}} > \varepsilon/2\right\}$$
$$\subset \{k \in \mathbb{N} : aP(F) > \varepsilon/2\} \cup \left\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_{\{|X_k|>a\}} > \varepsilon/2\right\}$$
$$= \left\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_{\{|X_k|>a\}} > \varepsilon/2\right\},$$

because $\{k \in \mathbb{N} : aP(F) > \varepsilon/2\} = \varnothing$. Thus, we obtain by (2) that

$$0 \le \delta_A\left(\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_F > \varepsilon\}\right) \le \delta_A\left(\left\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_{\{|X_k|>a\}} > \varepsilon/2\right\}\right) = 0,$$

which implies $\delta_A\left(\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_F > \varepsilon\}\right) = 0$. Therefore, (ii) holds.

Conversely, let (i) and (ii) hold. If $\sup_{k\in\mathbb{N}}{}_{st_A}\mathbb{E}\,|X_k| = M$ then by Remark 1 we get

$$\delta_A\left(\{k \in \mathbb{N} : \mathbb{E}\,|X_k| > M + \varepsilon\}\right) = 0. \tag{3}$$

On the other hand, from Markov's inequality we can write for any $c > 0$ that

$$\left\{k \in \mathbb{N} : P(|X_k| > c) > \frac{M+\varepsilon}{c}\right\} \subset \{k \in \mathbb{N} : \mathbb{E}\,|X_k| > M + \varepsilon\}.$$

Thus, by (3) we obtain

$$\delta_A\left(\left\{k \in \mathbb{N} : P(|X_k| > c) > \frac{M+\varepsilon}{c}\right\}\right) = 0. \tag{4}$$

Now, let $c > \frac{M+\varepsilon}{\nu}$. Then, we get

$$\{k \in \mathbb{N} : P(|X_k| > c) > \nu\} \subset \left\{k \in \mathbb{N} : P(|X_k| > c) > \frac{M+\varepsilon}{c}\right\}. \tag{5}$$

Therefore, we have by (4) and (5) that

$$\delta_A\left(\{k \in \mathbb{N} : P(|X_k| > c) > \nu\}\right) = 0. \tag{6}$$

On the other hand, by (ii) we can write

$$\left\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_{\{|X_k|>c\}} > \varepsilon\right\} \subset \{k \in \mathbb{N} : P(|X_k| > c) > \nu\}.$$

Thus, by (6) we get $\delta_A\left(\left\{k \in \mathbb{N} : \mathbb{E}\,|X_k|\,I_{\{|X_k|>c\}} > \varepsilon\right\}\right) = 0$ which yields

$$\sup_{k\in\mathbb{N}}{}_{st_A}\mathbb{E}\,|X_k|\,I_{\{|X_k|>c\}} \le \varepsilon.$$

Hence, proof is completed. □

Motivating from the de la Vallée Poussin characterization of uniform integrability (see, [46]) we give the following characterization of A-statistical uniform integrability:

Theorem 2. *A sequence of random variables* $\{X_k\}$ *is* A*-statistically uniformly integrable if and only if there exists a measurable function* $\phi:(0,\infty)\to(0,\infty)$ *such that* $\lim\limits_{t\to\infty}\dfrac{\phi(t)}{t}=\infty$ *and*

$$\sup_{k\in\mathbb{N}}{}_{st_A}\mathbb{E}\phi\left|X_k\right|<\infty.$$

Proof. Assume that $\{X_k\}$ is A-statistically uniformly integrable. Then, we can choose a sequence of positive integers $\{n_j\}$ such that for any $j\in\mathbb{N}$

$$\sup_{k\in\mathbb{N}}{}_{st_A}\mathbb{E}\left|X_k\right|I_{\{|X_k|>n_j\}}<\frac{1}{2^j}.$$

Therefore, for any $j\in\mathbb{N}$ we have

$$\delta_A\left(\left\{k\in\mathbb{N}:\mathbb{E}\left|X_k\right|I_{\{|X_k|>n_j\}}>\frac{1}{2^j}\right\}\right)=0. \tag{7}$$

Moreover, it is easy to see that there exists $j_0\in\mathbb{N}$ such that

$$\left\{k\in\mathbb{N}:\sum_{j=1}^{\infty}\mathbb{E}\left|X_k\right|I_{\{|X_k|>n_j\}}>\sum_{j=1}^{\infty}\frac{1}{2^j}\right\}\subset\left\{k\in\mathbb{N}:\mathbb{E}\left|X_k\right|I_{\{|X_k|>n_{j_0}\}}>\frac{1}{2^{j_0}}\right\}. \tag{8}$$

Considering that $\sum\limits_{j=1}^{\infty}\dfrac{1}{2^j}=1$, by (7) and (8) we obtain

$$\delta_A\left(\left\{k\in\mathbb{N}:\sum_{j=1}^{\infty}\mathbb{E}\left|X_k\right|I_{\{|X_k|>n_j\}}>1\right\}\right)=0. \tag{9}$$

On the other hand, there exists a measurable function (see, [1, 2]) $\phi:(0,\infty)\to(0,\infty)$ such that $\lim\limits_{t\to\infty}\dfrac{\phi(t)}{t}=\infty$ and for any $k\in\mathbb{N}$

$$\mathbb{E}\phi\left(|X_k|\right)\le\sum_{j=1}^{\infty}\sum_{i=n_j}^{\infty}P\left(|X_k|>i\right)$$

which implies (see, [2])

$$\begin{aligned}\{k\in\mathbb{N}:\mathbb{E}\phi\left(|X_k|\right)>1\}&\subset\left\{k\in\mathbb{N}:\sum_{j=1}^{\infty}\sum_{i=n_j}^{\infty}P\left(|X_k|>i\right)>1\right\}\\&\subset\left\{k\in\mathbb{N}:\sum_{j=1}^{\infty}\mathbb{E}\left|X_k\right|I_{\{|X_k|>n_j\}}>1\right\}.\end{aligned} \tag{10}$$

Now, by (9) and (10) we get$\delta_A\left(\{k\in\mathbb{N}:\mathbb{E}\phi\left(|X_k|\right)>1\}\right)=0$, which yields $\sup\limits_{k\in\mathbb{N}}{}_{st_A}\mathbb{E}\phi\left|X_k\right|\le 1$.

Conversely, assume that such a function ϕ exists and let $\varepsilon>0$. Then, then by Remark 1 we get

$$\delta_A\left(\{k\in\mathbb{N}:\mathbb{E}\phi\left(|X_k|\right)>M+\varepsilon\}\right)=0, \tag{11}$$

where $M:=\sup\limits_{k\in\mathbb{N}}{}_{st_A}\mathbb{E}\phi\left|X_k\right|$ and there exists $a>0$ such that $\dfrac{\phi(t)}{t}>\dfrac{M+\varepsilon+1}{\varepsilon}$ whenever $t>a$. Thus,

$$\left\{k\in\mathbb{N}:\mathbb{E}\left|X_k\right|I_{\{|X_k|>a\}}>\varepsilon\right\}$$

$$\subset \left\{k \in \mathbb{N} : \tfrac{\varepsilon}{M+\varepsilon+1}\mathbb{E}\phi\left(|X_k|\right) I_{\{|X_k|>a\}} > \varepsilon\right\}$$
$$= \left\{k \in \mathbb{N} : \tfrac{1}{M+\varepsilon+1}\mathbb{E}\phi\left(|X_k|\right) I_{\{|X_k|>a\}} > 1\right\}$$
$$\subset \left\{k \in \mathbb{N} : \tfrac{1}{M+\varepsilon+1} > \tfrac{1}{M+\varepsilon}\right\} \cup \left\{k \in \mathbb{N} : \mathbb{E}\phi\left(|X_k|\right) I_{\{|X_k|>a\}} > M+\varepsilon\right\}$$
$$= \left\{k \in \mathbb{N} : \mathbb{E}\phi\left(|X_k|\right) I_{\{|X_k|>a\}} > M+\varepsilon\right\},$$

because $\left\{k \in \mathbb{N} : \tfrac{1}{M+\varepsilon+1} > \tfrac{1}{M+\varepsilon}\right\} = \varnothing$. Hence, (11) implies $\delta_A\left(\left\{k \in \mathbb{N} : \mathbb{E}\left|X_k\right| I_{\{|X_k|>a\}} > \varepsilon\right\}\right) = 0$ which means $\{X_k\}$ is A-statistically uniformly integrable. □

4. LAW OF LARGE NUMBERS

The concept of uniform integrability is a useful tool to establish the law of large numbers with mean convergence [1, 2, 45]. For instance, uniform integrability of a sequence $\{X_k\}$ of pairwise independent random variables implies the following law of large numbers (see, [1]):

$$\lim_{n\to\infty} \mathbb{E}\left|\sum_{k=1}^{\infty} a_{nk}(X_k - \mathbb{E}X_k)\right| = 0. \tag{12}$$

After introducing the concept of A-uniform integrability, we can consider a result about the law of large numbers with mean convergence in the statistical sense. The A-statistical version of mean convergence of a sequence of random variables $\{X_k\}$ to random variable X can be defined by

$$st_A - \lim_{k\to\infty} \mathbb{E}\left|X_k - X\right| = 0.$$

Various notions on convergence of sequences of random variables in the statistical sense may be found in [47, 48]. The following example shows that convergence of the law of large numbers with mean convergence in the statistical sense can fail for sequences of pairwise independent random variables whereas statistical uniform integrability is still valid.

Example 3. Let $\{\varepsilon_j\}$ be a sequence of symmetric pairwise independent identically distributed Bernoulli random variables with $P(\varepsilon_j = 1) = P(\varepsilon_j = -1) = 1/2$ for any $j \in \mathbb{N}$. Next, consider a sequence of random variables $\{X_k\}$ defined by

$$X_k = \begin{cases} \varepsilon_j 2^{\sqrt{k}} k, & \text{if } k = j^2, \\ 0, & \text{otherwise.} \end{cases}$$

Similar to Example 2, $\{X_k\}$ is A-statistically uniformly integrable sequence of random variables that is not uniformly integrable. Obviously, $\{X_k\}$ is a sequence of pairwise independent random variables.

We can show that there is no law of large numbers for this sequence with mean convergence in statistical sense: We can write for any $n \in \mathbb{N}$ that

$$\mathbb{E}\left|\frac{1}{n}\sum_{k=1}^{n}(X_k - EX_k)\right| = \frac{1}{n}\mathbb{E}\left|\sum_{k=1}^{n} X_k\right| = \frac{1}{n}\mathbb{E}(Y_n),$$

where random variable

$$Y_n = \left|\sum_{k=1}^{n} X_k\right| = \left|\sum_{j=1}^{[\sqrt{n}]} 2^j j^2 \epsilon_j\right|$$

and $[a]$ denotes the greatest integer which is not greater than the real number a. Note that, the random variable Y_n takes at most $2^{[\sqrt{n}]}$ values, which are all the possible combinations of signs $+$ and $-$ in the sum:

$$\left|\sum_{j=1}^{[\sqrt{n}]} \pm 2^j j^2\right|$$

with the same probability $\frac{1}{2^{[\sqrt{n}]}}$ for any $n \in \mathbb{N}$.

Now, by the definition of the expectation of a random variable we have for any $n \in \mathbb{N}$ that

$$\mathbb{E}(Y_n) = \sum \left| \sum_{j=1}^{[\sqrt{n}]} \pm 2^j j^2 \right| \frac{1}{2^{[\sqrt{n}]}},$$

where the first sum is calculated over all possible combinations of $\pm$. All the values of the random variable are positive, so if we drop all terms in the sum except one, when all signs are $+$'s, then we obtain a smaller value. Thus we have

$$\mathbb{E}(Y_n) \geq \left(\sum_{j=1}^{[\sqrt{n}]} 2^j j^2 \right) \frac{1}{2^{[\sqrt{n}]}}.$$

In the last sum all terms are positive again, so if we take only the last one, again we get a smaller value:

$$\mathbb{E}(Y_n) \geq 2^{[\sqrt{n}]} [\sqrt{n}]^2 \frac{1}{2^{[\sqrt{n}]}} = [\sqrt{n}]^2.$$

Therefore, we obtain

$$\mathbb{E} \left| \frac{1}{n} \sum_{k=1}^{n} (X_k - \mathbb{E}X_k) \right| = \frac{1}{n} \mathbb{E}(Y_n) \geq 1$$

which implies the sequence $\left\{ \mathbb{E} \left| \frac{1}{n} \sum_{k=1}^{n} (X_k - \mathbb{E}\mathbb{X}_k) \right| \right\}$ does not have any sub-sequence that is convergent to zero. Therefore, it is not statistically convergent to zero.

Due to this last example, we cannot prove a result similar to (12) by using A-statistical uniform integrability; however, we can prove the following about the law of large numbers with mean convergence for a sub-sequence that is indexed with a density 1 set:

Theorem 3. *If $\{X_k\}$ is a A-statistically uniformly integrable sequence of pairwise independent random variables then there exists a set $K = \{i_j : j \in \mathbb{N}\}$ such that $\delta_A(K) = 1$ and*

$$\lim_{n \to \infty} \mathbb{E} \left| \frac{1}{n} \sum_{j=1}^{n} (X_{i_j} - \mathbb{E}X_{i_j}) \right| = 0.$$

Proof. Let $\varepsilon > 0$. As $\{X_k\}$ is a A-statistically uniformly integrable there exists $a > 0$ such that

$$\sup_{k \in \mathbb{N}}{}_{st_A} \mathbb{E} |X_k| I_{\{|X_k| > a\}} < \varepsilon/4$$

which yields

$$\delta_A \left(\left\{ k \in \mathbb{N} : \mathbb{E} |X_k| I_{\{|X_k| > a\}} \geq \varepsilon/4 \right\} \right) = 0.$$

Let $K := \mathbb{N} - \left\{ k \in \mathbb{N} : \mathbb{E} |X_k| I_{\{|X_k| > a\}} \geq \varepsilon/4 \right\}$. It is obvious that $\delta_A(K) = 1$. If we numerate K with $\{i_j : j \in \mathbb{N}\}$ then the sequence $\left\{ X_{i_j} \right\}$ is uniformly integrable. Now, we define for each $j \in \mathbb{N}$, $U_j = X_{i_j} I_{\left\{ \left| X_{i_j} \right| > a \right\}}$ and $V_j = X_{i_j} I_{\left\{ \left| X_{i_j} \right| \leq a \right\}}$.

Note that $\{U_j - \mathbb{E}U_j\}$ is a sequence of pairwise independent random variables with a uniform bound $2a$. Therefore, by pairwise independence we obtain for any $n \in \mathbb{N}$ that

$$0 \leq \mathbb{E} \left(\frac{1}{n} \sum_{j=1}^{n} (U_j - \mathbb{E}U_j) \right)^2 = \frac{1}{n^2} \sum_{j=1}^{n} \mathbb{E} \left(U_j - \mathbb{E}U_j \right)^2 \leq \frac{4a^2}{n}.$$

Thus, $\left\{\mathbb{E}\left(\frac{1}{n}\sum_{j=1}^{n}(U_j-\mathbb{E}U_j)\right)\right\}$ is convergent in L_2, hence in mean. So, there exists $n_0\in\mathbb{N}$ such that

$$\frac{1}{n}\mathbb{E}\left(\sum_{j=1}^{n}(U_j-\mathbb{E}U_j)\right)<\frac{\varepsilon}{2} \tag{13}$$

whenever $n\geq n_0$. On the other hand, using the definition of the set K we can write for any $n\in\mathbb{N}$ that

$$\frac{1}{n}\mathbb{E}\left(\sum_{j=1}^{n}|V_j-\mathbb{E}V_j|\right)\leq\frac{2}{n}\sum_{j=1}^{n}\mathbb{E}X_{i_j}I_{\left\{\left|X_{i_j}\right|>a\right\}}\leq\varepsilon/2. \tag{14}$$

Hence, by (13) and (14) the proof is completed. □

FUNDING

The research of M. Ünver was done while he was visiting University of Regina, Canada and the research has been supported by The Scientific and Technological Research Council of Turkey (TÜBİTAK) Grant 1059B191800534. A part of research of A. Volodin has been done during the visit to the University of Pisa in December 2016. The authors would like to express their appreciation to the INDAM for supporting this visit.

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