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# On Almost Sure Convergence of Series of Random Variables Irrespective of Their Joint Distributions

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*In this article, necessary and sufficient conditions are presented for a series of random variables to converge absolutely almost surely irrespective of their joint distributions. The summands are not assumed to be integrable. Illustrative examples and counterexamples are presented.*

**Keywords** Absolute convergence of series; Almost sure convergence; Irrespective of the joint distributions; Kolmogorov three-series criterion; Series of random variables.

**Mathematics Subject Classification** Primary 60F15.

## 1. Introduction

One of the most interesting problems of classical probability theory concerns determining whether a series of random variables converges almost surely (a.s.). Pertaining to a sequence of *independent* random variables $\{X_n, n \geq 1\}$, a celebrated result of Kolmogorov (see, e.g., [1, p. 117]) provides a triumvirate of conditions which are both necessary and sufficient for the a.s. convergence of the series $\sum_{n=1}^{\infty} X_n$. This so-called *three-series criterion* is stated as follows: For $t \in (0, \infty)$, consider the three series

$$\sum_{n=1}^{\infty} P(|X_n| > t), \tag{1.1}$$

$$\sum_{n=1}^{\infty} E\left(X_n I(|X_n| \leq t)\right), \tag{1.2}$$

and

$$\sum_{n=1}^{\infty} \mathrm{Var}(X_n I(|X_n| \leq t)). \tag{1.3}$$

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If for some $t \in (0, \infty)$ the three series in (1.1), (1.2), and (1.3) converge, then the series $\sum_{n=1}^{\infty} X_n$ converges a.s. Conversely, if the series $\sum_{n=1}^{\infty} X_n$ converges a.s., then for all $t \in (0, \infty)$ the three series in (1.1), (1.2), and (1.3) converge.

At the origin of the current investigation is the following result of Smit and Vervaat [2] concerning the a.s. convergence of a series of nonnegative random variable irrespective of their joint distributions (i.j.d.).

**Proposition 1.1.** (Smit and Vervaat [2]). *Let $\{X_n, n \geq 1\}$ be a sequence of nonnegative random variables. The following three statements are equivalent.*

*(i)* $\sum_{n=1}^{\infty} X_n < \infty$ *a.s. i.j.d. of the* $\{X_n, n \geq 1\}$.
*(ii)* $P(\sum_{n=1}^{\infty} X_n < \infty) > 0$ *i.j.d. of the* $\{X_n, n \geq 1\}$.
*(iii)* $\sum_{n=1}^{\infty} E(\min\{X_n, 1\}) < \infty$.

A striking consequence of Proposition 1.1 and its proof is the following observation made by Smit and Vervaat [2]. If a series of independent nonnegative random variables converges a.s., then it does so without the independence hypothesis. In other words and more precisely, if $\sum_{n=1}^{\infty} X_n < \infty$ a.s. where $\{X_n, n \geq 1\}$ is a sequence of independent nonnegative random variables and if $\{Y_n, n \geq 1\}$ is a sequence of random variables with $X_n$ and $Y_n$ being identically distributed for each $n \geq 1$, then $\sum_{n=1}^{\infty} Y_n < \infty$ a.s. i.j.d. of the $\{Y_n, n \geq 1\}$.

Smit and Vervaat [2] presented an example showing that $P(\sum_{n=1}^{\infty} |X_n| < \infty)$ can assume any value in $[0, 1]$ depending on the joint distributions of the random variables $\{X_n, n \geq 1\}$. We now present another example that also illustrates this and is considerably simpler than the example of Smit and Vervaat [2].

**Example 1.1.** Let $0 \leq p \leq 1$ and let $A$ be an event with $P(A) = 1 - p$. Let $X_n = I(A)$, $n \geq 1$. Then

$$P\left(\sum_{n=1}^{\infty} |X_n| < \infty\right) = P(A^c) = p.$$

In this article, we present in Theorem 3.1 (the main result) and in Corollary 3.1 new versions of Proposition 1.1. More specifically, for a sequence of random variables $\{X_n, n \geq 1\}$, we provide for $t_0 \in (0, \infty)$ three equivalent conditions which, in turn, are each equivalent to the assertion that the series $\sum_{n=1}^{\infty} X_n$ converges absolutely a.s. i.j.d. of the $\{X_n, n \geq 1\}$. Our method of proof is different from that of Smit and Vervaat [2]. The summands $\{X_n, n \geq 1\}$ are not assumed to be integrable. Of course, if the summands are integrable with $\sum_{n=1}^{\infty} E|X_n| < \infty$, then it is immediate that the series $\sum_{n=1}^{\infty} X_n$ converges absolutely a.s. i.j.d. of the $\{X_n, n \geq 1\}$.

According to Lemma 2.2 (resp., Lemma 2.3, Lemma 2.4) below, the single condition (3.1) (resp., (3.2), (3.3)) of Theorem 3.1 is equivalent to the pair of conditions (2.1) and (2.2). Now (2.1) is the condition (1.1) of the Kolmogorov three-series criterion whereas (2.2) is of the same spirit as condition (1.2) of Kolmogorov three-series criterion but is stronger. The single condition (3.2) of Theorem 3.1 is of the same spirit as condition (1.1) of Kolmogorov three-series criterion but is stronger.

The plan of this article is as follows. Preliminary lemmas are presented in Section 2. The main result is established in Section 3 and illustrative examples and counterexamples are presented in Section 4.

## 2. Preliminary Lemmas

To establish Theorem 3.1, we use the following lemmas.

**Lemma 2.1.** *Let* X *be a random variable and let* $t_0 \in (0, \infty)$. *Then*

$$\frac{E(|X|I(|X| \leq t_0))}{2t_0} + \frac{P(|X| > t_0)}{2} \leq E\left(\frac{|X|}{t_0 + |X|}\right)$$
$$\leq \frac{E(|X|I(|X| \leq t_0))}{t_0} + P(|X| > t_0).$$

*Proof.* Note that

$$E\left(\frac{|X|}{t_0 + |X|}\right)$$
$$= E\left(\frac{|X|}{t_0 + |X|}I(|X| \leq t_0)\right) + E\left(\frac{|X|}{t_0 + |X|}I(|X| > t_0)\right)$$
$$\begin{cases} \leq \dfrac{E(|X|I(|X| \leq t_0))}{t_0} + P(|X| > t_0) \\ \geq \dfrac{E(|X|I(|X| \leq t_0))}{2t_0} + \dfrac{P(|X| > t_0)}{2}. \end{cases}$$

□

The next lemma is an immediate consequence of Lemma 2.1.

**Lemma 2.2.** *Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables and let* $t_0 \in (0, \infty)$. *Then*

$$\sum_{n=1}^{\infty} E\left(\frac{|X_n|}{t_0 + |X_n|}\right) < \infty$$

*if and only if*

$$\sum_{n=1}^{\infty} P(|X_n| > t_0) < \infty \tag{2.1}$$

*and*

$$\sum_{n=1}^{\infty} E(|X_n|I(|X_n| \leq t_0)) < \infty. \tag{2.2}$$

**Remark 2.1.** The arguments used to establish Lemmas 2.1 and 2.2 are virtually identical to those of Loève [3, p. 209], and Heyde [4].

**Lemma 2.3.** $\{X_n, n \geq 1\}$ *be a sequence of random variables and let* $t_0 \in (0, \infty)$. *Then*

$$\int_0^{t_0} \sum_{n=1}^{\infty} P(|X_n| > t)\, dt < \infty \tag{2.3}$$

*if and only if (2.1) and (2.2) hold.*

*Proof.* Note that for $n \geq 1$,

$$\begin{aligned} &E(|X_n| I(|X_n| \leq t_0)) \\ &\quad = \int_0^{\infty} P(|X_n| I(|X_n| \leq t_0) > t)\, dt \\ &\quad = \int_0^{t_0} P(|X_n| I(|X_n| \leq t_0) > t)\, dt \\ &\quad = \int_0^{t_0} P(t < |X_n| \leq t_0)\, dt. \end{aligned} \tag{2.4}$$

Thus,

$$\begin{aligned} &\int_0^{t_0} \sum_{n=1}^{\infty} P(|X_n| > t)\, dt \\ &\quad = \int_0^{t_0} \sum_{n=1}^{\infty} P(t < |X_n| \leq t_0)\, dt + \int_0^{t_0} \sum_{n=1}^{\infty} P(|X_n| > t_0)\, dt \\ &\quad = \sum_{n=1}^{\infty} \int_0^{t_0} P(t < |X_n| \leq t_0)\, dt + t_0 \sum_{n=1}^{\infty} P(|X_n| > t_0) \\ &\quad = \sum_{n=1}^{\infty} E(|X_n| I(|X_n| \leq t_0)) + t_0 \sum_{n=1}^{\infty} P(|X_n| > t_0) \text{ (by (2.4))}. \end{aligned}$$

Hence, (2.3) is equivalent to the pair of conditions (2.1) and (2.2). □

**Remark 2.2.** The condition

$$\int_0^{t_0} \sum_{n=1}^{\infty} P(|X_n| > t)\, dt < \infty \text{ for some } t_0 \in (0, \infty) \tag{2.5}$$

is equivalent to the condition

$$\int_0^{T_0} \sum_{n=1}^{\infty} P(|X_n| > t)\, dt < \infty \text{ for all } T_0 \in (0, \infty). \tag{2.6}$$

It is clear that (2.6) implies (2.5); to see that (2.5) implies (2.6), let $t_0$ satisfy (2.5). It is clear that (2.6) holds for all $T_0 \in (0, t_0]$. Let $T_0 \in (t_0, \infty)$. Then

$$\begin{aligned}\int_0^{T_0} \sum_{n=1}^{\infty} P(|X_n| > t)\, dt &= \int_0^{t_0} \sum_{n=1}^{\infty} P(|X_n| > t)\, dt + \int_{t_0}^{T_0} \sum_{n=1}^{\infty} P(|X_n| > t)\, dt \\ &\leq \text{Constant} + \left(\sum_{n=1}^{\infty} P(|X_n| > t_0)\right)(T_0 - t_0) \\ &< \infty\end{aligned}$$

recalling that it was shown in the proof of Lemma 2.3 that (2.3) implies (2.1).

**Lemma 2.4.** $\{X_n, n \geq 1\}$ *be a sequence of random variables and let* $t_0 \in (0, \infty)$. *Then*

$$\sum_{n=1}^{\infty} E(\min\{|X_n|, t_0\}) < \infty$$

*if and only if (2.1) and (2.2) hold.*

*Proof.* Note that for $n \geq 1$,

$$\begin{aligned}&E(\min\{|X_n|, t_0\}) \\ &\quad = E(\min\{|X_n|, t_0\} I(|X_n| \leq t_0)) + E(\min\{|X_n|, t_0\} I(|X_n| > t_0)) \\ &\quad = E(|X_n| I(|X_n| \leq t_0)) + t_0 P(|X_n| > t_0)\end{aligned}$$

and the result follows. □

## 3. The Main Result

With the preliminaries accounted for, Theorem 3.1 may now be presented. We note that the three conditions (3.1), (3.2), and (3.3) in Theorem 3.1 involve only the marginal distributions of the $\{X_n, n \geq 1\}$ and again we note that the $\{X_n, n \geq 1\}$ are not assumed to be integrable.

**Theorem 3.1.** *Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables and let* $t_0 \in (0, \infty)$. *Then the following three statements are equivalent:*

$$\sum_{n=1}^{\infty} E\left(\frac{|X_n|}{t_0 + |X_n|}\right) < \infty, \tag{3.1}$$

$$\int_0^{t_0} \sum_{n=1}^{\infty} P(|X_n| > t)\, dt < \infty, \tag{3.2}$$

$$\sum_{n=1}^{\infty} E(\min\{|X_n|, t_0\}) < \infty. \tag{3.3}$$

*Any one of (3.1), (3.2), or (3.3) ensures that the series* $\sum_{n=1}^{\infty} X_n$ *converges absolutely a.s. i.j.d. of the* $\{X_n, n \geq 1\}$. *Conversely, if the* $\{X_n, n \geq 1\}$ *are independent and if*

$$P\left(\sum_{n=1}^{\infty} X_n \text{ converges absolutely}\right) > 0, \tag{3.4}$$

*then*

$$\text{the series } \sum_{n=1}^{\infty} X_n \text{ converges absolutely a.s.} \tag{3.5}$$

*and the three statements (3.1), (3.2), and (3.3) hold.*

*Proof.* It follows from Lemmas 2.2, 2.3, and 2.4 that (3.1), (3.2), and (3.3) are each equivalent to the pair of conditions (2.1) and (2.2), and, hence, the three conditions (3.1), (3.2), and (3.3) are equivalent.

Next, any one of (3.1), (3.2), and (3.3) ensures that

$$E\left(\sum_{n=1}^{\infty} |X_n| I(|X_n| \leq t_0)\right) = \sum_{n=1}^{\infty} E\left(|X_n| I(|X_n| \leq t_0)\right) < \infty$$

by (2.2), whence

$$\sum_{n=1}^{\infty} |X_n| I(|X_n| \leq t_0) < \infty \text{ a.s.} \tag{3.6}$$

Moreover, any one of (3.1), (3.2), or (3.3) ensures that the sequences $\{|X_n|, n \geq 1\}$ and $\{|X_n| I(|X_n| \leq t_0), n \geq 1\}$ are equivalent in the sense of Khintchine since

$$\sum_{n=1}^{\infty} P(|X_n| \neq |X_n| I(|X_n| \leq t_0)) = \sum_{n=1}^{\infty} P(|X_n| > t_0) < \infty$$

by (2.1) and so by the Borel-Cantelli lemma

$$P(\liminf_{n\to\infty}[|X_n| = |X_n| I(|X_n| \leq t_0)]) = 1. \tag{3.7}$$

Thus $\sum_{n=1}^{\infty} |X_n| < \infty$ a.s. by (3.6) and (3.7).

Conversely, suppose the $\{X_n, n \geq 1\}$ are independent and (3.4) holds. Then by the Kolmogorov 0-1 law (3.5) holds and so by the Kolmogorov three-series criterion (2.1) and (2.2) hold. As argued above, the pair of conditions (2.1) and (2.2) is equivalent to each of (3.1), (3.2), and (3.3). □

**Remark 3.1.** If $\{X_n, n \geq 1\}$ is a sequence of random variables such that

$$\sum_{n=1}^{\infty} P(|X_n| > t_n) < \infty \tag{3.8}$$

for some sequence $\{t_n, n \geq 1\}$ in $(0, \infty)$ with

$$\sum_{n=1}^{\infty} t_n < \infty, \tag{3.9}$$

then the conditions (3.1), (3.2), and (3.3) of Theorem 3.1 are satisfied for all $t_0 \in (0, \infty)$ (and, hence, by Theorem 3.1 the series $\sum_{n=1}^{\infty} X_n$ converges absolutely a.s. i.j.d. of the $\{X_n, n \geq 1\}$).

*Proof.* Since (3.9) ensures that $t_n \to 0$, we may suppose without loss of generality that $t_n < t_0$ for all $n \geq 1$. Then by (3.8)

$$\sum_{n=1}^{\infty} P(|X_n| > t_0) \leq \sum_{n=1}^{\infty} P(|X_n| > t_n) < \infty$$

establishing (2.1), and by (3.9) and (3.8)

$$\begin{aligned}
&\sum_{n=1}^{\infty} E(|X_n| I(|X_n| \leq t_0)) \\
&\quad = \sum_{n=1}^{\infty} E(|X_n| I(|X_n| \leq t_n) + \sum_{n=1}^{\infty} E(|X_n| I(t_n < |X_n| \leq t_0)) \\
&\quad \leq \sum_{n=1}^{\infty} t_n + t_0 \sum_{n=1}^{\infty} P(|X_n| > t_n) \\
&\quad < \infty
\end{aligned}$$

establishing (2.2). Thus, by Lemma 2.2 and Theorem 3.1, the conditions (3.1), (3.2), and (3.3) hold. □

**Remark 3.2.** We will see in Example 4.4 below that the conditions (3.8) and (3.9) are compatible with each other.

**Corollary 3.1.** *Let $\{X_n, n \geq 1\}$ be a sequence of random variables and let $t_0 \in (0, \infty)$. Then the following five statements are equivalent:*

$$\sum_{n=1}^{\infty} E\left(\frac{|X_n|}{t_0 + |X_n|}\right) < \infty, \tag{3.10}$$

$$\int_0^{t_0} \sum_{n=1}^{\infty} P(|X_n| > t)\, dt < \infty, \tag{3.11}$$

$$\sum_{n=1}^{\infty} E(\min\{|X_n|, t_0\}) < \infty, \tag{3.12}$$

$$P\left(\sum_{n=1}^{\infty} X_n \textit{ converges absolutely}\right) > 0 \textit{ i.j.d. of the } \{X_n, n \geq 1\}, \tag{3.13}$$

$$\textit{the series } \sum_{n=1}^{\infty} X_n \textit{ converges absolutely a.s. i.j.d. of the } \{X_n, n \geq 1\}. \tag{3.14}$$

*Proof.* By Theorem 3.1, the statements (3.10), (3.11), and (3.12) are equivalent to each other and each of them implies (3.14) which, in turn, implies (3.13). It only needs to be shown that (3.13) implies (3.10), (3.11), or (3.12). Now if (3.13) holds, then in particular

$$P\left(\sum_{n=1}^{\infty} X_n \text{ converges absolutely}\right) > 0$$

when $\{X_n, n \geq 1\}$ is a sequence of independent random variables. Then (3.10), (3.11), and (3.12) hold by the converse half of Theorem 3.1. □

**Remark 3.3.** Versions of Theorem 3.1 and Corollary 3.1 hold for a sequence of random elements $\{V_n, n \geq 1\}$ taking values in a real separable Banach space. The arguments are identical; just replace $|X_n|$ by $||V_n||$ throughout.

For a sequence of independent *symmetric* random variables, we obtain in the following corollary a new version of the convergence half of Theorem 3.1. Examples will be given in Section 4 showing that (3.15) and (3.16) do not imply each other without the symmetry assumption.

**Corollary 3.2.** *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent symmetric random variables. Then*

$$\textit{the series} \sum_{n=1}^{\infty} X_n \textit{ converges a.s.} \tag{3.15}$$

*if and only if*

$$\textit{the series} \sum_{n=1}^{\infty} X_n^2 \textit{ converges a.s.} \tag{3.16}$$

*If either*

$$P\left(\sum_{n=1}^{\infty} X_n \textit{ converges}\right) > 0 \tag{3.17}$$

*or*

$$P\left(\sum_{n=1}^{\infty} X_n^2 \textit{ converges}\right) > 0, \tag{3.18}$$

*then (3.15), (3.16), and the following three statements all hold:*

$$\sum_{n=1}^{\infty} E\left(\frac{X_n^2}{t_0 + X_n^2}\right) < \infty \textit{ for all } t_0 \in (0, \infty), \tag{3.19}$$

$$\int_0^{t_0} \sum_{n=1}^{\infty} P\big(X_n^2 > t\big)\, dt < \infty \textit{ for all } t_0 \in (0, \infty), \tag{3.20}$$

$$\sum_{n=1}^{\infty} E\big(\min\big\{X_n^2, t_0\big\}\big) < \infty \textit{ for all } t_0 \in (0, \infty). \tag{3.21}$$

*Proof.* Assume that (3.15) holds. Then by the Kolmogorov three-series criterion and the symmetry hypothesis,

$$\sum_{n=1}^{\infty} P(|X_n| > 1) < \infty$$

and

$$\begin{aligned} E\left(\sum_{n=1}^{\infty} X_n^2 I(|X_n| \le 1)\right) &= \sum_{n=1}^{\infty} E\left(X_n^2 I(|X_n| \le 1)\right) \\ &= \sum_{n=1}^{\infty} \operatorname{Var}(X_n I(|X_n| \le 1)) \\ &< \infty. \end{aligned}$$

Arguing as in the proof of Theorem 3.1, we obtain (3.16).

Conversely, assume that (3.16) holds. Then by the symmetry hypothesis and the Kolmogorov three-series criterion,

$$\sum_{n=1}^{\infty} E\left(X_n I(|X_n| \le 1)\right) = \sum_{n=1}^{\infty} 0 = 0,$$

$$\sum_{n=1}^{\infty} P(|X_n| > 1) = \sum_{n=1}^{\infty} P\big(X_n^2 > 1\big) < \infty,$$

and

$$\sum_{n=1}^{\infty} \operatorname{Var}(X_n I(|X_n| \le 1)) = \sum_{n=1}^{\infty} E\left(X_n^2 I(X_n^2 \le 1)\right) < \infty.$$

Again applying the Kolmogorov three-series criterion we obtain (3.15).

Next, (3.17) and (3.18) each imply both (3.15) and (3.16) by the Kolmogorov 0-1 law and the established equivalence between (3.15) and (3.16). Thus, (3.17) and (3.18) each imply (3.19), (3.20), and (3.21) by the converse half of Theorem 3.1 applied to the sequence of random variables $\{X_n^2, n \ge 1\}$. □

**Remark 3.4.** The proof of the implication ((3.16) implies (3.15)) follows essentially the same argument as in the proof of Theorem 2.13.5 (i) of Stout [5, p. 107].

## 4. Some Interesting Examples/Counterexamples

The first example illustrates Theorem 3.1.

**Example 4.1.** Let $\{X_n, n \geq 1\}$ be a sequence of random variables such that for each $n \geq 1$, the distribution function of $|X_n|$ is

$$F_n(t) = \begin{cases} 0, & t < 0 \\ (n^2 - 1)t, & 0 \leq t < \dfrac{1}{n^2} \\ 1 - \dfrac{1}{n^2}, & \dfrac{1}{n^2} \leq t \leq n^2 \\ 1 - \dfrac{1}{t}, & t > n^2. \end{cases}$$

We note that for all $n \geq 1$,

$$E|X_n| = \int_0^\infty (1 - F_n(t))\, dt \geq \int_{n^2}^\infty \frac{1}{t}\, dt = \infty.$$

Now

$$\begin{aligned} \int_0^1 \sum_{n=1}^\infty P(|X_n| > t)\, dt &= \sum_{n=1}^\infty \int_0^1 P(|X_n| > t)\, dt \\ &= \sum_{n=1}^\infty \left( \int_0^{n^{-2}} (1 - (n^2 - 1)t)\, dt + \int_{n^{-2}}^1 \frac{1}{n^2}\, dt \right) \\ &= \sum_{n=1}^\infty \left( \frac{3}{2n^2} - \frac{1}{2n^4} \right) \\ &< \infty \end{aligned}$$

and so by Theorem 3.1 with $t_0 = 1$ the series $\sum_{n=1}^\infty X_n$ converges absolutely a.s. i.j.d. of the $\{X_n, n \geq 1\}$.

In the second example, we show that $\sum_{n=1}^\infty |X_n| < \infty$ a.s. can hold for a sequence of random variables $\{X_n, n \geq 1\}$ when the conditions (3.1), (3.2), and (3.3) of Theorem 3.1 are not satisfied.

**Example 4.2.** Let $\{A_n, n \geq 1\}$ be a sequence of events such that $P(A_n) = \frac{1}{n}$ and $A_n \downarrow \emptyset$. Let $\{Y_n, n \geq 1\}$ be a sequence of random variables such that $Y_n$ and $I(A_n)$ are independent for all $n \geq 1$ and

$$\alpha(t) \equiv \inf\{P(|Y_n| > t) : n \geq 1\} > 0 \text{ for all } t \in (0, \infty).$$

Set $X_n = Y_n I(A_n)$, $n \geq 1$. Now $P(A_n \text{ i.o.}(n)) = 0$ and so

$$P\left(\liminf_{n\to\infty}[X_n = 0]\right) = 1.$$

Thus $\sum_{n=1}^\infty |X_n| < \infty$ a.s.

Next, for all $t_0 \in (0, \infty)$ and $n \geq 1$,

$$\begin{aligned} P(|X_n| > t_0) &= P([Y_n > t_0] \cap [I(A_n) = 1]) \\ &= P(|Y_n| > t_0) \cdot P(A_n) \\ &\geq \frac{\alpha(t_0)}{n} \end{aligned}$$

and so

$$\sum_{n=1}^{\infty} P(|X_n| > t_0) = \infty.$$

Thus by Lemma 2.2 and Theorem 3.1, the conditions (3.1), (3.2), and (3.3) fail for all $t_0 \in (0, \infty)$.

**Remark 4.1.** Let $\{A_n, n \geq 1\}, \{Y_n, n \geq 1\}$, and $\{X_n, n \geq 1\}$ be as in Example 4.2 and suppose in addition that

$$|Y_n| > t_0 \text{ a.s.}, n \geq 1 \text{ for some } t_0 \in (0, \infty).$$

Then for all $n \geq 1$,

$$\begin{aligned} E(|X_n| I(|X_n| \leq t_0)) &= E(|Y_n| I(A_n) I(|X_n| \leq t_0)) \\ &= E(|Y_n| I(|Y_n| \leq t_0) I(A_n)) \\ &= E(0) \\ &= 0. \end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} E(|X_n| I(|X_n| \leq t_0)) < \infty$$

and so (2.2) holds. Thus, by Lemmas 2.2, 2.3, and 2.4, the failure of the conditions (3.1), (3.2), and (3.3) of Theorem 3.1 is due only to the failure of (2.1).

In the third example, we show that if

$$\sum_{n=1}^{\infty} P(|X_n| > t_n) < \infty$$

for some sequence $0 < t_n \to 0$, then $\sum_{n=1}^{\infty} |X_n| < \infty$ a.s. can fail and, *a fortiori*, the conditions (3.1), (3.2), and (3.3) of Theorem 3.1 do not hold. Thus Remark 3.1 is not valid if the condition $\sum_{n=1}^{\infty} t_n < \infty$ for some sequence $\{t_n, n \geq 1\}$ in $(0, \infty)$ is weakened to $0 < t_n \to 0$.

**Example 4.3.** Let $\{X_n, n \geq 1\}$ be a sequence of random variables with

$$P\left(X_n = \frac{1}{n}\right) = P\left(X_n = -\frac{1}{n}\right) = \frac{1}{2}, n \geq 1.$$

Let $t_n = \frac{1}{n}, n \geq 1$. Then $0 < t_n \to 0$, $\sum_{n=1}^{\infty} t_n = \infty$,

$$\sum_{n=1}^{\infty} P(|X_n| > t_n) = \sum_{n=1}^{\infty} 0 = 0,$$

but

$$\sum_{n=1}^{\infty} |X_n| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \text{ a.s.}$$

In the fourth example, we show apropos of Remark 3.1 that the conditions (3.8) and (3.9) are compatible with each other.

**Example 4.4.** Let $0 < \alpha < 1$ and let $\{Y_n, n \geq 1\}$ be a sequence of identically distributed random variables where $Y_1$ has probability density function

$$f(y) = \begin{cases} \frac{\alpha}{y^{\alpha+1}}, & y \geq 1 \\ 0, & y < 1. \end{cases} \tag{4.1}$$

Let $X_n = Y_n/n^p, n \geq 1$, where $p > (\alpha + 1)/\alpha$. We note that $EX_n = \infty, n \geq 1$. Let $t_n = 1/n^{1+\varepsilon}, n \geq 1$ where $0 < \varepsilon < p - (\alpha + 1)/\alpha$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} P(|X_n| > t_n) &= \sum_{n=1}^{\infty} P(Y_1 > n^{p-1-\varepsilon}) \\ &= \sum_{n=1}^{\infty} \int_{n^{p-1-\varepsilon}}^{\infty} \frac{\alpha}{y^{\alpha+1}}\, dy \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{p\alpha-\alpha-\varepsilon\alpha}} \\ &< \infty \end{aligned}$$

and

$$\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} < \infty.$$

Thus the conditions (3.8) and (3.9) hold for the sequence $\{t_n, n \geq 1\}$ and so by Remark 3.1 the series $\sum_{n=1}^{\infty} X_n$ converges absolutely a.s. i.j.d. of the $\{X_n, n \geq 1\}$.

In the fifth example, we show that the converse of Remark 3.1 is not valid. Specifically, we demonstrate in this example that the conditions (3.1), (3.2), and (3.3) can hold for a sequence of random variables $\{X_n, n \geq 1\}$ (hence, the series $\sum_{n=1}^{\infty} X_n$ converges absolutely a.s. i.j.d. of the $\{X_n, n \geq 1\}$ by Theorem 3.1) but there does not exist a sequence of positive constants $\{t_n, n \geq 1\}$ satisfying both (3.8) and (3.9).

**Example 4.5.** Let $0 < \alpha < 1$ and let $\{Y_n, n \geq 1\}$ be a sequence of identically distributed random variables where $Y_1$ has probability density function given in (4.1). Let $X_n = Y_n/n^p, n \geq 1$ where $p \in (1/\alpha, (\alpha + 1)/\alpha]$. We note that $EX_n = \infty, n \geq 1$. Let $t_0 = 1$. To verify that (3.1), (3.2), and (3.3) hold, it suffices to verify that (2.1) and (2.2) hold in view of Lemma 2.2 and Theorem 3.1.

Now

$$\begin{aligned}\sum_{n=1}^{\infty} P(|X_n| > t_0) &= \sum_{n=1}^{\infty} P(Y_1 > n^p) \\ &= \sum_{n=1}^{\infty} \int_{n^p}^{\infty} \frac{\alpha}{y^{\alpha+1}}\, dy \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{p\alpha}} \\ &< \infty\end{aligned}$$

since $p\alpha > 1$. Thus, (2.1) holds.

Next,

$$\begin{aligned}&\sum_{n=1}^{\infty} E(|X_n| I(|X_n| \le t_0)) \\ &\quad= \sum_{n=1}^{\infty} \frac{E(Y_1 I(Y_1 \le n^p))}{n^p} \\ &\quad= \sum_{n=1}^{\infty} \frac{\int_1^{n^p} y \frac{\alpha}{y^{\alpha+1}}\, dy}{n^p} \\ &\quad= \sum_{n=1}^{\infty} \frac{\alpha}{1-\alpha}\left(\frac{1}{n^{p\alpha}} - \frac{1}{n^p}\right) \\ &\quad< \infty\end{aligned}$$

since $0 < \alpha < 1 < p\alpha < p$. Thus, (2.2) holds.

Suppose there exists a sequence of positive constants $\{t_n, n \ge 1\}$ satisfying both (3.8) and (3.9). Now

$$\sum_{n=1}^{\infty} P(Y_1 > n^p t_n) = \sum_{n=1}^{\infty} P(|X_n| > t_n) < \infty$$

and so $n^p t_n \to \infty$. Thus, for some positive integer $n_0$,

$$\infty > \sum_{n=1}^{\infty} P(Y_1 > n^p t_n) \ge \sum_{n=n_0}^{\infty} \int_{n^p t_n}^{\infty} \frac{\alpha}{y^{\alpha+1}}\, dy = \sum_{n=n_0}^{\infty} \frac{1}{n^{p\alpha} t_n^{\alpha}}. \tag{4.2}$$

Let

$$A = \left\{n \ge 1 : t_n \le \frac{1}{n}\right\} \text{ and } B = \left\{n \ge 1 : t_n > \frac{1}{n}\right\}.$$

Then by (3.9)

$$\sum_{n\in B}\frac{1}{n} \le \sum_{n\in B} t_n \le \sum_{n=1}^{\infty} t_n < \infty$$

and since

$$\sum_{n\in A}\frac{1}{n} + \sum_{n\in B}\frac{1}{n} = \sum_{n=1}^{\infty}\frac{1}{n} = \infty,$$

we have

$$\sum_{n\in A}\frac{1}{n} = \infty. \tag{4.3}$$

Now by (4.2) and $\alpha(p-1) \le 1$,

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty}\frac{1}{n^{p\alpha}t_n^{\alpha}}\\ &\ge \sum_{n\in A}\frac{1}{n^{p\alpha}t_n^{\alpha}}\\ &\ge \sum_{n\in A}\frac{1}{n^{p\alpha}(1/n)^{\alpha}}\\ &= \sum_{n\in A}\frac{1}{n^{\alpha(p-1)}}\\ &\ge \sum_{n\in A}\frac{1}{n}\end{aligned}$$

contradicting (4.3). Thus, there does not exist a sequence of positive constants $\{t_n, n \ge 1\}$ satisfying both (3.8) and (3.9).

In the sixth example, we show that (3.15) does not necessarily imply (3.16) if the independent random variables $\{X_n, n \ge 1\}$ are not symmetric.

**Example 4.6.** Let $\{U_n, n \ge 1\}$ be a sequence of independent and identically distributed random variables with

$$P(U_1 = 1) = P(U_1 = -1) = \frac{1}{2}, n \ge 1.$$

Set

$$X_n = \frac{(-1)^n}{\sqrt{n}} + \frac{U_n}{n}, n \ge 1.$$

Now

$$\sum_{n=1}^{\infty}\frac{U_n}{n} \text{ converges a.s.}$$

by the Khintchine-Kolmogorov convergence theorem (see, e.g., [1, p. 113]) and since

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ converges,}$$

it follows that

$$\text{the series} \sum_{n=1}^{\infty} X_n \text{ converges a.s.}$$

Now $\{X_n^2, n \geq 1\}$ is a sequence of uniformly bounded independent random variables and since

$$\sum_{n=1}^{\infty} EX_n^2 = \sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{n^2}\right) = \infty,$$

it follows from Corollary 5.1.1 of Chow and Teicher [1, p. 116] that the series $\sum_{n=1}^{\infty} X_n^2$ does not converge a.s. whence, by Theorem 3.1 applied to the sequence of random variables $\{X_n^2, n \geq 1\}$, we conclude that (3.19), (3.20), and (3.21) all fail for all $t_0 \in (0, \infty)$.

In the seventh and final example, we show that (3.16) does not necessarily imply (3.15) if the independent random variables $\{X_n, n \geq 1\}$ are not symmetric.

**Example 4.7.** Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with

$$P\left(X_n = \frac{1}{n}\right) = 1 - P\left(X_n = -\frac{1}{n}\right) = p, n \geq 1$$

where $\frac{1}{2} < p < 1$. Then

$$\sum_{n=1}^{\infty} X_n^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \text{ a.s.}$$

Now $\{X_n, n \geq 1\}$ is a sequence of uniformly bounded independent random variables and since

$$\sum_{n=1}^{\infty} EX_n = \sum_{n=1}^{\infty} \frac{2p-1}{n} = \infty,$$

it follows from Corollary 5.1.1 of Chow and Teicher [1, p. 116] that the series $\sum_{n=1}^{\infty} X_n$ does not converge a.s. Thus, the series $\sum_{n=1}^{\infty} X_n$ does not converge absolutely a.s. whence by Theorem 3.1, we conclude that (3.1), (3.2), and (3.3) all fail for all $t_0 \in (0, \infty)$.

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