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Strong Limit Theorems for Weighted Sums of Random Elements in Banach Spaces

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Abstract—We propose an approach to the strong law of large numbers for weighted sums of random elements, where the result of Jajte [4] will be extended to the Banach space setting. Some typical applications of the main results are given.

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1. INTRODUCTION

As is well known that the Kolmogorov strong law of large numbers (SLLN) and Marcinkiewicz–Zygmund SLLN play important roles in probability limit theory and mathematical statistics, which have been studied by many authors. It is more interesting to consider a general case.

Jajte [4] studied a large class of summability methods defined as follows: it is said that a sequence $\{X_n, n \geqslant 1\}$ of random variables is almost surely (a.s.) summable to random variable X by the method (h, g) if

$$\frac{1}{g(n)} \sum_{k=1}^{n} \frac{X_k}{h(k)} \to 0 \quad \text{a.s.} \quad \text{as} \quad n \to \infty. \tag{1}$$

Note that the SLLN of the form (1) embraces the Kolmogorov SLLN ($g(n) = n, h(n) = 1$) and the Marcinkiewicz–Zygmund SLLN ($g(n) = n^{1/r}, h(n) = 1, 1 < r < 2$).

For a sequence $\{X_n, n \geqslant 1\}$ of independent and identically distributed (i.i.d.) random variables, Jajte [4] proved that $\{X_n - \mathbb{E}(X_n \mathbb{I}(|X_n| \leqslant \phi(n))), n \geqslant 1\}$ is almost surely summable to 0 by the method (h, g) if and only if $\mathbb{E}(\phi^{-1}(|X|)) < \infty$, where ϕ^{-1} is the inverse of ϕ, and ϕ, g, h are functions satisfying the conditions of the following hypothesis.

Hypothesis A. Let g be a positive, increasing function with $\lim_{x\to\infty} g(x) = \infty$ and h a positive function such that $\phi(x) = g(x)h(x)$ satisfies the following conditions:

(i) For some $d \geqslant 0$, ϕ is strictly increasing on $[d, \infty)$ with range $[0, \infty)$;

(ii) There exist c and a positive integer k_0 such that $\frac{\phi(x+1)}{\phi(x)} \leqslant c, \quad x \geqslant k_0$;

(iii) There exist constants a and b such that

$$\phi^2(s) \int_s^\infty \frac{1}{\phi^2(x)} dx \leqslant as + b, \quad s > d. \tag{2}$$

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Inspired by Jajte [4], Jing and Liang [5] and Wang [17] extended the result of Jajte [4] to negatively associated random variables. The result of Jajte [4] was extended to $\tilde{\rho}$-mixing random variables by Meng and Lin [9] and to the random field setting by Lagodowski and Matuła [7]. Sung [16] gave some sufficient conditions to prove the SLLN for weighted sums of random variables. Recently, the result of Jajte [4] was studied by Miao et al. [10] and Son et al. [15] for martingale differences and negatively superadditive dependent random vectors in Hilbert spaces, respectively. The proofs of the above results are based on the Kolmogorov convergence criterion or the Kolmogorov three series theorem.

The main purpose of the present paper is to extend the result of Jajte [4] to the case of random elements in Banach spaces. We provide necessary and sufficient conditions so that the SLLN for weighted sums would hold for an arbitrary sequence of random elements without imposing any geometric condition on the Banach space. Some typical applications of the main results are given.

Throughout the paper, the symbol C will denote a generic positive constant which is not necessarily the same one in each appearance. $\mathbb{I}(A)$ denotes the indicator function of the event A. The definition of stochastic domination will be used in the paper as follows.

A sequence $\{X_n, n \geqslant 1\}$ of random elements is said to be strongly stochastically dominated by a random element X if there exist positive constants c_1 and c_2 such that

$$c_1\mathbb{P}(||X|| > t) \leqslant \mathbb{P}(||X_n|| > t) \leqslant c_2\mathbb{P}(||X|| > t) \quad \text{for all} \quad t \geqslant 0, \quad n \geqslant 1. \tag{3}$$

If only the right-hand side of (3) is satisfied, then the sequence $\{X_n, n \geqslant 1\}$ is said to be stochastically dominated by X. Note that (3) is, of course, automatic with $X = X_1$ and $c_1 = c_2 = 1$ if $\{X_n, n \geqslant 1\}$ is a sequence of identically distributed random elements.

2. MAIN RESULTS

Based on Hypothesis A, we will state the main results under the following hypothesis, where condition (2) of Jajte [4] is replaced by condition (4).

Hypothesis B. Let $p \geqslant 1$, let g be a positive, increasing function with $\lim_{x\to\infty} g(x) = \infty$ and h a positive function such that $\phi(x) = g(x)h(x)$ satisfies the following conditions:

(i) For some $d \geqslant 0$, ϕ is strictly increasing on $[d, \infty)$ with range $[0, \infty)$;

(ii) There exist c and a positive integer k_0 such that $\frac{\phi(x+1)}{\phi(x)} \leqslant c, \quad x \geqslant k_0$;

(iii) There exist constants a and b such that

$$\phi^p(s)\int_s^\infty \frac{1}{\phi^p(x)}dx \leqslant as + b, \quad s > d. \tag{4}$$

Theorem 2.1. *Let $p \geqslant 1$, let ϕ be a function satisfying the conditions of Hypothesis B, and let $\{X_n, n \geqslant 1\}$ be a sequence of random elements in a real separable Banach space, which is strongly stochastically dominated by a random element X. For $n \geqslant 1$, set*

$$m_n = \mathbb{E}(X_n\mathbb{I}(||X_n|| \leqslant \phi(n))); \quad Y_n = \frac{X_n\mathbb{I}(||X_n|| \leqslant \phi(n))}{\phi(n)}; \quad S_n = \sum_{k=1}^n (Y_k - \mathbb{E}Y_k).$$

(i) If

$$\sum_{k=n}^\infty \mathbb{E}||Y_k - \mathbb{E}Y_k||^p \to 0 \quad \text{as} \quad n \to \infty \tag{5}$$

implies $\{S_n, n \geqslant 1\}$ converges a.s., then the condition

$$\mathbb{E}(\phi^{-1}(||X||)) < \infty \tag{6}$$

implies

$$\sum_{n=1}^\infty \frac{X_n - m_n}{\phi(n)} \quad \text{converges a.s.} \tag{7}$$

(ii) If $X_n/\phi(n) \to 0$ a.s. implies $X_n/\phi(n) \to 0$ c.c., then (7) implies (6).

Proof. (i) Assume that (6) holds. Note that

$$\sum_{n=1}^{\infty} \mathbb{E}||Y_n - \mathbb{E}Y_n||^p \leqslant C \sum_{n=1}^{\infty} \frac{\mathbb{E}||X_n \mathbb{I}(||X_n|| \leqslant \phi(n))||^p}{\phi^p(n)}$$

$$\leqslant C \sum_{n=1}^{\infty} \frac{\mathbb{E}||X\mathbb{I}(|X| \leqslant \phi(n))||^p}{\phi^p(n)} + C \sum_{n=1}^{\infty} \mathbb{P}(||X|| > \phi(n)). \tag{8}$$

Then by (6) and the conditions of Hypothesis B, we have

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}||X\mathbb{I}(|X| \leqslant \phi(n))||^p}{\phi^p(n)} = \mathbb{E}\left(\sum_{n=1}^{\infty} \frac{||X||^p}{\phi^p(n)} \mathbb{I}(||X|| \leqslant \phi(n))\right)$$

$$= C + \mathbb{E}\left(\sum_{n=k_0}^{\infty} \frac{||X||^p}{\phi^p(n)} \mathbb{I}(||X|| \leqslant \phi(n))\right) \quad (k_0 > d)$$

$$\leqslant C + C\mathbb{E}\left(||X||^p \sum_{n=k_0}^{\infty} \int_n^{n+1} \frac{\mathbb{I}(||X|| \leqslant \phi(n))}{\phi^p(n+1)} dx\right)$$

$$\leqslant C + C\mathbb{E}\left(||X||^p \int_{k_0}^{\infty} \frac{\mathbb{I}(\phi^{-1}(||X||) \leqslant x)}{\phi^p(x)} dx\right) \leqslant C + C\mathbb{E}\left(a\phi^{-1}(||X||) + b\right) < \infty,$$

and

$$\sum_{n=1}^{\infty} \mathbb{P}\left(||X|| > \phi(n)\right) < \infty. \tag{9}$$

Combining (8) and (9) yields (5). So that

$$\sum_{n=1}^{\infty} (Y_n - \mathbb{E}Y_n) \quad \text{converges a.s.} \tag{10}$$

Now by using (9) again, we get

$$\sum_{n=1}^{\infty} \mathbb{P}\left(||X_n|| > \phi(n)\right) \leqslant C \sum_{n=1}^{\infty} \mathbb{P}\left(||X|| > \phi(n)\right) < \infty.$$

Hence, by the Borel–Cantelli Lemma,

$$\mathbb{P}\left(\limsup(||X_n|| > \phi(n))\right) = 0. \tag{11}$$

This implies that $Y_n = X_n/\phi(n)$ for all sufficiently large n with probability one. Therefore, (7) follows immediately from (10).

(ii) Now assume that (7) holds. We have

$$0 \leqslant \frac{||m_n||}{\phi(n)} \leqslant \frac{\mathbb{E}(||X_n||\mathbb{I}(||X_n|| \leqslant \phi(n)))}{\phi(n)} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore (7) implies $X_n/\phi(n) \to 0$ a.s., and so $X_n/\phi(n) \to 0$ c.c. This ensures that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(||X_n|| \geqslant \phi(n)\right) < \infty.$$

Then we have

$$\mathbb{E}(\phi^{-1}(||X||)) \leqslant C + \sum_{n=1}^{\infty} \mathbb{P}\left(||X|| > \phi(n)\right) \leqslant C + C \sum_{n=1}^{\infty} \mathbb{P}\left(||X_n|| > \phi(n)\right) < \infty.$$

This ends the proof of theorem. □

Remark 2.2. Let $p \geqslant 1$, let ϕ be a function satisfying the conditions of Hypothesis B, and let $\{X_n, \mathcal{F}_n, n \geqslant 1\}$ be an adapted sequence in a real separable Banach space. Assume that $\{X_n, n \geqslant 1\}$ is stochastically dominated by a random element X. We set

$$m_n = \mathbb{E}\left(X_n \mathbb{I}(||X_n|| \leqslant \phi(n)) | \mathcal{F}_{n-1}\right);$$

$$Y_n = \frac{X_n \mathbb{I}(||X_n|| \leqslant \phi(n))}{\phi(n)}; \quad S_n = \sum_{k=1}^{n} \left(Y_k - \mathbb{E}(Y_k | \mathcal{F}_{k-1})\right), \quad n \geqslant 1.$$

Then, by the similar arguments as above, we can show that statement (i) of Theorem 2.1 holds if (5) is replaced by the following condition

$$\sum_{k=n}^{\infty} \mathbb{E}||Y_k - \mathbb{E}(Y_k|\mathcal{F}_{k-1})||^p \to 0 \quad \text{as} \quad n \to \infty.$$

Theorem 2.3. *Let $p \geqslant 1$, let ϕ, g, h be functions satisfying the conditions of Hypothesis B, and let $\{X_n, n \geqslant 1\}$ be a sequence of mean zero random elements in a real separable Banach space, which is strongly stochastically dominated by a random element X. For $n \geqslant 1$, set*

$$Y_n = \frac{X_n \mathbb{I}(||X_n|| \leqslant \phi(n))}{\phi(n)}; \quad S_n = \sum_{k=1}^{n} (Y_k - \mathbb{E}Y_k).$$

(i) If (5) implies $\{S_n, n \geqslant 1\}$ converges a.s., then condition (6) implies

$$\frac{1}{g(n)} \sum_{k=1}^{n} \frac{X_k}{h(k)} \to 0 \quad a.s. \quad as \quad n \to \infty. \tag{12}$$

(ii) If $X_n/\phi(n) \to 0$ a.s. implies $X_n/\phi(n) \to 0$ c.c., then (12) implies (6).

Proof. (i) Assume that (6) holds. Then by using (11), we have

$$\sum_{n=1}^{\infty} \left|\left|\mathbb{E}\left(\frac{X_n}{\phi(n)} \mathbb{I}(||X_n|| \leqslant \phi(n))\right)\right|\right| = \sum_{n=1}^{\infty} \left|\left|\mathbb{E}\left(\frac{X_n}{\phi(n)} \mathbb{I}(||X_n|| > \phi(n))\right)\right|\right| < \infty.$$

It follows from this and (7) that

$$\sum_{n=1}^{\infty} \frac{X_n}{\phi(n)} \quad \text{converges a.s.} \tag{13}$$

Then by the Kronecker lemma, we obtain (12).

(ii) We now assume that (12) holds. Set

$$\sigma_n = \frac{1}{g(n)} \sum_{k=1}^{n} \frac{X_k}{h(k)} = \frac{1}{g(n)} \left(\sum_{k=1}^{n-1} \frac{X_k}{h(k)} + \frac{X_n}{h(n)} \right).$$

Then we have $\frac{X_n}{\phi(n)} = \sigma_n - \frac{g(n-1)}{g(n)} \sigma_{n-1}$. This and (12) ensure that $X_n/\phi(n) \to 0$ a.s. Repeating the arguments given in the end of the proof of Theorem 2.1 shows that (6) holds. □

Remark 2.4. According to the proof of Theorem 2.3, it is easy to see that this result holds if (12) is replaced by (13). Moreover, we can also show that Theorem 2.1 still holds if (7) is replaced by the following:

$$\frac{1}{g(n)} \sum_{k=1}^{n} \frac{X_k - m_k}{h(k)} \to 0 \quad \text{a.s.} \quad \text{as} \quad n \to \infty. \tag{14}$$

3. APPLICATIONS

We observe that the techniques used in the proof of the main results are relatively simple. However, these results can be applied to many classes of dependent random sequences. In this section, we will focus on three typical applications. The first is apparently a new result about the SLLN for weighted sums of independent random elements in Rademacher type p Banach spaces. Before stating Theorem 3.2, we recall the concept of Rademacher type p Banach space.

Let $\mathbf{E}$ be a Banach space, let $\{Y_n, n \geqslant 1\}$ be a symmetric Bernoulli sequence; that is, $\{Y_n, n \geqslant 1\}$ is a sequence of i.i.d. random variables with $\mathbb{P}(Y_1 = \pm 1) = 1/2$. Let $\mathbf{E}^\infty = \mathbf{E} \times \mathbf{E} \times \mathbf{E} \times \dots$ and define

$$\mathcal{C}(\mathbf{E}) = \Big\{(v_1, v_2, \dots) \in \mathbf{E}^\infty : \sum_{n=1}^{\infty} Y_n v_n \quad \text{converges in probability}\Big\}.$$

Then $\mathbf{E}$ is said to be of *Rademacher type* p $(1 \leqslant p \leqslant 2)$ if there exists a positive constant C such that

$$\mathbb{E}\left|\left|\sum_{n=1}^{\infty} Y_n v_n\right|\right|^p \leqslant C \sum_{n=1}^{\infty} ||v_n||^p \quad \text{for all} \quad (v_1, v_2, \dots) \in \mathcal{C}(\mathbf{E}).$$

It is well known that if a real separable Banach space is of Rademacher type p for some $1 < p \leqslant 2$, then it is of Rademacher type q for all $1 \leqslant q < p$. Every real separable Banach space is of Rademacher type (at least) 1, while the $\mathcal{L}^p$-spaces and ℓ^p-spaces are of Rademacher type $\min\{2; p\}$ for $p \geqslant 1$. Every real separable Hilbert space and real separable finite-dimensional Banach space is of Rademacher type 2. In particular, the real line $\mathbb{R}$ is of Rademacher type 2.

Lemma 3.1 ([14], Lemma 2.1). *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent mean zero random elements in a real separable Rademacher type p $(1 \leqslant p \leqslant 2)$ Banach space. Then*

$$\mathbb{E}\left(\max_{1\leqslant k\leqslant n}\left|\left|\sum_{l=1}^{k} X_l\right|\right|^p\right) \leqslant C \sum_{k=1}^{n} \mathbb{E}||X_k||^p, \quad n \geqslant 1,$$

where the constant C is independent of n.

Theorem 3.2. *Let $1 \leqslant p \leqslant 2$, let ϕ, g, h be functions satisfying the conditions of Hypothesis B, and let $\{X_n, n \geqslant 1\}$ be a sequence of independent random elements in a real separable Rademacher type p Banach space. Assume that $\{X_n, n \geqslant 1\}$ is strongly stochastically dominated by a random element X. Set $m_n = \mathbb{E}(X_n \mathbb{I}(||X_n|| \leqslant \phi(n)))$, $n \geqslant 1$. Then (6), (7), and (14) are equivalent to each other. If we further assume that $\mathbb{E}X_n = 0$ for all $n \geqslant 1$, then (6), (12), and (13) are equivalent to each other.*

Proof. For $n \geqslant 1$, set

$$Y_n = \frac{X_n \mathbb{I}(||X_n|| \leqslant \phi(n))}{\phi(n)}; \quad S_n = \sum_{k=1}^{n} (Y_k - \mathbb{E}Y_k).$$

We assume that (5) holds. Then by the Markov inequality and Lemma 3.1,

$$\mathbb{P}\left(\sup_{k\geqslant n} ||S_k - S_n|| > \varepsilon\right) = \mathbb{P}\left(\bigcup_{m\geqslant n} (\max_{n\leqslant k\leqslant m} ||S_k - S_n|| > \varepsilon)\right)$$
$$= \lim_{m\to\infty} \mathbb{P}(\max_{n\leqslant k\leqslant m} ||S_k - S_n|| > \varepsilon) \leqslant C \lim_{m\to\infty} \mathbb{E}(\max_{n\leqslant k\leqslant m} ||S_k - S_n||)^p$$
$$\leqslant C \sum_{k=n}^{\infty} \mathbb{E}||Y_k - \mathbb{E}Y_k||^p \to 0 \quad \text{as} \quad n \to \infty.$$

This implies that $\{S_n, n \geqslant 1\}$ converges a.s.

On the other hand, since the sequence $\{X_n, n \geqslant 1\}$ is independent, $X_n/\phi(n) \to 0$ a.s. implies $X_n/\phi(n) \to 0$ c.c.

From the above arguments, Theorems 2.1, 2.3, and Remark 2.4 we finish the proof of theorem. □

Remark 3.3. Let us note that the real line $\mathbb{R}$ is of Rademacher type p for all $1 \leqslant p \leqslant 2$. Therefore, if $\{X_n, n \geqslant 1\}$ is a sequence of i.i.d. (real-valued) random variables, then Theorem 3.2 implies the main result of Jajte [4], where condition (2) is replaced by condition (4) for some $1 \leqslant p \leqslant 2$.

From Remark 2.2, by using Lemma 2.2. of Hu et al. [3] and some similar arguments as in the proof of Theorem 3.2, we can obtain the next theorem. This result establishes the SLLN for weighted sums of martingale differences in martingale type p Banach spaces.

A real separable Banach space $\mathbf{E}$ is said to be of martingale type p $(1 \leqslant p \leqslant 2)$ if there exists a positive constant C such that for all martingales $\{X_n, \mathcal{F}_n, n \geqslant 1\}$ with values in $\mathbf{E}$,

$$\sup_{n\geqslant 1} \mathbb{E}||X_n||^p \leqslant C \sum_{n=1}^{\infty} \mathbb{E}||X_n - X_{n-1}||^p,$$

where $X_0 = 0$. It follows from Hoffmann–Jørgensen and Pisier [2] characterization of Rademacher type p Banach spaces that if a Banach space is of martingale type p, then it is of Rademacher type p. But the notion of martingale type p is only superficially similar to that of Rademacher type p and has a geometric characterization in terms of smoothness. For more details, the reader may refer to Pisier [11, 12].

Theorem 3.4. *Let $1 \leqslant p \leqslant 2$, let ϕ, g, h be functions satisfying the conditions of Hypothesis B, and let $\{X_n, \mathcal{F}_n, n \geqslant 1\}$ be a martingale difference sequence in a martingale type p Banach space. Assume that $\{X_n, n \geqslant 1\}$ is stochastically dominated by a random element X. Set*

$$m_n = \mathbb{E}\left(X_n \mathbb{I}(||X_n|| \leqslant \phi(n)) | \mathcal{F}_{n-1}\right);$$

$$Y_n = \frac{X_n \mathbb{I}(||X_n|| \leqslant \phi(n))}{\phi(n)}; \quad S_n = \sum_{k=1}^{n} \left(Y_k - \mathbb{E}(Y_k|\mathcal{F}_{k-1})\right), \quad n \geqslant 1.$$

Then the condition $\mathbb{E}(\phi^{-1}(||X||)) < \infty$ implies

$$\frac{1}{g(n)} \sum_{k=1}^{n} \frac{X_k - m_k}{h(k)} \to 0 \quad \text{a.s.} \quad \text{as} \quad n \to \infty.$$

Remark 3.5. In the special case when $\{X, X_n, \mathcal{F}_n, n \geqslant 1\}$ is a real-valued, identically distributed martingale differences and $p = 2$, from Theorem 3.4 we get Theorem 2.1 of Miao et al. [10]. Note that there is a typo in the result of Miao et al. [10]. The condition $\lim_{x\to\infty} g(x) = \infty$ is missing, this is used to apply the Kronecker lemma in their proof.

In the next theorem of the paper, we provide an application of the main results for the case of negative association. This theorem generalizes Theorem 2.3 of Jing and Liang [5]. Note that the usual truncation technique preserves the independence property. However, this technique does not preserve the negative association property. In order to state and prove Theorem 3.8, we recall the concept of negatively associated random variables and give the following two lemmas.

A finite family $\{X_k, 1 \leqslant k \leqslant n\}$ of random variables is said to be *negatively associated* if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, ..., n\}$,

$$\mathrm{Cov}\left(f_1(X_k, k \in A_1), f_2(X_l, l \in A_2)\right) \leqslant 0$$

whenever f_1 and f_2 are coordinatewise nondecreasing and the covariance exists. An infinite family of random variables is *negatively associated* if every finite subfamily is negatively associated. This concept was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [6].

The following lemma was proved by Shao in [13, Theorem 2].

Lemma 3.6. *Let $1 \leqslant p \leqslant 2$, and let $\{X_n, n \geqslant 1\}$ be a sequence of negatively associated mean zero random variables. Then*

$$\mathbb{E}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{l=1}^{k} X_l\right|^p\right) \leqslant C \sum_{k=1}^{n} \mathbb{E}|X_k|^p, \quad n \geqslant 1,$$

where the constant C is independent of n.

The next lemma follows immediately from Lemma 1 of Matuła [8]. We can use this lemma to show the relationship between almost sure convergence and complete convergence for some kinds of assumption on the dependence structure: independence, pairwise independence, negative association, and pairwise negative quadrant dependence.

Lemma 3.7 (A zero-one law). *Suppose that the sequence of events $\{A_n, n \geqslant 1\}$ satisfies*

$$\mathbb{P}(A_m \cap A_n) \leqslant \mathbb{P}(A_m)\mathbb{P}(A_n) \quad \text{for all} \quad m \neq n.$$

Then

$$\mathbb{P}\left(\limsup A_n\right) = \begin{cases} 0, & \text{when} \quad \sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty; \\ 1, & \text{when} \quad \sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty. \end{cases}$$

Theorem 3.8. *Let $1 \leqslant p \leqslant 2$, let ϕ, g, h be functions satisfying the conditions of Hypothesis B, and let $\{X_n, n \geqslant 1\}$ be a sequence of negatively associated random variables. Assume that $\{X_n, n \geqslant 1\}$ is strongly stochastically dominated by a random variables X. Set $m_n = \mathbb{E}(X_n \mathbb{I}(|X_n| \leqslant \phi(n)))$, $n \geqslant 1$. Then statements (7) and (14) are equivalent to*

$$\mathbb{E}(\phi^{-1}(|X|)) < \infty. \tag{15}$$

If we further assume that $\mathbb{E}X_n = 0$ for all $n \geqslant 1$, then (12), (13), and (15) are equivalent to each other.

Proof. Set

$$Y_n = \frac{X_n}{\phi(n)}\mathbb{I}(|X_n| \leqslant \phi(n)); \quad S_n = \sum_{k=1}^{n}(Y_k - \mathbb{E}Y_k);$$

$$Y_n^* = Y_n + \mathbb{I}(X_n > \phi(n)) - \mathbb{I}(X_n < -\phi(n)); \quad S_n^* = \sum_{k=1}^{n}(Y_k^* - \mathbb{E}Y_k^*).$$

We assume that

$$\sum_{k=n}^{\infty} \mathbb{E}|Y_k - \mathbb{E}Y_k|^p \to 0 \quad \text{as} \quad n \to \infty. \tag{16}$$

Then by Lemma 3.6, for all $\varepsilon > 0$ and $m \geqslant n \geqslant 1$, we have

$$\begin{aligned}
&\mathbb{P}(\max_{n \leqslant k \leqslant m} |S_k - S_n| > \varepsilon) \\
&\leqslant \mathbb{P}(\max_{n \leqslant k \leqslant m} |S_k^* - S_n^*| > \varepsilon/2) + \mathbb{P}\left(\sum_{k=n}^{m}\left(\mathbb{I}(|X_k| > \phi(k)) + \mathbb{P}(|X_k| > \phi(k))\right) > \varepsilon/2\right) \\
&\leqslant \mathbb{P}(\max_{n \leqslant k \leqslant m} |S_k^* - S_n^*| > \varepsilon/2) + C\sum_{k=n}^{m}\mathbb{P}(|X_k| > \phi(k)) \\
&\leqslant C\mathbb{E}(\max_{n \leqslant k \leqslant m} |S_k^* - S_n^*|)^p + C\sum_{k=n}^{m}\mathbb{P}(|X_k| > \phi(k)) \\
&\leqslant C\sum_{k=n}^{m}\mathbb{E}|Y_k^* - \mathbb{E}Y_n^*|^p + C\sum_{k=n}^{m}\mathbb{P}(|X_k| > \phi(k)) \\
&\leqslant C\sum_{k=n}^{m}\mathbb{E}\left(|Y_k - \mathbb{E}Y_n| + \mathbb{I}(|X_k| > \phi(k)) + \mathbb{P}(|X_k| > \phi(k))\right)^p + C\sum_{k=n}^{m}\mathbb{P}(|X_k| > \phi(k)) \\
&\leqslant C\sum_{k=n}^{m}\mathbb{E}|Y_k - \mathbb{E}Y_n|^p + C\sum_{k=n}^{m}\mathbb{P}(|X_k| > \phi(k)).
\end{aligned}$$

Therefore, if (15) holds, then condition (16) implies $\{S_n, n \geqslant 1\}$ converges a.s.

By Theorems 2.1, 2.3, and Remark 2.4, it is sufficient to prove that $X_n/\phi(n) \to 0$ a.s. implies $X_n/\phi(n) \to 0$ c.c. Indeed, assume that $X_n/\phi(n) \to 0$ a.s. Hence, $X_n^{\pm}/\phi(n) \to 0$ a.s. Since

$\{X_n^{\pm}/\phi(n), n \geqslant 1\}$ is still a negatively associated sequence, it follows from Lemma 3.7 that $X_n^{\pm}/\phi(n) \to 0$ c.c. Hence

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\frac{X_n}{\phi(n)}\right| \geqslant \varepsilon\right) \leqslant \sum_{n=1}^{\infty} \mathbb{P}\left(\frac{X_n^+}{\phi(n)} \geqslant \frac{\varepsilon}{2}\right) + \sum_{n=1}^{\infty} \mathbb{P}\left(\frac{X_n^-}{\phi(n)} \geqslant \frac{\varepsilon}{2}\right) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

This implies that $X_n/\phi(n) \to 0$ c.c. as $n \to \infty$. □

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