ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2020, Vol. 41, No. 4, pp. 686–702. © Pleiades Publishing, Ltd., 2020.

Comparison of Accuracy Properties of Point Estimators for the Ratio of Binomial Proportions with the Inverse-Direct Sampling Scheme

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(Submitted by A. M. Elizarov)

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Received October 11, 2019; revised December 12, 2019; accepted December 23, 2019

Abstract—We continue our investigation into the estimation of the ratio of Binomial proportions. We concentrate on point estimation and its accuracy properties. A problem of the point estimation for a ratio of two proportions using data from two independent Bernoulli samples is considered. In this article we mostly discuss the case when the first sample is obtained using the Inverse sampling scheme and the second one using the Direct Binomial sampling scheme. Our goal is to show that the normal approximations, which are relatively simple, for estimates of the ratio are reliable for the construction of point estimators with reliable accuracy properties. The main criterion of our judgment is the bias and mean squared error. The main accuracy characteristics of estimators corresponding to all possible combinations of sampling schemes are investigated by the Monte-Carlo method. Mean values and mean squared errors of point estimators are collected in tables, and some recommendations for the application of each estimators are presented.

DOI: 10.1134/S1995080220040198

Keywords and phrases: *point estimators, ratio of binomial proportions, inverse binomial sampling, direct binomial sampling, normal asymptotic of an estimator.*

1. INTRODUCTION

The ratio of two binomial proportions arises in prospective studies, biological experiments, and the comparison of manufacturing processes for quality control in industry. Statistical procedures for the ratio of binomial proportions (often called the relative risk) are also quite common in clinical trials, epidemiological studies, and the pharmaceutical setting. In epidemiological problems, such as cohort studies in two groups, the risk ratio or odds ratio, is related to vaccine efficiency and attributable risk.

In various public health applications, people are interested in knowing whether or not certain pollutants in the air increase the chances of a disease by two fold. In clinical trials, one of the main goals is to test whether a new drug performs better than an existing drug (or placebo) for curing a certain disease. The problem can be set up as investigating whether the new drug has a cure rate of 1.5 times the existing drug (or placebo). Conversely, if we are evaluating the disease incidence rate (more appropriate

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in vaccine trials), then we are interested in testing whether or not the new treatment reduces the chances of occurrence of the disease by a specify percentage (50% for example).

In this article we continue our investigation into the estimation of the ratio of Binomial proportions started in Ngamkham et al (2016) [1], Ngamkham (2018) [2], and Ngamkham (2020) [3]. The literature on the estimation of the ratio of Binomial proportions is very extensive and this problem has attracted an attention of statisticians for more than 70 years. We refer readers to the extensive list of references given in Ngamkham et al (2016) [1].

The general problem of estimating the ratio of Binomial proportions can be formulated in the following way. Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be two independent Bernoulli sequences with probabilities of success p_1 and p_2, respectively. The observations are done according to the sequential sampling schemes with Markov stopping times ν_1 and ν_2. From the results of observations $X^{(\nu_1)} = (X_1, \ldots, X_{\nu_1})$ and $Y^{\nu_2} = (Y_1, \ldots Y_{\nu_2})$, it is necessary to identify the sampling scheme and corresponding statistic for the estimator of the ratio $\theta = p_1/p_2$. We now remind readers the definitions of the Direct and Inverse Binomial sample schemes. For details we refer to Ngamkham et al (2016) [1] and Ngamkham (2018) [2].

Direct Binomial sampling: a random vector $X^{(n)} = (X_1, \ldots, X_n)$ with Bernoulli components and fixed number of observations n is observed. In the case of direct Binomial sampling, the statistic $\overline{X}_n = \frac{T}{n}$, where $T = \sum_{k=1}^{n} X_k$ is asymptotically normal with the mean p and variance $p(1-p)/n$. There is no unbiased estimate for the parametric function p^{-1}. An estimate of p^{-1} with an exponentially decreasing rate of bias is $\frac{n+1}{n\overline{X}_n+1}$.

Inverse Binomial sampling: a Bernoulli sequence $Y^{(\nu)} = (Y_1, \ldots, Y_\nu)$ is observed with a stopping time $\nu = \min\{n : \sum_{k=1}^{n} Y_k \geq m\}$. That is, the components of the sequence $Y_1, Y_2, \ldots$ are observed until the given number m of successes appears. In the case of Inverse Binomial sampling the statistic $\overline{Y}_m = \frac{\nu}{m}$ is asymptotically normal with the mean $1/p$ and variance $(1-p)/mp^2$. Statistic $\frac{m-1}{\nu-1}$ is an unbiased estimate of p.

In the following, keep the notation $X_1, X_2, \ldots$ for a Bernoulli sequence obtained by the direct sampling scheme and $Y_1, Y_2, \ldots$ for a Bernoulli sequence obtained by the inverse sampling scheme.

In this article, we consider the Inverse-Direct sampling scheme, where the first sample is obtained by the Inverse Binomial sampling scheme with the probability of success p_1 and stopping time that is defined by the fixed number of success m. The second sample is obtained by the Direct Binomial sampling scheme with the probability of success p_2 and a fixed sample size n.

Ngamkham (2020) [3] mentions that the Inverse-Direct sample scheme estimator performs the worst compared to other sampling schemes. It appears that the Mean Squared Error (MSE) for the Inverse-Direct sample scheme estimator is approximately ten times larger than the MSE for the Special Case Direct-Inverse estimator. However, there are two Special Cases of the Inverse-Direct sampling schemes that were not considered in Ngamkham (2020) [3]. Both Inverse-Direct Special Cases were considered, albeit briefly, and without any actual derivation of formulae, in Ngamkham et al (2016) [1] and Ngamkham (2018) [2]. All these issues are discussed in the current article.

In this article we concentrate most on the Special case of the Inverse-Direct sampling scheme where the first sample is obtained by the Inverse sampling scheme and the second sample is obtained by the Direct sampling scheme where the number of trials n is the same as the number of observations in the first experiment.

The main statistical tool that we are using in the article is the famous Delta method. It can be explained briefly in the following way; for details we refer interested reader to any advanced textbook on Mathematical Statistics, for example Lehmann (2004) [4].

Let $g(s)$ be a differentiable scalar function. Let the estimator $\mathfrak{T} = g(S)$ be a function of the basic statistic S. Statistic S usually has a simple form and is asymptotically normal. The asymptotic distribution of an estimator $\mathfrak{T}$ can be found by using the Delta method, a procedure of stochastic representation of $\mathfrak{T}$ with a desired accuracy.

In the Delta method, the Taylor expansion is used for the function g at the point $\mu = \mathrm{E}S$:

$$g(S) = g(\mu) + \frac{dg(\mu)}{ds}(S-\mu) + \text{Reminder}.$$

Since the Reminder converges to zero in probability as the sample size n increases to infinity sufficiently fast, we have that $\sqrt{n}[g(S) - g(\mu)]$ is asymptotically normal with zero mean and variance $\left(\frac{dg(\mu)}{ds}\right)^2 \mathrm{Var}(S)$. Therefore, the statistic $\mathfrak{T}$ is asymptotically normal with a mean $g(\mu)$ and variance of the form Const$/n$.

Now we remind readers of point estimators for the ratio of Binomial proportions for the Special case of the Inverse-Direct sampling scheme. There are two such estimators, and both appear in the following framework.

Special cases of the Inverse-Direct sampling scheme. The first sample is obtained by the Inverse sampling method with parameters (m, p_1). A proposed sampling plan for the second sample follows. Let ν be the (random) sample size for the first sample: the value achieved after m successes. This value, ν, from the first sample is used in the planning of the second sample. For the second sample, the number of trials, n, is the same as the number of observations in the first experiment; set $n = \nu$. Denote $T_\nu = \sum_{k=1}^{\nu} X_k$.

The random variable ν does not depend on $X_1, X_2, \ldots$; therefore it is possible to calculate the mean value and variance of T_ν and its distribution. Since there is a typographical error in the formulae for the variance in Ngamkham (2018) [2] for these calculations, and no actual calculations are presented in Ngamkham et al (2016) [1], we present the correct derivation.

Note that $T_\nu = \sum_{i=1}^{\nu} X_i$ is the sum of n independent identically distributed Bernoulli random variables $X_1, X_2, \ldots$ with parameters p_2 and ν, which has the Pascal (Negative Binomial) distribution with parameters m and p_1. Hence, $\mathrm{E}X_1 = p_2, \mathrm{Var}X_1 = p_2(1-p_2), \mathrm{E}\nu = \frac{m}{p_1}$, and $\mathrm{Var}(\nu) = \frac{m(1-p_1)}{p_1^2}$ (see, for example, Section 2.1 in Ngamkham (2018) [2]).

By Theorem III.6.2 of Gut (2013) [5]:

$$\mathrm{E}T_\nu = \mathrm{E}\nu \cdot \mathrm{E}X_1 = \frac{mp_2}{p_1} = \frac{m}{\theta},$$

$$\mathrm{Var}T_\nu = \mathrm{E}\nu \cdot \mathrm{Var}\mathrm{X}_1 + (\mathrm{E}X_1)^2 \cdot \mathrm{Var}\nu = \frac{m}{p_1} p_2(1-p_2) + p_2^2 \frac{m(1-p_1)}{p_1^2} = m\left(\frac{p_2}{p_1} + \frac{p_2^2}{p_1^2}(1-2p_1)\right).$$

First Special Case of the Inverse-Direct sampling scheme: Note that $\frac{T_\nu}{m}$ is an unbiased estimate for the ratio $\frac{p_2}{p_1} = \frac{1}{\theta}$.

Second Special Case of the Inverse-Direct sampling scheme: The suggested estimate for the parameter θ is

$$\widehat{\theta} = \frac{(m-1)(\nu+1)}{(\nu-1)(T_\nu+1)} \approx \frac{m}{T_\nu+1}.$$

We apply the Delta method to prove its asymptotically normality as $m \to \infty$ and to find the asymptotic mean and variance.

We apply the Delta-method for the estimator

$$\widehat{\theta} = g(S) = \frac{m}{T_\nu + 1} = \frac{1}{S + 1/m},$$

where $S = \frac{T_\nu}{m}$. Note that $\mathrm{E}(S) = \frac{1}{\theta}, \mathrm{Var}(S) = \frac{1}{m}\left(\frac{p_2}{p_1} + \frac{p_2^2}{p_1^2}(1-2p_1)\right)$ and $g(\theta) = \frac{1}{\frac{1}{\theta}+\frac{1}{m}} = \frac{\theta m}{\theta + m}$. The first derivative is

$$\frac{dg(s)}{ds} = -\frac{1}{(s+1/m)^2} \quad \text{and hence} \quad \frac{dg(\theta)}{ds} = -\frac{1}{(1/\theta + 1/m)^2}.$$

For a large m, note that:

$$\frac{dg(\theta)}{ds} \approx -\frac{1}{(1/\theta)^2} = -\theta^2.$$

By the Taylor expansion

$$\widehat{\theta} \approx \frac{\theta m}{\theta + m} - \frac{1}{\left(\frac{1}{\theta} + \frac{1}{m}\right)^2}\left(\frac{T_\nu}{m} - \frac{1}{\theta}\right) = \frac{\theta m}{\theta + m} - \left(\frac{\theta m}{\theta + m}\right)^2\left(\frac{T_\nu}{m} - \frac{1}{\theta}\right).$$

From this, $\mathrm{E}\left(\widehat{\theta}\right) = \frac{\theta m}{\theta + m} \approx \theta$ for $m \to \infty$, and

$$\mathrm{Var}(\widehat{\theta}) = \left(\frac{\theta m}{\theta + m}\right)^4 \frac{1}{m}\left(\frac{p_2}{p_1} + \frac{p_2^2}{p_1^2}(1 - 2p_1)\right).$$

2. NUMERICAL MODELING OF THE ACCURACY PROPERTIES OF THE ESTIMATORS

For each case, 10^5 simulations of the corresponding estimators have been performed. According to the results of the simulations, the mean value and Mean Squared Errors (MSE) were calculated. Tables that contain the values of the theoretical variances and MSE were constructed.

2.1. Modeling of the Bias of Estimators

Results for the first special case Inverse-Direct sampling scheme

In Tables 1—3 below, the following characteristics are presented: (1) mean value of unbiased estimator $\frac{T_\nu}{m}$, (2) the true value of $\theta^{-1} = \frac{p_2}{p_1}$, and (3) the bias of $\frac{T_\nu}{m}$.

According to the results presented in Tables 1—3, we can conclude that when the number of successes in the first sample is increasing, the mean value of the estimate approaches the true value of the probability ratio. When the probabilities are increasing, the mean value of estimator has faster rate of convergence to the desired parameter $\theta^{-1} = p_2/p_1$.

Results for the second special case Inverse-Direct sampling scheme

In Tables 4—6 below, the following characteristics are presented: (1) mean value of estimator $\widehat{\theta}$, (2) the true value of $\theta = p_1/p_2$, and (3) the bias of $\widehat{\theta}$.

According to the results presented in Tables 4-6, we can conclude that when the number of successes in the first sample is increasing, the mean value of the estimate approaches the true value of the probability ratio. When the probabilities are increasing, the mean value of the estimator has a faster rate of convergence to the desired parameter $\theta = p_1/p_2$.

2.2. Modeling of the Mean Squared Error for the Estimators

Results for the first special case Inverse-Direct sampling scheme

In Tables 7—9, the following characteristics are presented: (1) the variance of unbiased estimator $\mathrm{Var}\frac{T_\nu}{m}$ (theoretical variance), (2) true variance of this estimator (simulated variance), and (3) MSE.

According to the results presented in Tables 7—9, it is possible to conclude that the MSE decreases when the number of successes in the first sample increases. Similarly, when both probabilities increase, the MSE decreases.

Results for the Second Special case Inverse-Direct sampling scheme

In Tables 10—12, the following characteristics are presented: (1) the variance of unbiased estimator $\mathrm{Var}\frac{T_\nu}{m}$ (theoretical variance), (2) true variance of this estimator (simulated variance), and (3) MSE.

According to the results presented in Tables 10—12, it is possible to conclude that the MSE decreases when the number of successes in the first sample increases. Similarly, when both probabilities increase, the MSE decreases.

Table 1. Number of successes in the first sample $m = 50p_1$

p_2	p_1									
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	1.0049	0.5006	0.2502	0.1668	0.1251	0.1001	0.0835	0.0715	0.0626	0.0554
	1.0000	0.5000	0.2500	0.1667	0.1250	0.1000	0.0833	0.0714	0.0625	0.0556
	0.0049	0.0006	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	-0.0001
0.1	2.0089	1.0011	0.5006	0.3338	0.2502	0.2002	0.1668	0.1430	0.1251	0.1110
	2.0000	1.0000	0.5000	0.3333	0.2500	0.2000	0.1667	0.1429	0.1250	0.1111
	0.0089	0.0011	0.0006	0.0004	0.0002	0.0002	0.0001	0.0001	0.0001	-0.0001
0.2	4.0139	2.0025	1.0012	0.6675	0.5002	0.4003	0.3336	0.2859	0.2502	0.2221
	4.0000	2.0000	1.0000	0.6667	0.5000	0.4000	0.3333	0.2857	0.2500	0.2222
	0.0139	0.0025	0.0012	0.0008	0.0002	0.0003	0.0002	0.0002	0.0002	-0.0001
0.3	6.0217	3.0022	1.5013	1.0010	0.7504	0.6003	0.5004	0.4288	0.3753	0.3332
	6.0000	3.0000	1.5000	1.0000	0.7500	0.6000	0.5000	0.4286	0.3750	0.3333
	0.0217	0.0022	0.0013	0.0010	0.0004	0.0003	0.0004	0.0002	0.0003	-0.0001
0.4	8.0267	4.0051	2.0033	1.3348	1.0007	0.8003	0.6671	0.5717	0.5004	0.4443
	8.0000	4.0000	2.0000	1.3333	1.0000	0.8000	0.6667	0.5714	0.5000	0.4444
	0.0267	0.0051	0.0033	0.0015	0.0007	0.0003	0.0004	0.0003	0.0004	-0.0001
0.5	10.0331	5.0031	2.5014	1.6686	1.2508	1.0011	0.8340	0.7146	0.6253	0.5555
	10.0000	5.0000	2.5000	1.6667	1.2500	1.0000	0.8333	0.7143	0.6250	0.5556
	0.0331	0.0031	0.0014	0.0020	0.0008	0.0011	0.0007	0.0003	0.0003	-0.0001
0.6	12.0446	6.0026	3.0012	2.0017	1.5004	1.2005	1.0003	0.8572	0.7503	0.6673
	12.0000	6.0000	3.0000	2.0000	1.5000	1.2000	1.0000	0.8571	0.7500	0.6667
	0.0446	0.0026	0.0012	0.0017	0.0004	0.0005	0.0003	0.0001	0.0003	0.0006
0.7	14.0493	7.0056	3.5032	2.3354	1.7507	1.4005	1.1670	1.0001	0.8754	0.7784
	14.0000	7.0000	3.5000	2.3333	1.7500	1.4000	1.1667	1.0000	0.8750	0.7778
	0.0493	0.0056	0.0032	0.0021	0.0007	0.0005	0.0003	0.0001	0.0004	0.0006
0.8	16.0582	8.0061	4.0032	2.6690	2.0009	1.6006	1.3338	1.1430	1.0005	0.8895
	16.0000	8.0000	4.0000	2.6667	2.0000	1.6000	1.3333	1.1429	1.0000	0.8889
	0.0582	0.0061	0.0032	0.0024	0.0009	0.0006	0.0005	0.0002	0.0005	0.0006
0.9	18.0624	9.0066	4.5039	3.0027	2.2510	1.8007	1.5006	1.2859	1.1256	1.0006
	18.0000	9.0000	4.5000	3.0000	2.2500	1.8000	1.5000	1.2857	1.1250	1.0000
	0.0624	0.0066	0.0039	0.0027	0.0010	0.0007	0.0006	0.0002	0.0006	0.0006

Table 2. Number of successes in the first sample $m = 100p_1$

p_2	p_1									
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0.9992	0.5003	0.2502	0.1668	0.1251	0.1000	0.0834	0.0715	0.0626	0.0556
	1.0000	0.5000	0.2500	0.1667	0.1250	0.1000	0.0833	0.0714	0.0625	0.0556
	−0.0008	0.0003	0.0002	0.0002	0.0001	0.0000	0.0001	0.0001	0.0001	0.0000
0.1	1.9978	1.0004	0.5005	0.3335	0.2502	0.2001	0.1667	0.1430	0.1251	0.1112
	2.0000	1.0000	0.5000	0.3333	0.2500	0.2000	0.1667	0.1429	0.1250	0.1111
	−0.0022	0.0004	0.0005	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001
0.2	3.9959	2.0006	1.0008	0.6670	0.5004	0.4002	0.3335	0.2858	0.2501	0.2223
	4.0000	2.0000	1.0000	0.6667	0.5000	0.4000	0.3333	0.2857	0.2500	0.2222
	−0.0041	0.0006	0.0008	0.0004	0.0004	0.0002	0.0001	0.0001	0.0001	0.0001
0.3	5.9926	2.9994	1.5012	0.9997	0.7506	0.6000	0.5004	0.4286	0.3750	0.3332
	6.0000	3.0000	1.5000	1.0000	0.7500	0.6000	0.5000	0.4286	0.3750	0.3333
	−0.0074	−0.0006	0.0012	-0.0003	0.0006	0.0000	0.0004	0.0000	0.0000	−0.0001
0.4	7.9933	4.0016	2.0011	1.3335	1.0007	0.8001	0.6668	0.5715	0.4999	0.4449
	8.0000	4.0000	2.0000	1.3333	1.0000	0.8000	0.6667	0.5714	0.5000	0.4444
	−0.0067	0.0016	0.0011	0.0002	0.0007	0.0001	0.0001	0.0001	-0.0001	0.0004
0.5	9.9882	5.0010	2.5015	1.6676	1.2510	1.0006	0.8335	0.7143	0.6253	0.5560
	10.0000	5.0000	2.5000	1.6667	1.2500	1.0000	0.8333	0.7143	0.6250	0.5556
	−0.0118	0.0010	0.0015	0.0009	0.0010	0.0006	0.0001	0.0001	0.0003	0.0004
0.6	11.9841	6.0012	3.0028	2.0012	1.5007	1.2007	1.0002	0.8572	0.7503	0.6664
	12.0000	6.0000	3.0000	2.0000	1.5000	1.2000	1.0000	0.8571	0.7500	0.6667
	−0.0159	0.0012	0.0028	0.0012	0.0007	0.0007	0.0002	0.0001	0.0003	−0.0003
0.7	13.9825	7.0036	3.5018	2.3348	1.7512	1.4003	1.1668	1.0002	0.8751	0.7782
	14.0000	7.0000	3.5000	2.3333	1.7500	1.4000	1.1667	1.0000	0.8750	0.7778
	−0.0175	0.0036	0.0018	0.0014	0.0012	0.0003	0.0002	0.0002	0.0001	0.0004
0.8	15.9803	7.9996	4.0032	2.6677	2.0011	1.6005	1.3335	1.1429	1.0001	0.8890
	16.0000	8.0000	4.0000	2.6667	2.0000	1.6000	1.3333	1.1429	1.0000	0.8889
	−0.0197	−0.0004	0.0032	0.0010	0.0011	0.0005	0.0001	0.0001	0.0001	0.0001
0.9	17.9792	9.0024	4.5034	3.0012	2.2513	1.8007	1.5002	1.2858	1.1251	1.0001
	18.0000	9.0000	4.5000	3.0000	2.2500	1.8000	1.5000	1.2857	1.1250	1.0000
	−0.0208	0.0024	0.0034	0.0012	0.0013	0.0007	0.0002	0.0001	0.0001	0.0001

Table 3. Number of successes in the first sample $m = 200p_1$

p_2	p_1									
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.5	1.0013	0.5005	0.2502	0.1668	0.1251	0.1001	0.0834	0.0715	0.0625	0.0556
	1.0000	0.5000	0.2500	0.1667	0.1250	0.1000	0.0833	0.0714	0.0625	0.0556
	0.0013	0.0005	0.0002	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000
0.1	2.0031	1.0009	0.5004	0.3336	0.2502	0.2001	0.1667	0.1429	0.1250	0.1111
	2.0000	1.0000	0.5000	0.3333	0.2500	0.2000	0.1667	0.1429	0.1250	0.1111
	0.0031	0.0009	0.0004	0.0003	0.0002	0.0001	0.0001	0.0001	0.0000	0.0000
0.2	4.0034	2.0014	1.0004	0.6667	0.5006	0.3999	0.3334	0.2857	0.2499	0.2223
	4.0000	2.0000	1.0000	0.6667	0.5000	0.4000	0.3333	0.2857	0.2500	0.2222
	0.0034	0.0014	0.0004	0.0000	0.0006	−0.0001	0.0001	0.0000	−0.0001	0.0000
0.3	6.0061	3.0019	1.5004	1.0004	0.7503	0.6004	0.5001	0.4288	0.3752	0.3334
	6.0000	3.0000	1.5000	1.0000	0.7500	0.6000	0.5000	0.4286	0.3750	0.3333
	0.0061	0.0019	0.0004	0.0004	0.0003	0.0004	0.0001	0.0002	0.0002	0.0000
0.4	8.0097	4.0028	2.0002	1.3331	1.0002	0.8007	0.6665	0.5717	0.5002	0.4442
	8.0000	4.0000	2.0000	1.3333	1.0000	0.8000	0.6667	0.5714	0.5000	0.4444
	0.0097	0.0028	0.0002	−0.0002	0.0002	0.0007	−0.0001	0.0003	0.0002	−0.0002
0.5	10.0120	5.0025	2.5009	1.6670	1.2503	1.0007	0.8332	0.7146	0.6252	0.5555
	10.0000	5.0000	2.5000	1.6667	1.2500	1.0000	0.8333	0.7143	0.6250	0.5556
	0.0120	0.0025	0.0009	0.0003	0.0003	0.0007	−0.0001	0.0003	0.0002	−0.0001
0.6	12.0118	6.0037	3.0029	2.0021	1.5009	1.1999	1.0004	0.8571	0.7498	0.6669
	12.0000	6.0000	3.0000	2.0000	1.5000	1.2000	1.0000	0.8571	0.7500	0.6667
	0.0118	0.0037	0.0029	0.0021	0.0009	−0.0001	0.0004	−0.0001	−0.0002	0.0002
0.7	14.0158	7.0031	3.5022	2.3349	1.7507	1.4003	1.1668	1.0000	0.8749	0.7778
	14.0000	7.0000	3.5000	2.3333	1.7500	1.4000	1.1667	1.0000	0.8750	0.7778
	0.0158	0.0031	0.0022	0.0016	0.0007	0.0003	0.0001	0.0000	−0.0001	0.0000
0.8	16.0201	8.0051	4.0022	2.6681	2.0008	1.6007	1.3336	1.1430	1.0001	0.8890
	16.0000	8.0000	4.0000	2.6667	2.0000	1.6000	1.3333	1.1429	1.0000	0.8889
	0.0201	0.0051	0.0022	0.0014	0.0008	0.0007	0.0002	0.0002	0.0001	0.0001
0.9	18.0181	9.0056	4.5028	3.0016	2.2509	1.8005	1.5002	1.2858	1.1250	1.0000
	18.0000	9.0000	4.5000	3.0000	2.2500	1.8000	1.5000	1.2857	1.1250	1.0000
	0.0181	0.0056	0.0028	0.0016	0.0009	0.0005	0.0002	0.0001	0.0000	0.0000

Table 4. Number of successes in the first sample $m = 50p_1$

p_2	p_1									
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	1.1045	1.9693	3.7762	5.5738	7.3742	9.1799	10.9690	12.7685	14.5695	16.4369
	1.0000	2.0000	4.0000	6.0000	8.0000	10.0000	12.0000	14.0000	16.0000	18.0000
	0.1045	−0.0307	−0.2238	−0.4262	−0.6258	−0.8201	−1.0310	−1.2315	−1.4305	−1.5631
0.1	0.6444	1.1302	2.0837	3.0417	4.0015	4.9560	5.9231	6.8701	7.8380	8.8250
	0.5000	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000
	0.1444	0.1302	0.0837	0.0417	0.0015	−0.0440	−0.0769	−0.1299	−0.1620	−0.1750
0.2	0.3409	0.5853	1.0576	1.5367	2.0166	2.4959	2.9775	3.4569	3.9387	4.4276
	0.2500	0.5000	1.0000	1.5000	2.0000	2.5000	3.0000	3.5000	4.0000	4.5000
	0.0909	0.0853	0.0576	0.0367	0.0166	−0.0041	−0.0225	−0.0431	−0.0613	−0.0724
0.3	0.2299	0.3917	0.7058	1.0238	1.3442	1.6643	1.9850	2.3049	2.6258	2.9496
	0.1667	0.3333	0.6667	1.0000	1.3333	1.6667	2.0000	2.3333	2.6667	3.0000
	0.0633	0.0584	0.0391	0.0238	0.0108	−0.0024	−0.0150	−0.0285	−0.0409	−0.0504
0.4	0.1731	0.2940	0.5292	0.7680	1.0083	1.2481	1.4886	1.7286	1.9688	2.2117
	0.1250	0.2500	0.5000	0.7500	1.0000	1.2500	1.5000	1.7500	2.0000	2.2500
	0.0481	0.0440	0.0292	0.0180	0.0083	−0.0019	−0.0114	−0.0214	−0.0312	−0.0383
0.5	0.1382	0.2352	0.4237	0.6146	0.8065	0.9979	1.1906	1.3829	1.5756	1.7690
	0.1000	0.2000	0.4000	0.6000	0.8000	1.0000	1.2000	1.4000	1.6000	1.8000
	0.0382	0.0352	0.0237	0.0146	0.0065	−0.0021	−0.0094	−0.0171	−0.0244	−0.0310
0.6	0.1153	0.1960	0.3531	0.5122	0.6723	0.8325	0.9923	1.1530	1.3129	1.4726
	0.0833	0.1667	0.3333	0.5000	0.6667	0.8333	1.0000	1.1667	1.3333	1.5000
	0.0319	0.0293	0.0198	0.0122	0.0056	−0.0009	−0.0077	−0.0136	−0.0204	−0.0274
0.7	0.0989	0.1680	0.3026	0.4390	0.5763	0.7134	0.8507	0.9882	1.1252	1.2623
	0.0714	0.1429	0.2857	0.4286	0.5714	0.7143	0.8571	1.0000	1.1429	1.2857
	0.0275	0.0251	0.0169	0.0104	0.0049	−0.0009	−0.0065	−0.0118	−0.0176	−0.0234
0.8	0.0865	0.1469	0.2648	0.3841	0.5042	0.6242	0.7443	0.8647	0.9846	1.1046
	0.0625	0.1250	0.2500	0.3750	0.5000	0.6250	0.7500	0.8750	1.0000	1.1250
	0.0240	0.0219	0.0148	0.0091	0.0042	−0.0008	−0.0057	−0.0103	−0.0154	−0.0204
0.9	0.0769	0.1306	0.2353	0.3414	0.4482	0.5549	0.6616	0.7686	0.8752	0.9819
	0.0556	0.1111	0.2222	0.3333	0.4444	0.5556	0.6667	0.7778	0.8889	1.0000
	0.0213	0.0195	0.0131	0.0081	0.0037	−0.0007	−0.0051	−0.0092	−0.0137	−0.0181

Table 5. Number of successes in the first sample $m = 100p_1$

p_2	p_1									
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	1.1514	2.1246	4.0743	6.0196	7.9716	9.9161	11.8652	13.8239	15.7643	17.7291
	1.0000	2.0000	4.0000	6.0000	8.0000	10.0000	12.0000	14.0000	16.0000	18.0000
	0.1514	0.1246	0.0743	0.0196	−0.0284	−0.0839	−0.1348	−0.1761	−0.2357	−0.2709
0.1	0.6012	1.0827	2.0574	3.0375	4.0149	4.9955	5.9765	6.9571	7.9330	8.9183
	0.5000	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000
	0.1012	0.0827	0.0574	0.0375	0.0149	−0.0045	−0.0235	−0.0429	−0.0670	−0.0817
0.2	0.3025	0.5421	1.0290	1.5185	2.0071	2.4977	2.9884	3.4788	3.9682	4.4596
	0.2500	0.5000	1.0000	1.5000	2.0000	2.5000	3.0000	3.5000	4.0000	4.5000
	0.0525	0.0421	0.0290	0.0185	0.0071	−0.0023	−0.0116	−0.0212	−0.0318	−0.0404
0.3	0.2020	0.3615	0.6859	1.0135	1.3380	1.6660	1.9919	2.3193	2.6467	2.9744
	0.1667	0.3333	0.6667	1.0000	1.3333	1.6667	2.0000	2.3333	2.6667	3.0000
	0.0353	0.0282	0.0192	0.0135	0.0047	−0.0007	−0.0081	−0.0141	−0.0200	−0.0256
0.4	0.1516	0.2709	0.5146	0.7597	1.0040	1.2495	1.4945	1.7396	1.9857	2.2283
	0.1250	0.2500	0.5000	0.7500	1.0000	1.2500	1.5000	1.7500	2.0000	2.2500
	0.0266	0.0209	0.0146	0.0097	0.0040	−0.0005	−0.0055	−0.0104	−0.0143	−0.0217
0.5	0.1211	0.2167	0.4117	0.6074	0.8031	0.9992	1.1955	1.3919	1.5874	1.7829
	0.1000	0.2000	0.4000	0.6000	0.8000	1.0000	1.2000	1.4000	1.6000	1.8000
	0.0211	0.0167	0.0117	0.0074	0.0031	−0.0008	−0.0045	−0.0081	−0.0126	−0.0171
0.6	0.1010	0.1807	0.3430	0.5061	0.6694	0.8326	0.9963	1.1598	1.3228	1.4874
	0.0833	0.1667	0.3333	0.5000	0.6667	0.8333	1.0000	1.1667	1.3333	1.5000
	0.0176	0.0140	0.0096	0.0061	0.0027	−0.0007	−0.0037	−0.0069	−0.0105	−0.0126
0.7	0.0866	0.1548	0.2941	0.4338	0.5737	0.7141	0.8540	0.9940	1.1341	1.2737
	0.0714	0.1429	0.2857	0.4286	0.5714	0.7143	0.8571	1.0000	1.1429	1.2857
	0.0151	0.0120	0.0084	0.0053	0.0023	−0.0002	−0.0031	−0.0060	−0.0087	−0.0120
0.8	0.0758	0.1355	0.2573	0.3797	0.5021	0.6247	0.7473	0.8699	0.9924	1.1150
	0.0625	0.1250	0.2500	0.3750	0.5000	0.6250	0.7500	0.8750	1.0000	1.1250
	0.0133	0.0105	0.0073	0.0047	0.0021	−0.0003	−0.0027	−0.0051	−0.0076	−0.0100
0.9	0.0673	0.1204	0.2287	0.3375	0.4463	0.5552	0.6643	0.7732	0.8821	0.9911
	0.0556	0.1111	0.2222	0.3333	0.4444	0.5556	0.6667	0.7778	0.8889	1.0000
	0.0118	0.0093	0.0064	0.0042	0.0018	−0.0003	−0.0024	−0.0046	−0.0067	−0.0089

Table 6. Number of successes in the first sample $m = 200p_1$

p_2	p_1									
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	1.0938	2.0798	4.0536	6.0339	8.0159	9.9953	11.9704	13.9538	15.9334	17.9121
	1.0000	2.0000	4.0000	6.0000	8.0000	10.0000	12.0000	14.0000	16.0000	18.0000
	0.0938	0.0798	0.0536	0.0339	0.0159	−0.0047	−0.0296	−0.0462	−0.0666	−0.0879
0.1	0.5475	1.0401	2.0280	3.0174	4.0075	4.9973	5.9867	6.9780	7.9679	8.9585
	0.5000	1.0000	2.0000	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000	9.0000
	0.0475	0.0401	0.0280	0.0174	0.0075	−0.0027	−0.0133	−0.0220	−0.0321	−0.0415
0.2	0.2740	0.5200	1.0147	1.5095	2.0026	2.5011	2.9936	3.4910	3.9866	4.4788
	0.2500	0.5000	1.0000	1.5000	2.0000	2.5000	3.0000	3.5000	4.0000	4.5000
	0.0240	0.0200	0.0147	0.0095	0.0026	0.0011	−0.0064	−0.0090	−0.0134	−0.0212
0.3	0.1826	0.3467	0.6764	1.0059	1.3358	1.6656	1.9960	2.3254	2.6554	2.9863
	0.1667	0.3333	0.6667	1.0000	1.3333	1.6667	2.0000	2.3333	2.6667	3.0000
	0.0159	0.0134	0.0098	0.0059	0.0024	−0.0010	−0.0040	−0.0080	−0.0113	−0.0137
0.4	0.1370	0.2601	0.5074	0.7551	1.0022	1.2488	1.4977	1.7442	1.9917	2.2410
	0.1250	0.2500	0.5000	0.7500	1.0000	1.2500	1.5000	1.7500	2.0000	2.2500
	0.0120	0.0101	0.0074	0.0051	0.0022	−0.0012	−0.0023	−0.0058	−0.0083	−0.0090
0.5	0.1096	0.2081	0.4058	0.6038	0.8017	0.9993	1.1981	1.3954	1.5933	1.7922
	0.1000	0.2000	0.4000	0.6000	0.8000	1.0000	1.2000	1.4000	1.6000	1.8000
	0.0096	0.0081	0.0058	0.0038	0.0017	−0.0007	−0.0019	−0.0046	−0.0067	−0.0078
0.6	0.0913	0.1734	0.3380	0.5027	0.6679	0.8334	0.9978	1.1635	1.3286	1.4928
	0.0833	0.1667	0.3333	0.5000	0.6667	0.8333	1.0000	1.1667	1.3333	1.5000
	0.0080	0.0067	0.0046	0.0027	0.0012	0.0000	−0.0022	−0.0032	−0.0047	−0.0072
0.7	0.0783	0.1486	0.2898	0.4311	0.5726	0.7141	0.8556	0.9971	1.1387	1.2799
	0.0714	0.1429	0.2857	0.4286	0.5714	0.7143	0.8571	1.0000	1.1429	1.2857
	0.0068	0.0058	0.0041	0.0025	0.0011	−0.0002	−0.0016	−0.0029	−0.0042	−0.0058
0.8	0.0685	0.1300	0.2536	0.3772	0.5010	0.6247	0.7486	0.8723	0.9962	1.1199
	0.0625	0.1250	0.2500	0.3750	0.5000	0.6250	0.7500	0.8750	1.0000	1.1250
	0.0060	0.0050	0.0036	0.0022	0.0010	−0.0003	−0.0014	−0.0027	−0.0038	−0.0051
0.9	0.0609	0.1156	0.2254	0.3353	0.4453	0.5554	0.6654	0.7755	0.8855	0.9955
	0.0556	0.1111	0.2222	0.3333	0.4444	0.5556	0.6667	0.7778	0.8889	1.0000
	0.0053	0.0045	0.0032	0.0020	0.0009	−0.0002	−0.0012	−0.0023	−0.0033	−0.0045

Table 7. Number of successes in the first sample $m = 50p_1$

p_2	p_1									
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0.6333	0.1400	0.0288	0.0119	0.0064	0.0040	0.0027	0.0020	0.0015	0.0012
	0.6435	0.1401	0.0290	0.0119	0.0064	0.0040	0.0027	0.0020	0.0015	0.0012
	0.6435	0.1401	0.0290	0.0119	0.0064	0.0040	0.0027	0.0020	0.0015	0.0012
0.1	1.8667	0.3600	0.0650	0.0252	0.0131	0.0080	0.0054	0.0038	0.0029	0.0022
	1.8987	0.3602	0.0654	0.0253	0.0131	0.0080	0.0054	0.0038	0.0029	0.0023
	1.8988	0.3602	0.0654	0.0253	0.0131	0.0080	0.0054	0.0038	0.0029	0.0023
0.2	6.1333	1.0400	0.1600	0.0563	0.0275	0.0160	0.0104	0.0072	0.0053	0.0041
	6.2264	1.0471	0.1610	0.0566	0.0275	0.0160	0.0104	0.0072	0.0053	0.0041
	6.2265	1.0471	0.1610	0.0566	0.0275	0.0160	0.0104	0.0072	0.0053	0.0041
0.3	12.8000	2.0400	0.2850	0.0933	0.0431	0.0240	0.0150	0.0101	0.0073	0.0054
	12.9766	2.0435	0.2849	0.0934	0.0431	0.0240	0.0151	0.0101	0.0073	0.0055
	12.9769	2.0435	0.2849	0.0934	0.0431	0.0240	0.0151	0.0101	0.0073	0.0055
0.4	21.8667	3.3600	0.4400	0.1363	0.0600	0.0320	0.0193	0.0126	0.0088	0.0064
	22.1559	3.3786	0.4416	0.1367	0.0604	0.0318	0.0194	0.0125	0.0088	0.0064
	22.1564	3.3786	0.4416	0.1367	0.0604	0.0318	0.0194	0.0125	0.0088	0.0064
0.5	33.3333	5.0000	0.6250	0.1852	0.0781	0.0400	0.0231	0.0146	0.0098	0.0069
	33.7381	5.0270	0.6288	0.1859	0.0783	0.0400	0.0233	0.0145	0.0098	0.0069
	33.7389	5.0269	0.6288	0.1859	0.0783	0.0400	0.0233	0.0145	0.0098	0.0069
0.6	47.2000	6.9600	0.8400	0.2400	0.0975	0.0480	0.0267	0.0161	0.0103	0.0069
	47.8586	6.9751	0.8448	0.2394	0.0972	0.0483	0.0264	0.0161	0.0103	0.0070
	47.8601	6.9750	0.8448	0.2394	0.0972	0.0483	0.0264	0.0161	0.0103	0.0070
0.7	63.4667	9.2400	1.0850	0.3007	0.1181	0.0560	0.0298	0.0171	0.0104	0.0065
	64.3294	9.2867	1.0943	0.3010	0.1186	0.0560	0.0296	0.0171	0.0104	0.0066
	64.3312	9.2866	1.0943	0.3010	0.1186	0.0560	0.0296	0.0171	0.0104	0.0066
0.8	82.1333	11.8400	1.3600	0.3674	0.1400	0.0640	0.0326	0.0177	0.0100	0.0057
	83.3318	11.8865	1.3667	0.3667	0.1404	0.0640	0.0323	0.0177	0.0100	0.0058
	83.3344	11.8864	1.3667	0.3667	0.1404	0.0640	0.0323	0.0177	0.0100	0.0058
0.9	103.2000	14.7600	1.6650	0.4400	0.1631	0.0720	0.0350	0.0178	0.0091	0.0044
	104.5758	14.7979	1.6739	0.4396	0.1635	0.0719	0.0347	0.0178	0.0091	0.0045
	104.5786	14.7978	1.6739	0.4396	0.1635	0.0719	0.0347	0.0178	0.0091	0.0045

Table 8. Number of successes in the first sample $m = 100p_1$

p_2	p_1									
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0.3800	0.0700	0.0144	0.0059	0.0032	0.0020	0.0014	0.0010	0.0008	0.0006
	0.3786	0.0699	0.0145	0.0059	0.0032	0.0020	0.0014	0.0010	0.0008	0.0006
	0.3786	0.0699	0.0145	0.0059	0.0032	0.0020	0.0014	0.0010	0.0008	0.0006
0.1	1.1200	0.1800	0.0325	0.0126	0.0066	0.0040	0.0027	0.0019	0.0014	0.0011
	1.1184	0.1795	0.0327	0.0126	0.0066	0.0040	0.0027	0.0019	0.0014	0.0011
	1.1184	0.1795	0.0327	0.0126	0.0066	0.0040	0.0027	0.0019	0.0014	0.0011
0.2	3.6800	0.5200	0.0800	0.0281	0.0138	0.0080	0.0052	0.0036	0.0027	0.0020
	3.6558	0.5203	0.0799	0.0281	0.0137	0.0080	0.0052	0.0036	0.0027	0.0020
	3.6558	0.5203	0.0799	0.0281	0.0137	0.0080	0.0052	0.0036	0.0027	0.0020
0.3	7.6800	1.0200	0.1425	0.0467	0.0216	0.0120	0.0075	0.0051	0.0036	0.0027
	7.6326	1.0186	0.1419	0.0469	0.0214	0.0120	0.0075	0.0050	0.0036	0.0027
	7.6326	1.0185	0.1419	0.0469	0.0214	0.0120	0.0075	0.0050	0.0036	0.0027
0.4	13.1200	1.6800	0.2200	0.0681	0.0300	0.0160	0.0096	0.0063	0.0044	0.0032
	13.0381	1.6786	0.2199	0.0684	0.0300	0.0160	0.0096	0.0063	0.0044	0.0032
	13.0381	1.6786	0.2199	0.0684	0.0300	0.0160	0.0096	0.0063	0.0044	0.0032
0.5	20.0000	2.5000	0.3125	0.0926	0.0391	0.0200	0.0116	0.0073	0.0049	0.0034
	19.7634	2.4812	0.3113	0.0923	0.0390	0.0201	0.0115	0.0073	0.0049	0.0034
	19.7633	2.4812	0.3113	0.0923	0.0390	0.0201	0.0115	0.0073	0.0049	0.0034
0.6	28.3200	3.4800	0.4200	0.1200	0.0488	0.0240	0.0133	0.0080	0.0052	0.0035
	28.0323	3.4795	0.4181	0.1194	0.0486	0.0239	0.0133	0.0080	0.0052	0.0035
	28.0323	3.4794	0.4181	0.1194	0.0486	0.0239	0.0133	0.0080	0.0052	0.0035
0.7	38.0800	4.6200	0.5425	0.1504	0.0591	0.0280	0.0149	0.0086	0.0052	0.0033
	37.7771	4.6191	0.5400	0.1501	0.0591	0.0282	0.0149	0.0085	0.0052	0.0033
	37.7770	4.6190	0.5400	0.1501	0.0591	0.0282	0.0149	0.0085	0.0052	0.0033
0.8	49.2800	5.9200	0.6800	0.1837	0.0700	0.0320	0.0163	0.0089	0.0050	0.0029
	48.8837	5.9026	0.6780	0.1829	0.0702	0.0320	0.0163	0.0088	0.0050	0.0029
	48.8836	5.9025	0.6780	0.1829	0.0702	0.0320	0.0163	0.0088	0.0050	0.0029
0.9	61.9200	7.3800	0.8325	0.2200	0.0816	0.0360	0.0175	0.0089	0.0046	0.0022
	61.3251	7.3790	0.8279	0.2198	0.0815	0.0361	0.0175	0.0088	0.0046	0.0022
	61.3249	7.3789	0.8279	0.2198	0.0815	0.0361	0.0175	0.0088	0.0046	0.0022

Table 9. Number of successes in the first sample $m = 200p_1$

p_2	p_1									
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0.1900	0.0350	0.0072	0.0030	0.0016	0.0010	0.0007	0.0005	0.0004	0.0003
	0.1911	0.0351	0.0072	0.0030	0.0016	0.0010	0.0007	0.0005	0.0004	0.0003
	0.1911	0.0351	0.0072	0.0030	0.0016	0.0010	0.0007	0.0005	0.0004	0.0003
0.1	0.5600	0.0900	0.0163	0.0063	0.0033	0.0020	0.0013	0.0010	0.0007	0.0006
	0.5569	0.0900	0.0162	0.0063	0.0033	0.0020	0.0013	0.0010	0.0007	0.0006
	0.5569	0.0900	0.0162	0.0063	0.0033	0.0020	0.0013	0.0010	0.0007	0.0006
0.2	1.8400	0.2600	0.0400	0.0141	0.0069	0.0040	0.0026	0.0018	0.0013	0.0010
	1.8366	0.2589	0.0402	0.0140	0.0069	0.0040	0.0026	0.0018	0.0013	0.0010
	1.8366	0.2589	0.0402	0.0140	0.0069	0.0040	0.0026	0.0018	0.0013	0.0010
0.3	3.8400	0.5100	0.0713	0.0233	0.0108	0.0060	0.0038	0.0025	0.0018	0.0014
	3.8252	0.5086	0.0709	0.0232	0.0107	0.0060	0.0037	0.0025	0.0018	0.0013
	3.8252	0.5086	0.0709	0.0232	0.0107	0.0060	0.0037	0.0025	0.0018	0.0013
0.4	6.5600	0.8400	0.1100	0.0341	0.0150	0.0080	0.0048	0.0031	0.0022	0.0016
	6.5461	0.8385	0.1095	0.0341	0.0149	0.0080	0.0048	0.0032	0.0022	0.0016
	6.5461	0.8385	0.1095	0.0341	0.0149	0.0080	0.0048	0.0032	0.0022	0.0016
0.5	10.0000	1.2500	0.1563	0.0463	0.0195	0.0100	0.0058	0.0036	0.0024	0.0017
	9.9785	1.2412	0.1551	0.0462	0.0195	0.0100	0.0058	0.0036	0.0024	0.0017
	9.9786	1.2412	0.1551	0.0462	0.0195	0.0100	0.0058	0.0036	0.0024	0.0017
0.6	14.1600	1.7400	0.2100	0.0600	0.0244	0.0120	0.0067	0.0040	0.0026	0.0017
	14.1024	1.7346	0.2096	0.0597	0.0244	0.0120	0.0066	0.0040	0.0026	0.0017
	14.1024	1.7346	0.2096	0.0597	0.0244	0.0120	0.0066	0.0040	0.0026	0.0017
0.7	19.0400	2.3100	0.2713	0.0752	0.0295	0.0140	0.0075	0.0043	0.0026	0.0016
	18.9877	2.2975	0.2705	0.0751	0.0295	0.0140	0.0074	0.0043	0.0026	0.0016
	18.9877	2.2974	0.2705	0.0751	0.0295	0.0140	0.0074	0.0043	0.0026	0.0016
0.8	24.6400	2.9600	0.3400	0.0919	0.0350	0.0160	0.0081	0.0044	0.0025	0.0014
	24.5801	2.9506	0.3383	0.0915	0.0350	0.0159	0.0081	0.0044	0.0025	0.0014
	24.5802	2.9505	0.3383	0.0915	0.0350	0.0159	0.0081	0.0044	0.0025	0.0014
0.9	30.9600	3.6900	0.4163	0.1100	0.0408	0.0180	0.0088	0.0045	0.0023	0.0011
	30.8550	3.6768	0.4143	0.1094	0.0407	0.0179	0.0087	0.0044	0.0023	0.0011
	30.8550	3.6767	0.4143	0.1094	0.0407	0.0179	0.0087	0.0044	0.0023	0.0011

Table 10. Number of successes in the first sample $m = 50p_1$

p_2	p_1									
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0.2004	0.5831	1.9159	3.9983	6.8305	10.4123	14.7439	19.8251	25.6560	32.2366
	0.6056	1.6027	5.4619	11.4886	19.7112	30.3097	42.7446	57.6692	74.8698	95.4307
	0.6165	1.6036	5.5119	11.6702	20.1026	30.9819	43.8071	59.1853	76.9153	97.8731
0.1	0.0630	0.1736	0.5015	0.9838	1.6204	2.4113	3.3565	4.4560	5.7099	7.1181
	0.2887	0.6610	1.6526	3.1232	4.9699	7.1512	10.0402	12.9848	16.7487	21.2581
	0.3096	0.6779	1.6596	3.1249	4.9698	7.1531	10.0461	13.0016	16.7748	21.2885
0.2	0.0174	0.0444	0.1093	0.1947	0.3005	0.4269	0.5737	0.7411	0.9289	1.1372
	0.0900	0.1535	0.2577	0.4138	0.5851	0.7888	1.0599	1.3260	1.6682	2.0532
	0.0982	0.1608	0.2610	0.4151	0.5853	0.7888	1.0604	1.3279	1.6720	2.0585
0.3	0.0080	0.0195	0.0435	0.0721	0.1053	0.1431	0.1854	0.2323	0.2838	0.3399
	0.0406	0.0564	0.0812	0.1148	0.1599	0.2095	0.2756	0.3298	0.4012	0.4809
	0.0446	0.0598	0.0827	0.1153	0.1600	0.2095	0.2759	0.3306	0.4028	0.4834
0.4	0.0045	0.0108	0.0226	0.0355	0.0494	0.0643	0.0802	0.0972	0.1152	0.1342
	0.0223	0.0276	0.0373	0.0503	0.0664	0.0835	0.1031	0.1229	0.1454	0.1704
	0.0246	0.0295	0.0381	0.0506	0.0665	0.0835	0.1032	0.1233	0.1463	0.1718
0.5	0.0029	0.0068	0.0137	0.0205	0.0274	0.0342	0.0410	0.0479	0.0547	0.0615
	0.0126	0.0154	0.0205	0.0274	0.0342	0.0414	0.0491	0.0563	0.0649	0.0731
	0.0141	0.0166	0.0211	0.0277	0.0342	0.0414	0.0492	0.0566	0.0654	0.0741
0.6	0.0020	0.0047	0.0091	0.0132	0.0169	0.0203	0.0234	0.0261	0.0286	0.0307
	0.0083	0.0102	0.0130	0.0166	0.0201	0.0237	0.0263	0.0296	0.0322	0.0347
	0.0093	0.0110	0.0133	0.0168	0.0201	0.0237	0.0264	0.0298	0.0326	0.0355
0.7	0.0015	0.0034	0.0065	0.0091	0.0113	0.0130	0.0144	0.0153	0.0158	0.0159
	0.0059	0.0071	0.0089	0.0110	0.0130	0.0145	0.0157	0.0167	0.0172	0.0174
	0.0067	0.0077	0.0092	0.0111	0.0130	0.0145	0.0157	0.0168	0.0175	0.0180
0.8	0.0012	0.0026	0.0048	0.0066	0.0079	0.0088	0.0093	0.0094	0.0091	0.0083
	0.0044	0.0052	0.0065	0.0078	0.0089	0.0095	0.0098	0.0099	0.0095	0.0088
	0.0049	0.0057	0.0067	0.0078	0.0089	0.0095	0.0099	0.0100	0.0098	0.0092
0.9	0.0009	0.0021	0.0037	0.0050	0.0058	0.0063	0.0063	0.0060	0.0052	0.0041
	0.0033	0.0040	0.0049	0.0057	0.0063	0.0066	0.0065	0.0061	0.0053	0.0042
	0.0038	0.0044	0.0050	0.0058	0.0064	0.0066	0.0065	0.0062	0.0055	0.0045

Table 11. Number of successes in the first sample $m = 100p_1$

p_2	p_1									
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0.1833	0.5401	1.7747	3.7037	6.3272	9.6451	13.6574	18.3642	23.7654	29.8611
	0.7227	1.8990	5.7097	11.4592	19.2433	28.7092	40.5621	54.8219	69.9529	88.4374
	0.7456	1.9145	5.7151	11.4595	19.2440	28.7160	40.5798	54.8524	70.0078	88.5099
0.1	0.0478	0.1229	0.3552	0.6967	1.1475	1.7075	2.3769	3.1555	4.0434	5.0406
	0.1883	0.3304	0.7534	1.4138	2.2056	3.2570	4.4681	5.9258	7.4809	9.2153
	0.1985	0.3372	0.7566	1.4152	2.2058	3.2570	4.4686	5.9276	7.4853	9.2219
0.2	0.0118	0.0267	0.0658	0.1172	0.1810	0.2571	0.3455	0.4463	0.5594	0.6849
	0.0373	0.0514	0.0995	0.1651	0.2446	0.3431	0.4606	0.5927	0.7329	0.9014
	0.0401	0.0531	0.1003	0.1655	0.2446	0.3431	0.4607	0.5931	0.7339	0.9030
0.3	0.0052	0.0110	0.0247	0.0409	0.0598	0.0812	0.1052	0.1319	0.1611	0.1930
	0.0142	0.0185	0.0330	0.0519	0.0722	0.0971	0.1267	0.1556	0.1895	0.2263
	0.0154	0.0193	0.0334	0.0521	0.0722	0.0971	0.1267	0.1558	0.1899	0.2269
0.4	0.0029	0.0059	0.0125	0.0195	0.0272	0.0354	0.0442	0.0535	0.0634	0.0739
	0.0076	0.0092	0.0157	0.0233	0.0315	0.0405	0.0497	0.0601	0.0716	0.0832
	0.0083	0.0097	0.0159	0.0234	0.0315	0.0405	0.0498	0.0602	0.0718	0.0837
0.5	0.0018	0.0037	0.0074	0.0111	0.0148	0.0185	0.0222	0.0259	0.0296	0.0333
	0.0044	0.0056	0.0090	0.0127	0.0165	0.0204	0.0240	0.0283	0.0321	0.0361
	0.0049	0.0058	0.0092	0.0128	0.0165	0.0204	0.0241	0.0284	0.0323	0.0364
0.6	0.0013	0.0025	0.0049	0.0070	0.0090	0.0108	0.0125	0.0140	0.0153	0.0164
	0.0030	0.0037	0.0058	0.0079	0.0098	0.0116	0.0133	0.0147	0.0162	0.0175
	0.0033	0.0039	0.0059	0.0079	0.0098	0.0116	0.0133	0.0148	0.0163	0.0176
0.7	0.0009	0.0018	0.0034	0.0048	0.0060	0.0069	0.0076	0.0081	0.0084	0.0084
	0.0021	0.0026	0.0040	0.0053	0.0064	0.0073	0.0079	0.0084	0.0087	0.0088
	0.0024	0.0028	0.0041	0.0053	0.0064	0.0073	0.0080	0.0084	0.0088	0.0089
0.8	0.0007	0.0014	0.0025	0.0035	0.0042	0.0046	0.0049	0.0049	0.0048	0.0043
	0.0016	0.0020	0.0029	0.0037	0.0044	0.0048	0.0051	0.0050	0.0049	0.0045
	0.0018	0.0021	0.0030	0.0038	0.0044	0.0048	0.0051	0.0051	0.0050	0.0046
0.9	0.0006	0.0011	0.0019	0.0026	0.0030	0.0033	0.0033	0.0031	0.0027	0.0021
	0.0013	0.0015	0.0022	0.0028	0.0032	0.0034	0.0034	0.0031	0.0028	0.0022
	0.0014	0.0016	0.0023	0.0028	0.0032	0.0034	0.0034	0.0032	0.0028	0.0022

Table 12. Number of successes in the first sample $m = 200p_1$

p_2	p_1									
	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.05	0.1298	0.3825	1.2567	2.6228	4.4806	6.8301	9.6715	13.0046	16.8295	21.1461
	0.3705	0.8657	2.5546	5.2218	8.7890	13.2872	18.3100	24.9534	31.9602	40.0528
	0.3793	0.8721	2.5575	5.2229	8.7891	13.2871	18.3107	24.9553	31.9643	40.0601
0.1	0.0288	0.0740	0.2139	0.4196	0.6911	1.0284	1.4315	1.9004	2.4352	3.0358
	0.0577	0.1187	0.3053	0.5825	0.9506	1.3974	1.9231	2.5625	3.2565	4.0555
	0.0600	0.1203	0.3061	0.5828	0.9506	1.3974	1.9233	2.5629	3.2575	4.0572
0.2	0.0065	0.0147	0.0362	0.0645	0.0997	0.1416	0.1902	0.2457	0.3080	0.3771
	0.0109	0.0197	0.0443	0.0758	0.1161	0.1661	0.2174	0.2861	0.3523	0.4285
	0.0115	0.0201	0.0446	0.0759	0.1161	0.1661	0.2174	0.2861	0.3525	0.4290
0.3	0.0028	0.0059	0.0132	0.0218	0.0319	0.0433	0.0562	0.0704	0.0860	0.1030
	0.0044	0.0075	0.0152	0.0242	0.0349	0.0477	0.0608	0.0763	0.0937	0.1107
	0.0046	0.0077	0.0153	0.0242	0.0349	0.0477	0.0608	0.0764	0.0938	0.1109
0.4	0.0015	0.0031	0.0065	0.0103	0.0143	0.0186	0.0232	0.0281	0.0333	0.0388
	0.0023	0.0039	0.0073	0.0112	0.0153	0.0197	0.0246	0.0299	0.0352	0.0409
	0.0025	0.0040	0.0074	0.0113	0.0153	0.0197	0.0246	0.0300	0.0353	0.0410
0.5	0.0010	0.0019	0.0038	0.0058	0.0077	0.0096	0.0115	0.0135	0.0154	0.0173
	0.0014	0.0023	0.0042	0.0062	0.0081	0.0101	0.0121	0.0140	0.0159	0.0179
	0.0015	0.0024	0.0043	0.0062	0.0081	0.0101	0.0121	0.0140	0.0160	0.0180
0.6	0.0007	0.0013	0.0025	0.0036	0.0047	0.0056	0.0064	0.0072	0.0079	0.0085
	0.0010	0.0016	0.0027	0.0038	0.0049	0.0058	0.0066	0.0075	0.0082	0.0087
	0.0010	0.0016	0.0027	0.0038	0.0049	0.0058	0.0066	0.0075	0.0082	0.0087
0.7	0.0005	0.0009	0.0018	0.0025	0.0031	0.0035	0.0039	0.0042	0.0043	0.0043
	0.0007	0.0011	0.0019	0.0026	0.0032	0.0036	0.0040	0.0042	0.0044	0.0044
	0.0008	0.0011	0.0019	0.0026	0.0032	0.0036	0.0040	0.0043	0.0044	0.0044
0.8	0.0004	0.0007	0.0013	0.0018	0.0021	0.0024	0.0025	0.0025	0.0024	0.0022
	0.0005	0.0008	0.0014	0.0018	0.0022	0.0024	0.0025	0.0026	0.0025	0.0023
	0.0006	0.0009	0.0014	0.0018	0.0022	0.0024	0.0025	0.0026	0.0025	0.0023
0.9	0.0003	0.0006	0.0010	0.0013	0.0016	0.0017	0.0017	0.0016	0.0014	0.0011
	0.0004	0.0006	0.0011	0.0014	0.0016	0.0017	0.0017	0.0016	0.0014	0.0011
	0.0004	0.0007	0.0011	0.0014	0.0016	0.0017	0.0017	0.0016	0.0014	0.0011

Table 13. Comparison of the performance for all sample schemes

Sampling Scheme	Variance	MSE
Direct-Direct	1.8400	2.1429
Direct-Inverse	1.8400	1.9030
Inverse-Direct	1.8400	6.8242
Inverse-Inverse	1.8400	1.8792
Special Case for Direct-Inverse	0.7000	0.6998
First Special Case for Inverse-Direct	1.8400	1.8366
Second Special Case for Inverse-Direct	1.2567	2.5575

3. COMPARISON AND ANALYSIS OF THE ACCURACY PROPERTIES OF THE ESTIMATORS

From the simulation results, the First Special Case of the Inverse-Direct sampling scheme performs well in terms of variance and, thus, MSE, because of unbiasedness.

In Ngamkham (2020) [3] simulation results show that the best accuracy from the MSE is the Special case of the Direct-Inverse sampling scheme, where the first sample is obtained by the Direct Bernoulli sampling scheme, and the second sample is retrieved with the Inverse sampling scheme where the number of successes equals the number of successes in the first sample. The worst sampling scheme that has the largest MSE appears to be the regular Inverse-Direct sampling scheme, where the first sample is obtained with the Inverse sampling scheme and the second with the Direct sampling scheme.

All other sampling schemes have practically identical MSEs. In Ngamkham (2020) [3], a table of typical values of the MSEs for five sampling schemes is presented when $n_1 = n_2 = 200$ and $p_1 = 0.2$, $p_2 = 0.05$. For the First and Second Special cases of the Inverse-Direct scheme, it is equivalent to $m = 200p_1$.

The following, Table 13, provides the typical values for the MSE for all seven sampling schemes in the settings mentioned in the previous paragraph. The first five rows of this table are taken from Ngamkham (2020) [3].

These simulation results show that both the First and Second Special Cases of the Inverse-Direct sampling scheme strongly improve the accuracy in terms of variance and MSE compared to the regular Inverse-Direct sampling scheme.

In connection with this finding, our recommendation for practitioners follows. If it is required to run an experiment for an estimation of the ratio of Binomial proportions in the framework of the Inverse-Direct sampling scheme, then we recommend considering Special Cases of this scheme of the design. Specifically, the first sample is obtained by the Inverse sampling method with parameters that have a fixed number of successes, m. Let ν be the (random) sample size for the first sample: the value when we achieved m successes. This value ν from the first sample is used in the planning of the second sample. For the second sample, the number of trials, n, is the same as the number of observations in the first experiment; in other words, set $n = \nu$.

We would like to remind readers once more that in the First Special Case for the Inverse-Direct sampling scheme, we estimate not the parameter θ, but its reciprocal θ^{-1}.

FUNDING

The research of A. Volodin was funded by the subsidy allocated to Kazan Federal University for the state assignment in the sphere of scientific activities, project no. 1.13556.2019/13.1.

REFERENCES

1. T. Ngamkham, A. Volodin, and I. Volodin, "Confidence intervals for a ratio of Binomial proportions based on direct and inverse sampling schemes," Lobachevskii J. Math. **37** (4), 466–494 (2016).
2. T. Ngamkham, "Confidence interval estimation for the ratio of binomial proportions and random numbers generation for some statistical models," Ph. D. Thesis (Univ. of Regina, 2018).
3. T. Ngamkham, "Comparison of accuracy properties of point estimators for the ratio of binomial proportions with different sampling schemes," Thailand Statistician (2020, in press).
4. E. L. Lehmann, *Elements of Large Sample Theory* (Springer Science, New York, 2004).
5. A. Gut, *Probability: a Graduate Course* (Springer Science, New York, 2013).