ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2009, Vol. 30, No. 4, pp. 337–346. © Pleiades Publishing, Ltd., 2009.

Marcinkiewicz-Zygmund Type Law of Large Numbers for Double Arrays of Random Elements in Banach Spaces

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Received July 30, 2009

Abstract—In this paper we establish Marcinkiewicz-Zygmund type laws of large numbers for double arrays of random elements in Banach spaces. Our results extend those of Hong and Volodin [6].

2000 Mathematics Subject Classification: 60B11, 60B12, 60F15, 60G42

DOI: 10.1134/S1995080209040118

Key words and phrases: *Marcinkiewicz-Zygmund inequality, Rademacher type p Banach spaces, Martingale type p Banach spaces, Double arrays of random elements, Strong and L_p laws of large numbers.*

1. INTRODUCTION

Marcinkiewicz-Zygmund type strong laws of large numbers were studied by many authors. In 1981, Etemadi [3] proved that if $\{X_n; n \geq 1\}$ is a sequence of pairwise i.i.d. random variables with $EX_1 < \infty$, then $\lim \dfrac{1}{n} \sum_{i=1}^{n}(X_i - EX_1) = 0$ a.s.

Later, in 1985, Choi and Sung [2] have shown that if $\{X_n; n \geq 1\}$ are pairwise independent and are dominated in distribution by a random variable X with $E|X|^p(\log^+|X|)^2 < \infty$, $1 < p < 2$, then

$$\lim \frac{1}{n^{\frac{1}{p}}} \sum_{i=1}^{n}(X_i - EX_i) = 0 \text{ a.s.}$$

Recently, Hong and Hwang [5], Hong and Volodin [6] studied Marcinkiewicz-Zygmund strong law of large numbers for double sequence of random variables, Quang and Thanh [12] established the Marcinkiewicz-Zygmund strong law of large numbers for blockwise adapted sequence. In this paper, we extend the results of Hong and Volodin [6] to some special class of Banach spaces, so-called Banach spaces that satisfy the maximal Marcinkiewicz-Zygmund inequality with exponent p (see the definition below). This class includes Rademacher type p and martingale type p Banach spaces, $0 < p \leq 2$.

For $a, b \in \mathbb{R}$, $\max\{a, b\}$ will be denoted by $a \vee b$. Throughout this paper, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

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2. PRELIMINARIES

Technical definitions relevant to the current work will be discussed in this section.

The Banach space $\mathcal{X}$ is said to be of *Rademacher type* p $(1 \leq p \leq 2)$ if there exists a constant $C < \infty$ such that

$$E\left|\left|\sum_{j=1}^{n} V_j\right|\right|^p \leq C\sum_{j=1}^{n} E||V_j||^p$$

for all independent $\mathcal{X}$-valued random elements $V_1, \dots, V_n$ with mean 0.

We refer the reader to Pisier [10] and Woyczyński [16] for a detailed discussion of this notion.

Scalora [14] introduced the idea of the conditional expectation of a random element in a Banach space. For a random element V and sub σ-algebra $\mathcal{G}$ of $\mathcal{F}$, the conditional expectation $E(V|\mathcal{G})$ is defined analogously to that in the random variable case and enjoys similar properties.

A real separable Banach space $\mathcal{X}$ is said to be *martingale type* p $(1 \leq p \leq 2)$ if there exists a finite positive constant C such that for all martingales $\{S_n; n \geq 1\}$ with values in $\mathcal{X}$,

$$\sup_{n\geq 1} E||S_n||^p \leq C\sum_{n=1}^{\infty} E||S_n - S_{n-1}||^p.$$

It can be shown using classical methods from martingale theory that if $\mathcal{X}$ is of martingale type p, then for all $1 \leq r < \infty$ there exists a finite constant C such that

$$E\sup_{n\geq 1}||S_n||^r \leq CE\left(\sum_{n=1}^{\infty}||S_n - S_{n-1}||^p\right)^{\frac{r}{p}}.$$

Clearly every real separable Banach space is of martingale type 1 and the real line (the same as any Hilbert space) is of martingale type 2. If a real separable Banach space of martingale type p for some $1 < p \leq 2$ then it is of martingale type r for all $r \in [1, p)$.

It follows from the Hoffmann-Jørgensen and Pisier [4] characterization of Rademacher type p Banach spaces that if a Banach space is of martingale type p, then it is of Rademacher type p. But the notion of martingale type p is only superficially similar to that of Rademacher type p and has a geometric characterization in terms of smoothness. For proofs and more details, the reader may refer to Pisier [10, 11].

Definition. Let $0 < p \leq 2$. We say that a collection $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ of mn random elements taking values in a real separable Banach space $\mathcal{X}$ satisfies the *maximal Marcinkiewicz-Zygmund inequality with exponent* p if

$$E\max_{\substack{1\leq k\leq m\\1\leq l\leq n}}\left|\left|\sum_{i=1}^{k}\sum_{j=1}^{l} V_{ij}\right|\right|^p \leq C\sum_{i=1}^{m}\sum_{j=1}^{n} E||V_{ij}||^p, \tag{2.1}$$

where the constant C is independent of m and n.

It is clear that for $0 < p \leq 1$, every collection random elements $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent p. Now we provide an example of a collection of random elements $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ taking values in a real separable Banach space $\mathcal{X}$ that satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent p in the case of $1 < p \leq 2$.

Example. Suppose that $\{V_j; 1 \leq j \leq n\}$ is a collection of n independent mean 0 random elements taking values in a Rademacher type p $(1 < p \leq 2)$ Banach space. Then there exists a constant C such that

$$E\left|\left|\sum_{j=1}^{n} V_j\right|\right|^p \leq C\sum_{j=1}^{n} E||V_j||^p.$$

This follows from the following observation. If we set $V_{1j} = V_j$ and $V_{ij} = 0$ for all $2 \le i \le m$, $1 \le j \le n$, then $\{V_{ij}; 1 \le i \le m, 1 \le j \le n\}$ satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent p by the definition of the Rademacher type p.

Random elements $\{V_{mn}; m \ge 1, n \ge 1\}$ are said to be *stochastically dominated* by a random element V if for some constant $C < \infty$

$$P\{||V_{mn}|| > t\} \le CP\{||V|| > t\}, \qquad t \ge 0, \quad m \ge 1, \quad n \ge 1.$$

Denote d_k be the number of divisors of k. The following lemma can be found in Gut and Spătaru [7] or Gut [8].

Lemma 1.

$$\sum_{k=1}^{n} \frac{d_k}{k^\gamma} = O(n^{1-\gamma} \log n) \qquad (\gamma < 1), \tag{2.2}$$

$$\sum_{k=i+1}^{\infty} \frac{d_k}{k^\gamma} = O\left(\frac{\log i}{(i+1)^{\gamma-1}}\right) \qquad (\gamma > 1). \tag{2.3}$$

3. MAIN RESULTS

With the preliminaries accounted for, the main results may now be established. In the following we let $\{V_{mn}; m \ge 1, n \ge 1\}$ be an array of random elements in a real separable Banach space $\mathcal{X}$.

Theorem 1. *Let $0 < p \le 2$ and $\{V_{ij}; i \ge, j \ge 1\}$ be an array of random elements that satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent p. If*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E||V_{ij}||^p}{(i^\alpha j^\beta)^p} < \infty$$

for some $\alpha > 0$ and $\beta > 0$, then

$$\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n} V_{ij} \to 0 \quad a.s. \quad as \quad m \vee n \to \infty \tag{3.1}$$

and

$$\frac{1}{m^\alpha n^\beta}\sum_{i=1}^{m}\sum_{j=1}^{n} V_{ij} \to 0 \quad in \quad L_p \quad as \quad m \vee n \to \infty. \tag{3.2}$$

Proof. Set

$$S_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} V_{ij},$$

and

$$T_{kl} = \max_{\substack{2^k \le m < 2^{k+1} \\ 2^l \le m < 2^{l+1}}} \left|\left|\frac{S_{mn}}{m^\alpha n^\beta} - \frac{S_{2^k 2^l}}{2^{k\alpha} 2^{l\beta}}\right|\right|.$$

First, we prove (3.1). For arbitrary $\varepsilon > 0$, by the Markov and maximal Marcinkiewicz-Zygmund inequalities with exponent p we have

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} P\left\{\left|\left|\frac{S_{2^k 2^l}}{2^{k\alpha} 2^{l\beta}}\right|\right| > \varepsilon\right\} \le \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{(2^{k\alpha}2^{l\beta})^p \varepsilon^p} E||S_{2^k 2^l}||^p$$

$$\le \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{C}{(2^{k\alpha}2^{l\beta})^p \varepsilon^p} \sum_{i=1}^{2^k}\sum_{j=1}^{2^l} E||V_{ij}||^p \le \frac{C}{\varepsilon^p}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E||V_{ij}||^p}{(i^\alpha j^\beta)^p} < \infty.$$

It follows by the Borel-Cantelli lemma that

$$\lim_{k\vee l\to\infty}\frac{S_{2^k2^l}}{2^{k\alpha}2^{l\beta}}=0\quad\text{a.s.}\tag{3.3}$$

Again, let $\varepsilon>0$ be arbitrary, from the maximal Marcinkiewicz-Zygmund inequality with exponent p we obtain

$$P\{|T_{kl}|>\varepsilon\}\le P\left\{\frac{||S_{2^k2^l}||}{2^{k\alpha}2^{l\beta}}>\frac{\varepsilon}{2}\right\}+P\left\{\max_{\substack{2^k\le m<2^{k+1}\\ 2^l\le n<2^{l+1}}}\frac{||S_{mn}||}{m^\alpha n^\beta}>\frac{\varepsilon}{2}\right\}\le P\left\{\frac{||S_{2^k2^l}||}{2^{k\alpha}2^{l\beta}}>\frac{\varepsilon}{2}\right\}$$

$$+P\left\{\max_{\substack{2^k\le m<2^{k+1}\\ 2^l\le n<2^{l+1}}}\frac{||S_{mn}||}{2^{k\alpha}2^{l\beta}}>\frac{\varepsilon}{2}\right\}\le P\left\{||S_{2^k2^l}||>\frac{2^{k\alpha}2^{l\beta}\varepsilon}{2}\right\}+P\left\{\max_{\substack{1\le m\le 2^{k+1}\\ 1\le n\le 2^{l+1}}}||S_{mn}||>\frac{2^{k\alpha}2^{l\beta}\varepsilon}{2}\right\}$$

$$\le\frac{2^p}{(2^{k\alpha}2^{l\beta})^p\varepsilon^p}E||S_{2^k2^l}||^p+\frac{2^p}{(2^{k\alpha}2^{l\beta})^p\varepsilon^p}E\max_{\substack{1\le m\le 2^{k+1}\\ 1\le n\le 2^{l+1}}}||S_{mn}||^p\le\frac{2^pC}{(2^{k\alpha}2^{l\beta})^p\varepsilon^p}\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}E||V_{ij}||^p$$

$$+\frac{2^pC}{(2^{k\alpha}2^{l\beta})^p\varepsilon^p}\sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}}E||V_{ij}||^p\le\frac{C}{\varepsilon^p}\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}\frac{E||V_{ij}||^p}{(2^{k\alpha}2^{l\beta})^p}+\frac{C}{\varepsilon^p}\sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}}\frac{E||V_{ij}||^p}{(2^{(k+1)\alpha}2^{(l+1)\beta})^p}$$

$$\le\frac{C}{\varepsilon^p}\sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}}\frac{E||V_{ij}||^p}{(2^{(k+1)\alpha}2^{(l+1)\beta})^p}.$$

From this we obtain

$$\sum_{k=1}^\infty\sum_{l=1}^\infty P\{|T_{kl}|>\varepsilon\}\le\sum_{k=1}^\infty\sum_{l=1}^\infty\frac{C}{\varepsilon^p}\sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}}\frac{E||V_{ij}||^p}{(2^{(k+1)\alpha}2^{(l+1)\beta})^p}\le\frac{C}{\varepsilon^p}\sum_{i=1}^\infty\sum_{j=1}^\infty\frac{E||V_{ij}||^p}{(i^\alpha j^\beta)^p}<\infty.$$

Again by the Borel-Cantelli lemma, we have that

$$\lim_{k\vee l\to\infty}T_{kl}=0\quad\text{a.s.}\tag{3.4}$$

Note that for $2^k\le m<2^{k+1}$ and $2^l\le n<2^{l+1}$,

$$\frac{||S_{mn}||}{m^\alpha n^\beta}\le\left|\left|\frac{S_{mn}}{m^\alpha n^\beta}-\frac{S_{2^k2^l}}{2^{k\alpha}2^{l\beta}}\right|\right|+\frac{||S_{2^k2^l}||}{2^{k\alpha}2^{l\beta}}\le T_{kl}+\frac{||S_{2^k2^l}||}{2^{k\alpha}2^{l\beta}},\tag{3.5}$$

and so the conclusion (3.1) follows from (3.3) and (3.4).

Next, we will prove (3.2). Since convergence of the following series

$$\sum_{k=1}^\infty\sum_{l=1}^\infty\frac{1}{(2^{k\alpha}2^{l\beta})^p}E||S_{2^k2^l}||^p\le\sum_{k=1}^\infty\sum_{l=1}^\infty\frac{C}{(2^{k\alpha}2^{l\beta})^p}\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}E||V_{ij}||^p\le C\sum_{i=1}^\infty\sum_{j=1}^\infty\frac{E||V_{ij}||^p}{(i^\alpha j^\beta)^p}<\infty,$$

we get

$$\lim_{k\vee l\to\infty}E||\frac{S_{2^k2^l}}{2^{k\alpha}2^{l\beta}}||^p=0.\tag{3.6}$$

On the other hand, we have that

$$E|T_{kl}|^p\le CE\max_{\substack{2^k\le m<2^{k+1}\\ 2^l\le m<2^{l+1}}}\left|\left|\frac{S_{mn}}{m^\alpha n^\beta}\right|\right|^p+CE\frac{||S_{2^k2^l}||^p}{(2^{k\alpha}2^{l\beta})^p}\le\frac{C}{(2^{k\alpha}2^{l\beta})^p}E\max_{\substack{1\le m\le 2^{k+1}\\ 1\le n\le 2^{l+1}}}||S_{mn}||^p$$

$$+\frac{C}{(2^{k\alpha}2^{l\beta})^p}E||S_{2^k2^l}||^p\le\frac{C}{(2^{k\alpha}2^{l\beta})^p}\sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}}E||V_{ij}||^p+\frac{C}{(2^{k\alpha}2^{l\beta})^p}\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}E||V_{ij}||^p$$

$$\le C\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}\frac{E||V_{ij}||^p}{(2^{k\alpha}2^{l\beta})^p}+C\sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}}\frac{E||V_{ij}||^p}{(2^{(k+1)\alpha}2^{(l+1)\beta})^p}\le C\sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}}\frac{E||V_{ij}||^p}{(2^{(k+1)\alpha}2^{(l+1)\beta})^p}.$$

This implies

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}E|T_{kl}|^p\le\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}C\sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}}\frac{E||V_{ij}||^p}{(2^{(k+1)\alpha}2^{(l+1)\beta})^p}\le C\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E||V_{ij}||^p}{(i^\alpha j^\beta)^p}<\infty.$$

Hence we have

$$\lim_{k\vee l\to\infty}E|T_{kl}|^p=0. \tag{3.7}$$

It follows from (3.5) that for $2^k\le m<2^{k+1}$, $2^l\le n<2^{l+1}$,

$$E\left|\left|\frac{S_{mn}}{m^\alpha n^\beta}\right|\right|^p\le CE|T_{kl}|^p+CE\left|\left|\frac{S_{2^k2^l}}{(2^{k\alpha}2^{l\beta})}\right|\right|^p.$$

Thus, the conclusion (3.2) follows from (3.6) and (3.7). □

In the next two theorems, we obtain the Marcinkiewicz-Zygmund type law of large numbers for double array of random elements.

Theorem 2. *Let $1<r<p\le 2$ and $\{V_{mn};m\ge 1,n\ge 1\}$ be an array of random elements that satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent p. Suppose that $\{V_{mn};m\ge 1,n\ge 1\}$ is be stochastically dominated by a random element V such that $E||V||^r\log^+||V||^r<\infty$ if $1\le r<p$ then*

$$\frac{1}{(mn)^{\frac{1}{r}}}\sum_{i=1}^{m}\sum_{j=1}^{n}V_{ij}\to 0\quad a.s.\ and\ in\quad L_p\ as\quad m\vee n\to\infty. \tag{3.8}$$

Proof. Let F be the distribution of $||V||$. Set

$$V'_{ij}=V_{ij}I(||V_{ij}||\le(ij)^{\frac{1}{r}}),V''_{ij}=V_{ij}I(||V_{ij}||>(ij)^{\frac{1}{r}}).$$

Applying the equation (2.3) of Lemma 2.1 with $\gamma=\frac{p}{r}$ we obtain $\sum_{k=i+1}^{\infty}\frac{d_k}{k^{\frac{p}{r}}}=O(\frac{\log i}{(i+1)^{\frac{p}{r}-1}})$. Then we have

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E||V'_{ij}||^p}{(ij)^{\frac{p}{r}}}\le C\sum_{k=1}^{\infty}\frac{d_k}{k^{\frac{p}{r}}}\int_0^{k^{\frac{1}{r}}}x^pdF(x)\le C\sum_{i=0}^{\infty}\left(\sum_{k=i+1}^{\infty}\frac{d_k}{k^{\frac{p}{r}}}\right)\int_{i^{\frac{1}{r}}}^{(i+1)^{\frac{1}{r}}}x^pdF(x) \tag{3.9}$$

$$\le C\sum_{i=0}^{\infty}\frac{\log i}{(i+1)^{\frac{p}{r}-1}}\int_{i^{\frac{1}{r}}}^{(i+1)^{\frac{1}{r}}}x^pdF(x)$$

$$\le CE||V||^r\log^+||V||^r<\infty.$$

On the other hand, applying the equation (2.2) of Lemma 2.1 with $\gamma=\frac{1}{r}$ we obtain $\sum_{k=1}^{n}\frac{d_k}{k^{\frac{1}{r}}}=O(n^{1-\frac{1}{r}}\log n)$. Hence we have the following inequalities

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E||(V''_{ij}||}{(ij)^{\frac{1}{r}}}\le C\sum_{k=1}^{\infty}\frac{d_k}{k^{\frac{1}{r}}}\int_{k^{\frac{1}{r}}}^{\infty}|x|dF(x)\le C\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}\frac{d_i}{i^{\frac{1}{r}}}\right)\int_{i^{\frac{1}{r}}}^{(i+1)^{\frac{1}{r}}}|x|dF(x)$$

$$\leq C\sum_{i=1}^{\infty} i^{1-\frac{1}{r}}\log i \int\limits_{i^{\frac{1}{r}}}^{(i+1)^{\frac{1}{r}}} |x|dF(x) \leq C\int\limits_{1}^{\infty}|x|^r\log^+|x|^r dF(x) \leq CE||V||^r\log^+||V||^r < \infty.$$

This implies that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E||(V''_{ij}||^p}{(ij)^{\frac{p}{r}}} < \infty. \tag{3.10}$$

(3.9) and (3.10) yield

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E||(V_{ij}||^p}{(ij)^{\frac{p}{r}}} < \infty.$$

Applying Theorem 3.1 with $\alpha = \beta = \dfrac{1}{r}$ we obtain (3.8). □

Theorem 3. *Let* $0 < r < 1$. *If* $\{V_{mn}; m \geq 1, n \geq 1\}$ *is be stochastically dominated by a random element* V *such that* $E||V||^r\log^+||V||^r < \infty$, *then*

$$\frac{1}{(mn)^{\frac{1}{r}}}\sum_{i=1}^{m}\sum_{j=1}^{n}V_{ij} \to 0 \quad a.s.\ and\ in \quad L_1\ as \quad m\vee n\to\infty. \tag{3.11}$$

Proof. By (2.3) of Lemma 2.1 with $\gamma = \dfrac{1}{r}$ we obtain $\sum_{k=i+1}^{\infty}\dfrac{d_k}{k^{\frac{1}{r}}} = O\left(\dfrac{\log i}{(i+1)^{\frac{1}{r}-1}}\right)$. Hence we can show that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E||V_{ij}||}{(ij)^{\frac{1}{r}}} \leq E||V||^r\log^+||V||^r < \infty.$$

Applying Theorem 3.1 with $\alpha = \beta = \dfrac{1}{r}$ we obtain (3.11). □

In the next sections we present corollaries to the theorems.

4. MARTINGALE TYPE p BANACH SPACE CASE

The fact the a collection of a martingale difference random elements $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ taking values in martingale type p satisfy the maximal Marcinkiewicz-Zygmund inequality with exponent p, follows from the lemma below. We denote $\mathcal{F}_{kl}$ the σ-field generated by the family of random elements $\{V_{ij}; i < k \text{ or } j < l\}$, $\mathcal{F}_{1,1} = \{\emptyset; \Omega\}$.

Lemma 2. *Let* $1 \leq p \leq 2$ *and let* $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ *be a collection of* mn *random elements in a real separable martingale type* p *Banach space with* $E(V_{ij}|\mathcal{F}_{ij}) = 0$ *for all* $1 \leq i \leq m, 1 \leq j \leq n$. *Then* $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ *satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent* p.

Proof. The conclusion (2.1) is trivial in the case of $p = 1$. Hence we consider the case of $1 < p \leq 2$. Set $S_{kl} = \sum_{i=1}^{k}\sum_{j=1}^{l}V_{ij}, Y_l = \max_{1\leq k\leq m}||S_{kl}||$. If σ_l is a σ-field generated by $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq l\}$ then for each $l (1 \leq l \leq n)$, $\sigma_l \subset \mathcal{F}_{i,l+1}$ for all $i \geq 1$. This implies that $E(V_{i,l+1}|\sigma_l) = E(E(V_{i,l+1}|\mathcal{F}_{i,l+1})|\sigma_l) = 0$.

Thus, we have

$$E(S_{k,l+1}|\sigma_l) = E(S_{kl}|\sigma_l) + \sum_{i=1}^{k}E(V_{i,l+1}|\sigma_l) = S_{kl},$$

that is, $\{S_{kl}, \sigma_l; 1 \le l \le n\}$ is a martingale. Hence, $\{||S_{kl}||, \sigma_l; 1 \le l \le n\}$ is a nonnegative submartingale for each $k = 1, 2, \ldots, m$. By Lemma 2.2 of Thanh [13] it follows that $\{Y_l, \sigma_l; 1 \le l \le n\}$ is a nonnegative submartingale.

Applying Doob's inequality (see, e.g., Chow and Teicher [1, p. 255]), we obtain

$$E\left(\max_{\substack{1\le k\le m\\1\le l\le n}} ||S_{kl}||^p\right) = E\left(\max_{1\le l\le n} Y_l\right)^p \le CEY_n^p. \tag{2.2}$$

On the other hand, since $E(V_{ij}|\mathcal{F}_{ij}) = 0$ we have that $\{S_{kn}, \mathcal{G}_k = \mathcal{F}_{k+1,1}; 1 \le k \le m\}$ is a martingale. Thus

$$EY_n^p = E\max_{1\le k\le m} ||S_{kn}||^p \le C\sum_{k=1}^m E||\sum_{j=1}^n V_{kj}||^p. \tag{2.3}$$

Note again that for each $k(1 \le k \le m)$, $\{\sum_{j=1}^l V_{kj}, \mathcal{G}_{kl} = \mathcal{F}_{k,l+1}; 1 \le l \le n\}$ is a martingale. Hence,

$$E||\sum_{j=1}^n V_{kj}||^p \le E\max_{1\le l\le n} ||\sum_{j=1}^l V_{kj}||^p \le C\sum_{l=1}^n E||V_{kl}||^p. \tag{2.4}$$

Combining (2.2), (2.3) and (2.4) we obtain (2.1). □

The following theorem characterizes the martingale type p Banach spaces.

Theorem 4. *Let $\{V_{mn}; m \ge 1, n \ge 1\}$ be an array of random elements in a real separable Banach space $\mathcal{X}$. Then the following two statements are equivalent*:

(i) The Banach $\mathcal{X}$ is of martingale type p.

(ii) For every double arrays $\{V_{mn}; m \ge 1, n \ge 1\}$ of random elements in $\mathcal{X}$ with $E(V_{mn}|\mathcal{F}_{mn}) = 0$ for all $m \ge 1, n \ge 1$, the condition

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{E||V_{mn}||^p}{(m^\alpha n^\beta)^p} < \infty$$

for some $\alpha > 0$ and $\beta > 0$, implies that

$$\frac{1}{m^\alpha n^\beta}\sum_{i=1}^m\sum_{j=1}^n V_{ij} \to 0 \quad a.s. \quad as \quad m\vee n\to\infty.$$

Proof. First we prove $[(i) \Rightarrow (ii)]$.

The case of $1 < r < p \le 2$ follows from Lemma 5.1 and Theorem 3.1. Hence we only prove for the case of $r = 1$. Applying (2.3) from Lemma 2.1 with $\gamma = p$, we obtain $\sum_{k=i+1}^\infty \frac{d_k}{k^p} = O\left(\frac{\log i}{(i+1)^{p-1}}\right)$. Hence we have

$$\sum_{i=1}^\infty\sum_{j=1}^\infty \frac{E||V'_{ij} - E(V'_{ij}|\mathcal{F}_{ij})||^p}{(ij)^p} \le C\sum_{i=1}^\infty\sum_{j=1}^\infty \frac{E||V'_{ij}||^p}{(ij)^p} \le C\sum_{k=1}^\infty \frac{d_k}{k^p}\int_0^k x^p dF(x)$$

$$\le C\sum_{i=0}^\infty\left(\sum_{k=i+1}^\infty \frac{d_k}{k^p}\right)\int_i^{(i+1)} x^p dF(x) \le C\sum_{i=0}^\infty \frac{\log i}{(i+1)^{p-1}}\int_i^{(i+1)} x^p dF(x) \le CE||V||\log^+||V|| < \infty.$$

By Theorem 3.1 with $\alpha = \beta = 1$

$$\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n (V'_{ij} - E(V'_{ij}|\mathcal{F}_{ij})) \to 0 \quad \text{a.s.} \quad \text{as} \quad m\vee n\to\infty.$$

Next, for arbitrary $\varepsilon > 0$,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P(||V''_{ij}|| > \varepsilon) \leq \sum_{k=1}^{\infty} d_k P(||V|| > k) = \sum_{i=1}^{\infty}\left(\sum_{k=1}^{i} d_k\right)\int_{i}^{i+1} dF(x)$$

$$\leq C\sum_{i=1}^{\infty} i\log i \int_{i}^{i+1} dF(x) \leq CE||V||\log^+||V|| < \infty.$$

Hence by the Borel-Cantelli lemma

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} V''_{ij} \to 0 \quad \text{a.s.}$$

Finally, by the identity

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} V_{ij} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(V'_{ij} - E(V'_{ij}|\mathcal{F}_{ij}) + \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} V''_{ij} + \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} E(V''_{ij}|\mathcal{F}_{ij}),$$

it it enough to show that

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(E(V''_{ij}|\mathcal{F}_{ij}) \to 0 \quad \text{a.s.} \quad \text{as} \quad m \vee n \to \infty. \tag{4.4}$$

The proof of (4.4) is the same as that of $\sum_{i=1}^{m}\sum_{j=1}^{n}[E(X'_{ij}|\mathcal{F}_{ij}) - E(X''_{ij}|\mathcal{F}_{ij})] \to 0$ a.s. as $m \vee n \to \infty$ of Hong and Volodin for $p = 1$ (pp. 1139–1140) with noting that

$$||E(V''_{ij}|\mathcal{F}_{ij})|| \leq E(||V''_{ij}|||\mathcal{F}_{ij}) \leq 2E(||V||I(||V|| > ij)|\mathcal{F}_{ij}),$$

and we use $||V''_{ij}||$ and$||V||$ are instead of $|X''_{ij}|$ and $|X|$, respectively.

Now we prove $[(ii) \Rightarrow (i)]$. Assume that (ii) holds. Let $\{W_n, \mathcal{G}_n; n \geq 1\}$ be an arbitrary sequence of martingale difference in $\mathcal{X}$ such that

$$\sum_{n=1}^{\infty}\frac{E||W_n||^p}{n^p} < \infty$$

For $n \geq 1$, set

$$V_{mn} = W_n \quad \text{if} \quad m = 1 \quad \text{and} \quad V_{mn} = 0 \quad \text{if} \quad m \geq 2.$$

Then $\{V_{mn}; m \geq 1, n \geq 1\}$ is a double array of random elements in in $\mathcal{X}$ satisfies $E(V_{mn}|\mathcal{F}_{mn}) = 0$ for all $m \geq 1, n \geq 1$, and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{E||V_{mn}||^p}{(mn)^p} = \sum_{n=1}^{\infty}\frac{E||W_n||^p}{n^p} < \infty.$$

By (ii),

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} V_{ij} \to 0 \quad \text{a.s.} \quad \text{as} \quad m \vee n \to \infty.$$

Taking $m = 1$ and letting $n \to \infty$ we obtain

$$\frac{1}{n}\sum_{j=1}^{n} W_j \to 0 \quad \text{a.s.} \quad \text{as} \quad n \to \infty.$$

Then by Theorem 2.2 of Hoffmann-Jørgensen and Pisier [4], $\mathcal{X}$ is of martingale type p. □

The next corollary follows from Theorems 3.2 and 3.3.

Corollary 1. Let $1 \leq r < p \leq 2$ and $\mathcal{X}$ be a martingale type p Banach space. Suppose that $\{V_{mn}; m \geq 1, n \geq 1\}$ is be stochastically dominated by a random element V such that $E||V||^r \log^+ ||V||^r < \infty$ if $1 \leq r < p$

$$\frac{1}{(mn)^{\frac{1}{r}}} \sum_{i=1}^{m} \sum_{j=1}^{n} (V_{ij} - E(V_{ij}|\mathcal{F}_{ij})) \to 0 \quad \text{a.s.} \quad \text{and} \quad \text{in} \quad L_r \text{ as } m \vee n \to \infty.$$

5. RADEMACHER TYPE p BANACH SPACE CASE

The fact that a collection $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ of mn independent mean zero random elements spaces taking values in Rademacher type p satisfy the maximal Marcinkiewicz-Zygmund inequality with exponent p, follows from the following lemma. For the proof we refer to Lemma 2.3. of Thanh [13], while it is obvious.

Lemma 3. *Let $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ be a collection of mn independent mean 0 random elements in a real separable Rademacher type p Banach spaces. Then $\{V_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$ satisfies the maximal Marcinkiewicz-Zygmund inequality with exponent p.*

The following theorem can be proved in the same way as Theorem 4.2 and hence we omit its proof.

Theorem 5. *Let $\{V_{mn}; m \geq 1, n \geq 1\}$ be an array of independent random elements in a real separable Banach space $\mathcal{X}$. Then the following two statements are equivalent:*

(i) The Banach $\mathcal{X}$ is of Rademacher type p.

(ii) For every double arrays $\{V_{mn}; m \leq 1, n \leq 1\}$ of independent mean 0 random elements in $\mathcal{X}$, the condition

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{E||V_{ij}||^p}{(i^\alpha j^\beta)^p} < \infty$$

for some $\alpha > 0$ and $\beta > 0$ implies that

$$\frac{1}{m^\alpha n^\beta} \sum_{i=1}^{m} \sum_{j=1}^{n} V_{ij} \to 0 \quad a.s. \quad as \quad m \vee n \to \infty.$$

Note that the above result is more general than Theorem 3.1 (necessity part) of Rosalsky and Thanh [11].

An open problem. A relatively interesting case $r = 1$ is not considered in Theorem 3.2, while we propose that the result remains true in this case.

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