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The Topp–Leone Discrete Laplace Distribution and Its Applications

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Abstract—A new Topp–Leone generated family of distributions, which we call the Topp–Leone Discrete Laplace ($TL-DL$) distribution, is proposed. It has a shape parameter $\alpha > 0$ and a scale parameter $0 < p < 1$. The $TL-DL$ is an alternative distribution for discrete data that have an asymmetric distribution. Some mathematical properties of the proposed distribution are also derived. Namely, we present the quantile function and the moments for the $TL-DL$ distribution. The Maximum Likelihood procedure is applied for parameter estimation. An application study is presented using real data. We use two data sets for this part of the analysis to illustrate the applications of the $TL-DL$ distribution. For the first data set, the change of the stock price in comparison with the closing price for the previous day is considered. The second data set provides information about the comparison of production cycle times of employees before and after the improvement a slippery production line in the degreasing alkaline process by increasing the pressure of the nozzle. The $TL-DL$ distribution is applied to a real life data and it fits data more efficiently than the Discrete Laplace (DL) and Discrete Normal (DN) distributions.

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1. INTRODUCTION

The characterization of a probability distribution plays an important role in statistics and mathematical sciences. Before a model based on a particular probability distribution can be applied to fit real world data, it is essential to confirm whether the given probability distribution satisfies the underlying requirements of its characterization. Probability distributions are commonly used to describe real world phenomena. Their theory is widely studied and new distributions are developed. Many classes of distributions have been developed and applied to describe data in numerous fields of research, such as biology, economics, forestry, genetics, medicine, psychology, reliability, etc; for example see Alzaatreh et al. (2013) [1] and Ahsanullah and Shakil (2014) [2].

Eugene et al. (2002) [3] presented how to develop a new generalized class of distributions, which they called the beta-G distribution, based on the generated distribution of the beta distribution. Subsequently, some well-known G-distributions, for which G is a parent distribution, were developed; these include the exponential [16], Weibull [5], and gamma [6] distributions.

Recently, Alzaatreh et al. (2013) [1] and Kong et al. (2007) [6], proposed the technique of using the quantile function to construct a new distribution, the $T-X$ family, where T is a continuous random

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variable from the generator distribution and X is a parent distribution, which can be a continuous or discrete distribution. When X is continuous distribution, the resulting $T-X$ distribution family is a continuous distribution, too. Similarly, when X is a discrete distribution, the resulting $T-X$ distribution family is a discrete, too.

The Topp–Leone (TL) distribution was proposed by Topp and Leone (1955) [7]. The distribution is constructed for empirical data with a J-shaped histogram such as power tool failures, and automatic calculating machine failure. Many authors proposed $TL-G$ distributions such as the TL generalized exponential [8] and the TL exponential [9] distributions.

The Topp–Leone generator ($TL-G$) of distributions was investigated only for the continuous case until Rezaei et al. (2017) applied a discrete parent distribution to the $T-X$ family of distributions [10]. Namely, the T-geometric family containing the discrete analogues of continuous distributions. Alzaatreh et al. (2013) [1] defined this method as the discrete analogue of continuous distributions and suggested that their proposed distribution can be performed in the various fields of discrete data. Consequently, we propose the $T-X$ distribution family, where T has TL distribution and X has a discrete distribution.

Discrete distributions are useful in many applications, such as the difference between the self-reported and true number of visits to a doctor, the difference in chromosomal counts in healthy and diseased individuals, the difference in the counts of retail outlets before and after particular periods, counts for economical indicators of a country development, stock price change data, and currency exchange rates during financial transactions [11].

The theory and applications of discrete distributions have been widely studied and the literature on discrete distributions taking values in the set of integer numbers is exceptionally large. Examples of discrete distributions that can be applied for the asymptotic analysis of a data set are: the Discrete Normal (DN) distribution [11], the Discrete Laplace (DL) [12], and the Discrete Analogue of a Laplace (double exponential) distributions [13].

The Discrete Laplace distribution is closely connected with the Geometric distribution. Namely, the distribution of a discrete Laplace random variable is equal to the distribution of the difference of two independent and identically distributed geometric random variables. For applications of the Geometric distribution we note that the hydro climatic episodes such as droughts, floods, warm spells and cold spells are commonly quantified in terms of their duration and magnitude. The durations of episodes above and below the reference level, known as positive and negative episodes, respectively, are frequently modeled by the Geometric distribution (see [12]).

This article is organized as follows. In Section 2, we introduce some preliminary results that are important for our discussion. In Section 3, the new $TL-G$ distribution family, namely, the Topp–Leone Discrete Laplace distribution is proposed. Some probability properties of the proposed distribution are discussed, and the maximum likelihood method is used to estimate the unknown parameters of the Topp–Leone Discrete Laplace distribution. Finally, an application study of the proposed distribution is illustrated.

2. PRELIMINARY RESULTS

Here, we introduce and give examples of a discrete distribution such as the DN distribution [11] and the DL distribution [12].

2.1. The DN Distribution

We say that a random variable X has the DN distribution with parameters μ and σ, denoted as $X \sim DN(\mu,\sigma)$, if its probability mass function (pmf) is

$$g(x;\mu,\sigma) = P(X=x) = \Phi\left(\frac{x+1-\mu}{\sigma}\right) - \Phi\left(\frac{x-\mu}{\sigma}\right); x \in \mathrm{I}, \tag{2.1}$$

where $\mathrm{I} = \{\dots,-2,-1,0,1,2,\dots\}$, $-\infty < \mu < \infty$, $\sigma > 0$, and $\Phi(\cdot)$ is the standard normal cumulative distribution function (cdf). The pmf plots of the DN distribution in (2.1) are shown in Figure 1.

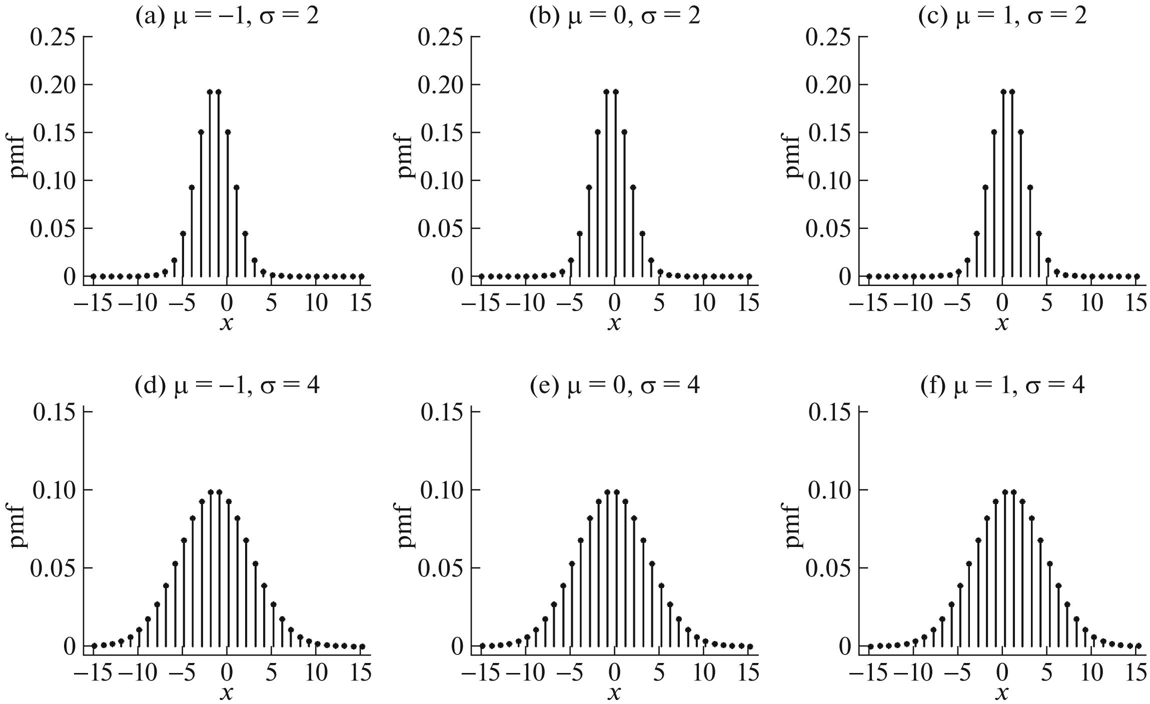

Fig. 1. The probability mass function plots of DN distribution with the specified parameters of μ and σ.

2.2. The DL Distribution

The DL distribution was proposed by Inusah and Kozubowski (2003) [13]. We say that a random variable X has the DL distribution with the parameter p, denoted as $X \sim DL(p)$, if its pmf is

$$g_{DL}(x;p) = P(X = x) = \frac{1-p}{1+p}p^{|x|}; x \in \mathrm{I}, \tag{2.2}$$

where $\mathrm{I} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and $0 < p < 1$. The associated cdf of X is

$$G_{DL}(x;p) = \begin{cases} \frac{p^{-x}}{1+p}; & x = -1, -2, \dots \\ 1 - \frac{p^{x+1}}{1+p}; & x = 0, 1, 2, \dots \end{cases} \tag{2.3}$$

Some plots of the DL's pmf from (2.2) are shown in Figure 2.

The moment generating function of the DL distribution (2.2) is

$$M_{DL}(t;p) = \frac{(1-p)^2}{(1-p\exp(t))(1-p\exp(-t))}; \quad \log(p) < t < -\log(p).$$

The mean and variance of the DL distribution are, respectively

$$\mu_{DL} = \frac{2p}{(1-p)(1+p)} \quad \text{and} \quad \sigma^2_{DL} = \frac{2p}{(1-p)^2}.$$

Since the DL distribution is a symmetric discrete distribution with cdf (2.3), we have the quantile function of the DL distribution, i.e.,

$$Q_{DL}(u;p) = \begin{cases} -\left\lfloor \frac{\log(1+p)+\log(u)}{\log(p)} \right\rfloor; & 0 \le u < 0.5, \\ \left\lfloor \frac{\log(1+p)+\log(1-u)}{\log(p)} \right\rfloor - 1; & 0.5 \le u \le 1, \end{cases} \tag{2.4}$$

where $\lfloor \cdot \rfloor$ is the floor function, that is the greatest interger less than or equal to the argument.

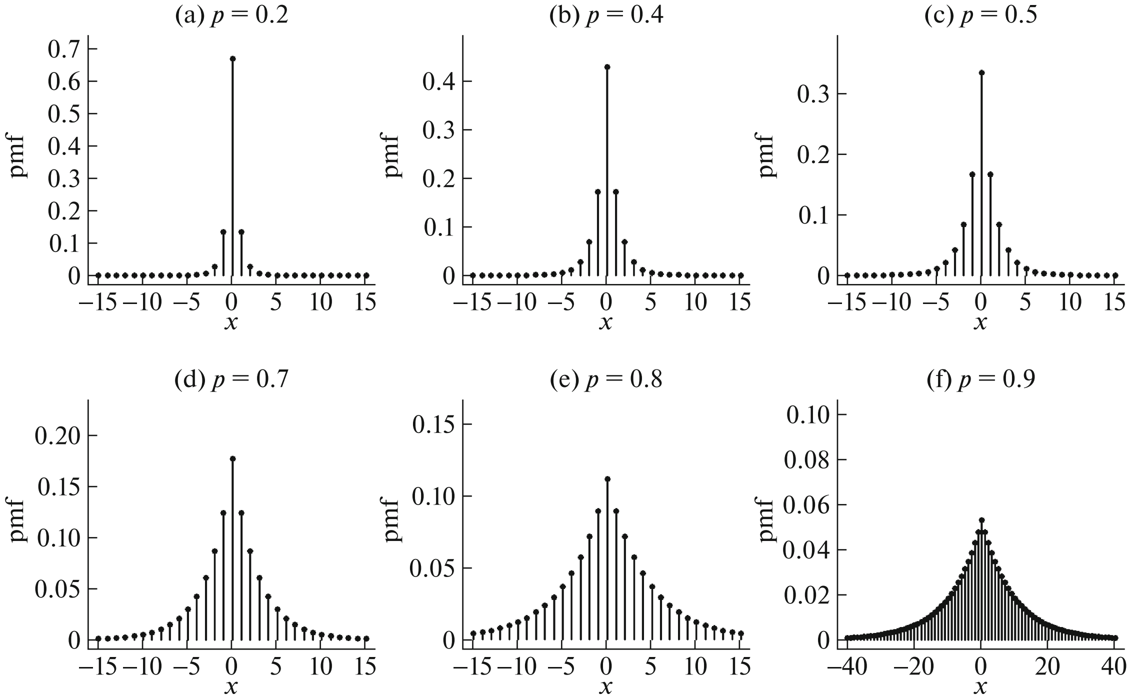

Fig. 2. Some pmf plots of the DL-distribution with the specified parameter p.

2.3. The $TL - G$ Family of Distributions

In order to define and investigate the $TL - G$ family of distributions, the TL distribution should be introduced. We say that a random variable T has the TL distribution with the parameter $\alpha > 0$, denoted as $T \sim TL(\alpha)$, if its pdf and cdf respectively are

$$f(t;\alpha) = 2\alpha t^{\alpha-1}(1-t)(2-t)^{\alpha-1} \quad \text{and} \quad F(t;\alpha) = t^{\alpha}(2-t)^{\alpha},$$

where $0 < t < 1$ and $\alpha > 0$.

Note that in general TL distribution can be defined with a support $[0, b]$ for $b < \infty$ (see [1, 9, 10, 14, 15]). In this article we focus primarily on the TL distribution with support $[0, 1]$ to avoid any additional function for creating a new generated family of distributions.

The quantile function of the TL distribution is (see [16])

$$Q(u;\alpha) = F^{-1}(u;\alpha) = 1 - \sqrt{1 - u^{1/\alpha}}, \tag{2.5}$$

where $0 \leq u \leq 1$.

The $TL - G$ family of distributions, that is the creating a new family of distributions, requires two principal components: a generator and a parent distribution. The probability density function of the generator is transformed into a new pmf through the cdf $G(x;\theta)$ of the parent distribution with the parameter θ as follows. Let T be a TL distributed random variable with the parameter α (generator) and X be a discrete random variable (parent distribution with the parameter of θ) that has the cdf of $G(x;\theta)$. The cdf of the $TL - G$ distribution is defined as

$$F(x;\alpha,\theta) = G^{\alpha}(x;\theta)\,(2 - G(x;\theta))^{\alpha},$$

where $\alpha > 0$ is a shape parameter (see [9, 10, 16]).

If $Q_G(u;\theta)$ is the quantile function of a parent distribution, then according to (2.5), the quantile function of the $TL - G$ distributed random variable X is:

$$Q_X(u) = Q_G\left(1 - \sqrt{1 - u^{1/\alpha}}\right), \tag{2.6}$$

where $0 \leq u \leq 1$.

The moments of the $TL-G$ distribution can be computed by the probability weighted moments of order (s,r) of the parent distribution. The (s,r) probability weighted moment of X is

$$E\left[X^s G^r(X)\right] = \tau_{s,r} = \int_{-\infty}^{\infty} x^s G^r(x) g(x)dx = \int_{0}^{1} Q_G^s(u) u^r dx,$$

where $-\infty < s < \infty$. In the case when parameter α is a positive integer, the moments of $TL-G$ distribution can be calculated by the formula:

$$E(X^s) = \sum_{i=1}^{\alpha} \binom{\alpha}{i} 2^{\alpha-i}(\alpha+i)\tau_{s,\alpha+i-1}, \tag{2.7}$$

see Topp and Leone (1955) [7]; Vicaria et al. (2008) [17].

3. RESULTS AND DISCUSSION

3.1. A New $TL-G$ Distribution

In this section, we propose the Topp–Leone Discrete Laplace $TL-DL$ distribution. We consider DL distribution as the generator. We present the pmf and cdf of the proposed distribution in Theorem 1.

Theorem 1. *Let X be a random variable which has the Topp–Leone Discrete Laplace ($TL-DL$) distribution with parameters $\alpha > 0$ and $0 < p < 1$, denoted as $X \sim TL-DL(\alpha, p)$. The cdf of X is*

$$F_{TL-DL}(x;\alpha,p) = \begin{cases} \left(\frac{p^{-x}}{1+p}\right)^{\alpha}\left(2-\frac{p^{-x}}{1+p}\right)^{\alpha}; & x=-1,-2,\ldots, \\ \left(1-\frac{p^{x+1}}{1+p}\right)^{\alpha}\left(1+\frac{p^{x+1}}{1+p}\right)^{\alpha}; & x=0,1,2,\ldots. \end{cases}$$

The pmf of X is

$$f_{TL-DL}(x;\alpha,p) = \begin{cases} \left(\frac{p^{-1-x}}{1+p}\right)^{\alpha}\left(2-\frac{p^{-1-x}}{1+p}\right)^{\alpha} - \left(\frac{p^{-x}}{1+p}\right)^{\alpha}\left(2-\frac{p^{-x}}{1+p}\right)^{\alpha}; & x=-1,-2,\ldots, \\ \left(1-\frac{p^{2(x+2)}}{(1+p)^2}\right)^{\alpha} - \left(1-\frac{p^{2(x+1)}}{(1+p)^2}\right)^{\alpha}; & x=0,1,2,\ldots. \end{cases}$$

The plots of pmf $TL-DL$ distribution are shown in Figure 3.

From the $TL-G$ quantile function (2.6) and the DL quantile function (2.4), we have the quantile function of the $TL-DL$ distribution:

$$Q_{TL-DL}(u;\alpha,p) = \begin{cases} -\left\lfloor \frac{\log(1-\sqrt{1-u^{1/\alpha}})+\log(1+p)}{\log(p)} \right\rfloor; & 0 \le u < 0.5, \\ \left\lfloor \frac{\log(1-\sqrt{1-u^{1/\alpha}})+\log(1+p)}{\log(p)} \right\rfloor - 1; & 0.5 \le u \le 1, \end{cases}$$

where $\lfloor\cdot\rfloor$ is the floor function. In the case when α is a positive integer, moments of the $TL-DL$ distribution can be calculated by formula (2.7) with

$$\tau_{s,\alpha+i-1} = \int_0^{0.5}\left\{-\frac{\log(1+p)+\log(1-\sqrt{1-u^{1/\alpha}})}{\log(p)}\right\}^s u^{\alpha+i-1}du$$
$$+\left\{\int_{0.5}^{1}\frac{\log(\sqrt{1-u^{1/\alpha}})+\log(1+p)}{\log(p)}-1\right\}^s u^{\alpha+i-1}du.$$

Table 1. Comparison of the observed and expected values of the change of the stock price compared to the closing price of the previous day

Class	Observed frequency	Expected frequency		
		DN	DL	TL − DL
< -6	4	5.61	9.61	7.08
−6	2	3.42	3.33	3.16
−5	8	4.85	4.47	4.44
−4	9	6.51	6.02	6.15
−3	10	8.27	8.1	8.35
−2	12	9.95	10.91	11.05
−1	9	11.33	14.68	14.14
0	14	12.23	19.75	17.27
1	19	12.49	14.68	14.51
2	7	12.07	10.91	11.65
3	10	11.05	8.1	9.09
4	9	9.58	6.02	6.95
5	4	7.87	4.47	5.25
6	6	6.12	3.33	3.92
7	4	4.5	2.47	2.91
8	2	3.14	1.84	2.15
9	3	2.07	1.36	1.58
> 9	2	2.94	3.95	4.35
Parameter estimates		$\hat{\mu} = 0.3852$	$\hat{p} = 0.7431$	$\hat{p} = 0.8543$
		$\hat{\sigma} = 4.2705$		$\hat{\alpha} = 2.6247$
D value of KS test		0.0970	0.0672	0.0373
$\log(L)$		387.5391	391.5113	388.7429
AIC		779.0782	785.0226	781.4858
BIC		779.3324	785.1497	781.7400

3.2. Parameter Estimation of the $TL - DL$ Distribution

Here we present the maximum likelihood estimation (MLE) of the parameters for the $TL - DL$ distribution. The likelihood function of the $TL - DL(\alpha, p)$ distribution is given by:

$$L(\alpha, p) = \prod_{i=1}^{n} \left\{ I_{(x_i<0)} \left[\left(\frac{p^{-1-x_i}}{1+p} \right)^{\alpha} \left(2 - \frac{p^{-1-x_i}}{1+p} \right)^{\alpha} - \left(\frac{p^{-x_i}}{1+p} \right)^{\alpha} \left(2 - \frac{p^{-x_i}}{1+p} \right)^{\alpha} \right] \right.$$
$$\left. + I_{(x_i \geq 0)} \left[\left(1 - \frac{p^{2(x_i+2)}}{(1+p)^2} \right)^{\alpha} - \left(1 - \frac{p^{2(x_i+1)}}{(1+p)^2} \right)^{\alpha} \right] \right\}.$$

The corresponding log-likelihood function is

$$\log L(\alpha, p; x) = \sum_{i=1}^{n} \left\{ I_{(x_i<0)} \left[\log \left(\left(\frac{p^{-1-x_i}}{1+p} \right)^{\alpha} \left(2 - \frac{p^{-1-x_i}}{1+p} \right)^{\alpha} - \left(\frac{p^{-x_i}}{1+p} \right)^{\alpha} \left(2 - \frac{p^{-x_i}}{1+p} \right)^{\alpha} \right) \right] \right.$$

Table 2. Comparison of the observed and expected values of the production cycle times (unit: seconds) of 133 employees before and after improvement of the production line

Time difference	Observed frequency (number of employees)	Expected frequency		
		DN	DL	TL − DL
< -5	10	6.08	10.98	7.47
-5	4	4.07	4.22	3.83
-4	6	5.87	5.84	5.60
-3	8	7.92	8.08	8.00
-2	11	10.01	11.18	11.11
-1	14	11.85	15.48	14.84
0	17	13.13	21.43	18.75
1	15	13.62	15.48	15.69
2	12	13.23	11.18	12.42
3	9	12.03	8.08	9.49
4	7	10.24	5.84	7.09
5	5	8.16	4.22	5.22
6	4	6.09	3.05	3.80
7	3	4.25	2.20	2.75
8	2	2.78	1.59	1.98
9	2	1.70	1.15	1.42
> 9	4	1.98	2.99	3.53
Parameter estimates		$\hat{\mu} = 0.3852$, $\hat{\sigma} = 4.2705$	$\hat{p} = 0.7431$	$\hat{p} = 0.8543$, $\hat{\alpha} = 2.6247$
D value of KS test		0.0902	0.0556	0.0226
$\log(L)$		369.1972	373.4848	370.2500
AIC		742.3944	748.9696	744.5000
BIC		742.6421	749.0935	744.7477

$$+ I_{(x_i \geq 0)} \left[\log \left(\left(1 - \frac{p^{2(x_i+2)}}{(1+p)^2} \right)^{\alpha} - \left(1 - \frac{p^{2(x_i+1)}}{(1+p)^2} \right)^{\alpha} \right) \right] \Bigg\} .$$

To estimate the unknown parameters θ and α, we take the partial derivatives of the log-likelihood function $\log L(\alpha, p; x)$ with respect to α and p and equate them to zero. That is, the score equations are $\frac{\partial \log L(\alpha,p;x)}{\partial \alpha} = 0$, $\frac{\partial \log L(\alpha,p;x)}{\partial p} = 0$.

It is not possible to solve these equations in closed form. Therefore, a simple iterative numerical procedure to approximate the MLE solutions can be used. The MLE solutions $\hat{\alpha}$ and $\hat{p}$ can be obtained by solving the score equations simultaneously using the *nlm* function in the statistical software package R [18].

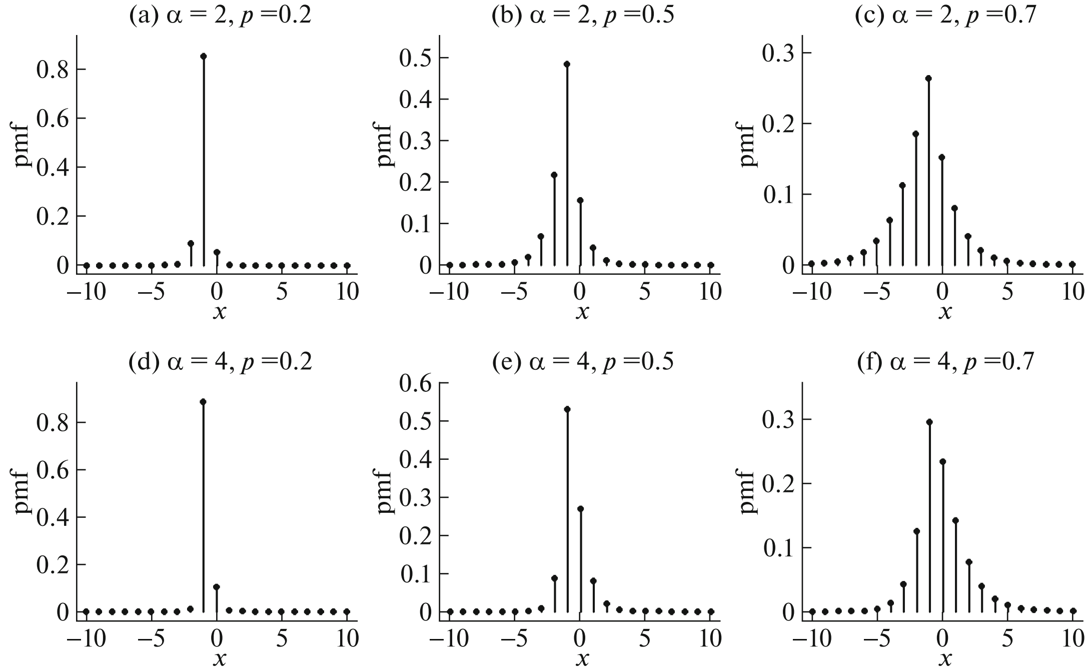

Fig. 3. Some pmf plots of the $TL-DL$ distribution for the specified parameters α and p.

4. APPLICATIONS STUDY

We used two data sets for this part of the analysis to illustrate the applications of the $TL-DL$ distribution. For the first data set, the change of the stock price (unit: Baht) data is recorded (Table 4). The data provide the change of the stock price in comparison with the closing price for the previous day. It was obtained from the Petroleum Authority of Thailand Public Company Limited (PTT) and was recorded for a period from April 1, 2014 to October 20, 2014. It can be seen on the official website of the Stock Exchange of Thailand (SET) PTT Public Company Limited (see [19]). The second data set provides information about the comparison of production cycle times (unit: seconds) of 133 employees [20] before and after the improvement a slippery production line in the degreasing alkaline process by increasing the pressure of the nozzle from 0.3 mpa to 0.4 mpa (Table 4).

We used the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) for the comparison of the considered distributions in order to decide which one fits our data better. Given a collection of models for the data, these criteria estimate the quality of each model, relative to each of the other models. Suppose that we have a statistical model of some data. Let k be the number of estimated parameters in the model, and n be the sample size. Let L be the maximum value of the likelihood function for the model. Then the AIC and BIC values of the model are: AIC $= -2\log(L) + 2k$, and BIC $= -2\log(L) + k\log(n)$. Moreover, the KS (Kolmogorov–Smirnov) test can be modified to serve as a goodness of fit test for the application study. The empirical distribution function $F_n(x)$ based on n independent and identically distributed ordered observations X_i is defined as $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,x)}(X_i)$, where $I_{(-\infty,x)}(X_i)$ is the indicator function, which is equal to 1 if $X_i \leq x$ and equal to 0 otherwise. The KS statistic for a given cumulative distribution function $F(x)$ is $\mathrm{D} = \sup_x |F_n(x) - F(x)|$, where supremum is taken over all real numbers x.

From the results in Table 1 and Table 2, the AIC and BIC values for the DN distribution is smaller than for the $TL-DL$ and DL distributions, respectively. When the K-S test is considered, the $TL-DL$ distribution has a K-S value close to zero. The $TL-DL$ distribution has a K-S value less than the corresponding values for the DL and DN distributions. Thus, the observed values of the two data sets have a better fit by the $TL-DL$ distribution than by the DL and DN distributions (see Figure 4).

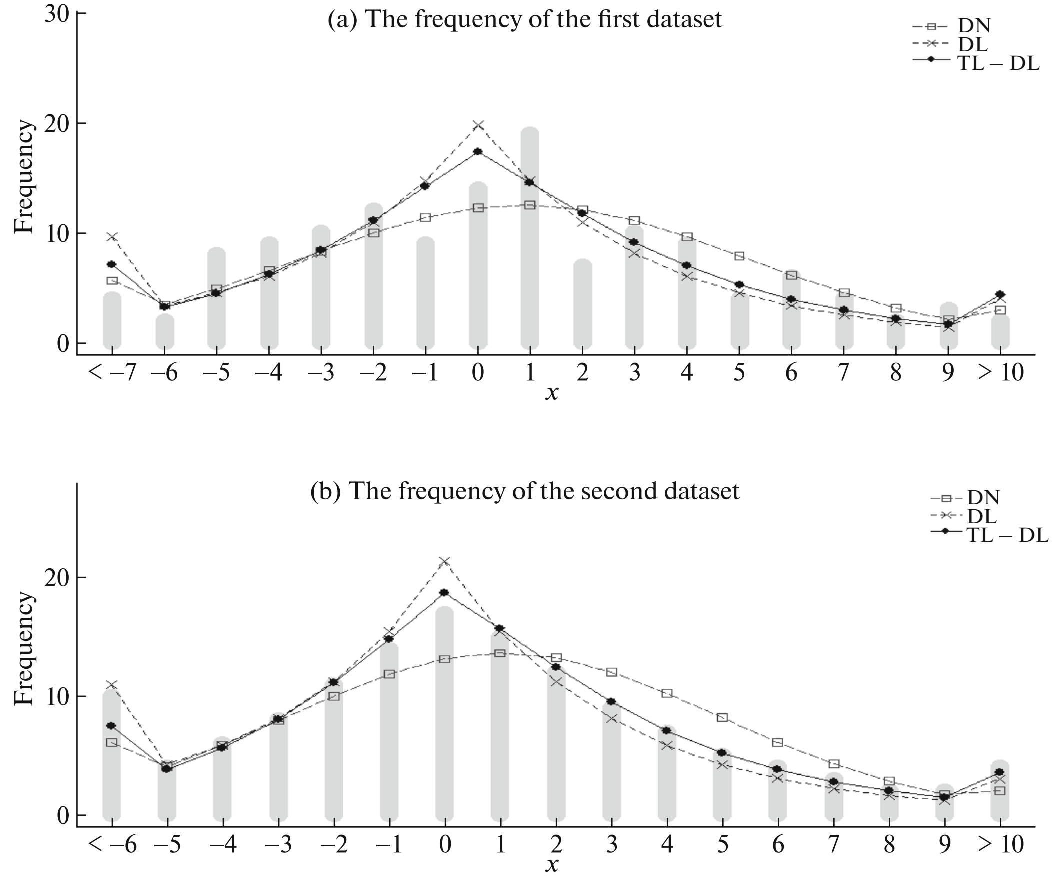

Fig. 4. Empirical and fitted distributions of the DN, DL, and $TL-DL$ distributions for two data sets.

5. CONCLUSION

In this article, we propose the Topp–Leone Discrete Laplace ($TL-DL$) distribution, which is a new $T-G$ distribution where T is distributed according to the Topp–Leone (TL) distribution and G is distributed according to the Discrete Laplace (DL) distribution. The $TL-DL$ distribution has two parameters: the shape parameter $\alpha > 0$ and the scale parameter $0 < p < 1$. The maximum likelihood estimation procedure is applied to estimate the $TL-DL$ parameters. Examples from two real data sets show the efficiency of the distribution for model fitting. We can conclude that the $TL-DL$ distribution is a flexible alternative for an analysis of the discrete data.

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