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Interval Estimation for the Shape and Scale Parameters of the Birnbaum–Saunders Distribution

N. Jantakoon[1*] **and A. Volodin**[2**]

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[1]*Department of Applied Statistics, Rajabhat Maha Sarakham University, Muang Maha Sarakham, 44000 Thailand*

[2]*Department of Mathematics and Statistics, University of Regina, Regina, SK. S4S 0A2 Canada*

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Abstract—Two-parameter Birnbaum–Saunders distribution has been widely studied in Reliability Theory due to its important Engineering applications. This article proposes a novel confidence intervals construction for the shape and scale parameters of the Birnbaum–Saunders distribution. We apply the following two methods: The generalized pivotal approach and the percentile bootstrap approach. The Monte Carlo simulations are used to evaluate the performance of the confidence intervals. We compare the coverage probability and average width of the proposed confidence intervals with already known. Simulation results have shown that the proposed confidence intervals perform well in terms of coverage probability and average length for various sample sizes. The illustrative example and some concluding remarks are finally presented.

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1. INTRODUCTION

The two-parameter Birnbaum–Saunders distribution was originally proposed by Birnbaum and Saunders [1] in 1969 as a lifetime distribution for fatigue failure caused under cyclic loading. This distribution is widely used as a lifetime distribution in the case when the failure is due to the development and growth of a dominant crack. The Birnbaum–Saunders distribution has been normally applied to reliability studies. This distribution has many interesting properties. For example, as a lifetime distribution it is positively skewed (asymmetry to the right) and a failure rate with upside-down bathtub shape [2, 3].

Because of important Engineering applications, many authors have considered estimation of its parameters. For the maximum likelihood estimation (MLE) we refer to Birnbaum and Saunders [1] and Englehardt et al. [4]. Moment estimations for the original parametrization of the Birnbaum–Saunders distribution were presented by Leiva et al. [5] and Balakrishnan et al. [6]. Note that the moment estimators may not always exist. In all of these cases, it is not possible to find explicit expressions for its estimators, so that numerical procedures must be used. For this reason, Ng et al. [7] introduced a modified moment (MME) method for estimating the parameters by an application of the bias-reduction method. From and Li [8] showed several estimation methods for the Birnbaum–Saunders distribution. Recent works on improving inference for this distribution are due to Lemonte et al. [9] and Cysneiros et al. [10]. We also refer to the article Ahmed et al. [11], where a new parametrization of the Birnbaum–Saunders distribution has been introduced and three types of the parameter estimation have been investigated: MLE, Method of Moments and Regression-Quantile.

*E-mail: nitaya.ja@rmu.ac.th

**E-mail: andrei.volodin@uregina.ca

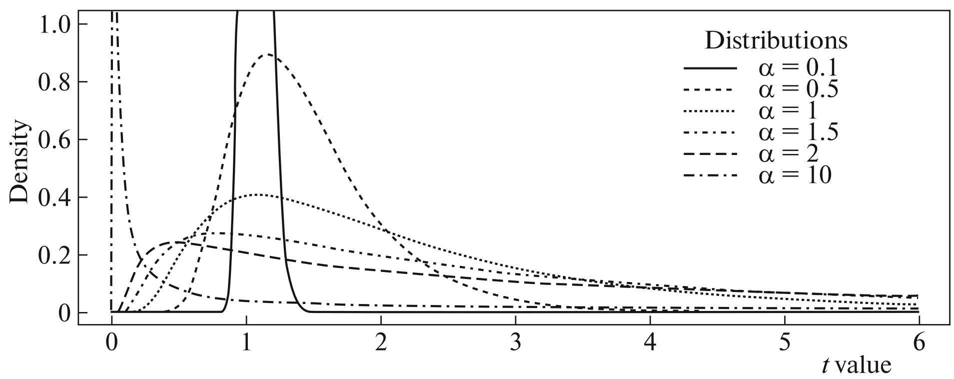

Fig. 1. Comparison of the Birnbaum–Saunders for some values α with $\beta = 1$.

It is well known that a point estimation has a disadvantage providing as a result only a single number and saying too much about the accuracy of the estimation. Contrary to this, a confidence interval provides additional information about the variability of the estimate. Therefore, an interval estimate provides more information about a population characteristic than does a point estimate. However, there are few researches that have investigated confidence intervals for parameters of the Birnbaum–Saunders distribution. For example, Ng et al. [7] and [12] proposed interval estimations based on the MLE's and MME's. The asymptotic distributions of the MLEs have been derived and have been used to construct asymptotic confidence intervals for the unknown parameters.

Therefore, in this article we are interested in new methods for a construction of confidence intervals for the shape and scale parameters of the Birnbaum–Saunders distribution. The first method of confidence interval construction is based on the generalized pivotal approach (GPA). The second method of confidence interval construction is based on the percentile bootstrap approach (PBA). Monte Carlo simulations have been carried out to examine the performance of the proposed methods comparing to MLE and MME methods.

The rest of this paper is organized as follows. In Section 2, we describe briefly the properties of Birnbaum–Saunders distribution. The methods of interval estimation for parameters are presented in Section 3. In Section 4, we provide Monte Carlo simulation results evaluating the performance of all methods. Finally, some concluding remarks are made in Section 5.

2. THE TWO-PARAMETER BIRNBAUM–SAUNDERS DISTRIBUTION

A continuous random variable T is said to have two-parameter *Birnbaum–Saunders distribution* (notation $T \sim BS(\alpha, \beta)$), if its probability density function is given by (see [1]):

$$f_T(t;\alpha,\beta) = \frac{1}{2\alpha\beta\sqrt{2\pi}}\left[\left(\frac{\beta}{t}\right)^{1/2} + \left(\frac{\beta}{t}\right)^{3/2}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)\right],$$

where $t > 0$, $\alpha > 0$ and $\beta > 0$.

The parameters α and β are the *shape* and *scale* parameters, respectively. The density of this distribution is skewed to the right. Nevertheless, the asymmetry of the distribution decreases with α. The Birnbaum–Saunders density for some values α with $\beta = 1$ is shown in Fig. 1.

The cumulative distribution function of the Birnbaum–Saunders distribution is:

$$F_T(t;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{1/2} - \left(\frac{t}{\beta}\right)^{1/2}\right\}\right], \quad t > 0, \quad \alpha > 0 \quad \text{and} \quad \beta > 0, \tag{1}$$

where $\Phi(\cdot)$ is the standard normal distribution function.

If $T \sim BS(\alpha, \beta)$ then the following monotone transformation

$$Z = \frac{1}{\alpha}\left[\left(\frac{T}{\beta}\right)^{1/2} - \left(\frac{T}{\beta}\right)^{-1/2}\right], \tag{2}$$

or

$$T = \beta\left(1 + 2Z^2 + 2Z\left(1+Z^2\right)^{1/2}\right);$$

transforms T to the normally distributed random variable Z with mean zero and variance $\frac{\alpha^2}{4}$. This fact easily follows from (1). Using the above transformation, and knowing moments of the standard normal random variable we can find moments of the Birnbaum–Saunders random variable T. Namely, for an integer r the following formulae have been derived in [13] and [14]:

$$E\left(T^r\right) = \beta^r \sum_{j=0}^{r} \binom{2r}{2j} \sum_{i=0}^{j} \binom{i}{j} \frac{(2r-2j+2i)!}{2^{r-j+i}(r-j+i)!} \left(\frac{\alpha}{2}\right)^{2r-2j+2i}. \tag{3}$$

From (3), the expected value, variance, and coefficients of skewness and kurtosis can be easily obtained as

$$E(T) = \beta\left(1+\frac{1}{2}\alpha^2\right), \quad \mathrm{Var}(T) = (\alpha\beta)^2\left(1+\frac{5}{4}\alpha^2\right),$$

$$\gamma = \frac{16\alpha^2\left(11\alpha^2+6\right)}{\left(5\alpha^2+3\right)^3}, \quad \kappa = 3 + \frac{6\alpha^2\left(93\alpha^2+41\right)}{\left(5\alpha^2+4\right)^2}.$$

The coefficient of variation of T given by

$$CV(T) = \tau = \frac{\sqrt{Var(T)}}{E(T)} = \frac{\sqrt{(\alpha\beta)^2\left(1+\frac{5}{4}\alpha^2\right)}}{\beta\left(1+\frac{1}{2}\alpha^2\right)} = \alpha\left(\frac{\sqrt{5\alpha^2+4}}{\alpha^2+2}\right),$$

which does not depend on the scale parameter β.

3. METHODS OF INTERVAL ESTIMATION FOR THE SHAPE AND SCALE PARAMETERS

The methods compared in this study fall into the following five categories. First two are based on the maximum likelihood estimators, next two are based on the moment estimators. These are the methods of confidence interval construction that have been known before. We suggest two new methods namely the approach based on the generalized pivotal method and the approach based on the percentile bootstrap method. Now we describe all 6 methods in more detail.

In the following we assume that $\mathbf{T} = \{T_1, \ldots, T_n\}$ is a random sample of size n from $BS(\alpha, \beta)$ and $\mathbf{t} = \{t_1, \ldots, t_n\}$ are the observed values. The sample mean and harmonic mean are

$$\bar{T} = \frac{1}{n}\sum_{i=1}^{n} T_i \quad \text{and} \quad H = \left[\frac{1}{n}\sum_{i=1}^{n} T_i^{-1}\right]^{-1}, \quad \text{respectively.}$$

The sample mean and harmonic mean based on the observed values are

$$s = \frac{1}{n}\sum_{i=1}^{n} t_i \quad \text{and} \quad r = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{-1}\right]^{-1}, \quad \text{respectively.}$$

3.1. Confidence Intervals Based on Maximum Likelihood (ML) Approach

This method of estimation of parameters α and β was suggested by Ng et al. [7] and can be described as follows. The ML estimator of β denoted as $\hat{\beta}$, is the positive root of the equation

$$\beta^2 - \beta(2H + K(\beta)) + H[\bar{T} + K(\beta)] = 0,$$

where

$$K(x) = \left[\frac{1}{n}\sum_{i=1}^{n}(x+T_i)^{-1}\right]^{-1}, \quad x \geq 0.$$

Once the MLE of β is obtained, then the MLE of α can then be obtained as

$$\hat{\alpha} = \left[\frac{\bar{T}}{\hat{\beta}} + \frac{\hat{\beta}}{H} - 2\right]^{1/2}. \quad (4)$$

The joint distribution of $\hat{\alpha}$ and $\hat{\beta}$ is bivariate normal,

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} \sim N\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\frac{\alpha^2}{2n} & 0\\ 0 & \frac{\beta^2}{n(0.25+\alpha^{-2}+I(\alpha))}\end{pmatrix}\right],$$

where

$$I(\alpha) = 2\int_0^\infty \left\{(1+g(\alpha x))^{-1} - 1/2\right\}^2 d\Phi(x),$$

with

$$g(y) = 1 + \frac{y^2}{2} + y\left(1+\frac{y^2}{4}\right)^{1/2}.$$

Note that $\hat{\alpha}$ and $\hat{\beta}$ are asymptotically independent.

Ng et al. [12] proposed the $(1-\nu)\,100\%$ two-sided confidence intervals for α and β based on the maximum likelihood approach as

$$CI(\alpha)_{ML} = [\alpha_L, \alpha_U] = \left[\hat{\alpha}\left(\frac{z_{\nu/2}}{\sqrt{2n}}+1\right)^{-1}, \hat{\alpha}\left(\frac{z_{1-\nu/2}}{\sqrt{2n}}+1\right)^{-1}\right]$$

and

$$CI(\beta)_{ML} = [\beta_L, \beta_U] = \left[\hat{\beta}\left(\frac{z_{\nu/2}}{\sqrt{nh_1(\hat{\alpha})}}+1\right)^{-1}, \hat{\beta}\left(\frac{z_{1-\nu/2}}{\sqrt{nh_1(\hat{\alpha})}}+1\right)^{-1}\right],$$

where $h_1(x) = 0.25 + x^{-2} + I(x)$. By z_ν we denote the νth quantile of the standard normal distribution.

3.2. Confidence Intervals Based on Modified Maximum Likelihood (MML) Approach

The maximum likelihood estimator of α is biased, especially when the sample sizes are small. The almost unbiased maximum likelihood estimators of α and β, denoted by $\hat{\alpha}^*$ and $\hat{\beta}^*$ were proposed by Ng et al. [7] as:

$$\hat{\alpha}^* = \left(\frac{n}{n-1}\right)\hat{\alpha} \quad \text{and} \quad \hat{\beta}^* = \left(1+\frac{(\hat{\alpha}^*)^2}{4n}\right)\hat{\beta}.$$

These are the bias-corrected estimators. Ng et al. [12] proposed the $(1-\nu)\,100\%$ two-sided confidence intervals for α and β based on the almost unbiased maximum likelihood estimators as

$$CI(\alpha)_{MML} = [\alpha_L, \alpha_U] = \left[\hat{\alpha}^*\left(\sqrt{\frac{n}{2}}\frac{z_{\nu/2}}{(n-1)}+1\right)^{-1}, \hat{\alpha}^*\left(\sqrt{\frac{n}{2}}\frac{z_{1-\nu/2}}{(n-1)}+1\right)^{-1}\right],$$

and

$$CI(\beta)_{MML} = [\beta_L, \beta_U] = \left[\hat{\beta}^*\left(\frac{n}{h_1(\hat{\alpha}^*)}\frac{4z_{\nu/2}}{(4n+(\hat{\alpha}^*)^2)}+1\right)^{-1}, \hat{\beta}^*\left(\frac{n}{h_1(\hat{\alpha}^*)}\frac{4z_{1-\nu/2}}{\left(4n+(\hat{\alpha}^*)^2\right)}+1\right)^{-1}\right].$$

3.3. Confidence Intervals Based on Method of Moments Estimator (ME) Approach

The method od moments estimators of α and β can be obtained by equating the sample mean and sample variance to the their true values. The moment estimators do not exist in the case of the sample coefficient of variation is greater than $\sqrt{5}$ or $\alpha > 2737.2$. If the sample coefficient of variation is less than $\sqrt{5}$, then the method of moments estimators exist. However, the method of moments estimator for β may not be unique. The method of moments estimators of α and β are obtained as follows in [7]

$$\tilde{\alpha} = \left\{2\left[\left(\frac{\bar{T}}{H}\right)^{1/2} - 1\right]\right\}^{1/2} \quad \text{and} \quad \tilde{\beta} = (\bar{T}H)^{1/2}.$$

Ng et al. [7] proved that the joint asymptotic distribution of $\tilde{\alpha}$ and $\tilde{\beta}$ is bivariate normal

$$\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix} \sim N\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\frac{\alpha^2}{2n} & 0\\ 0 & \frac{\alpha\beta^2}{n}\left(\frac{1+\frac{3}{4}\alpha}{\left(1+\frac{1}{2}\alpha^2\right)^2}\right)\end{pmatrix}\right].$$

It is easy to see that the $(1-\nu)\,100\%$ two-sided confidence intervals for α and β based on the method of moments approach [12] are

$$CI(\alpha)_{ME} = [\alpha_L, \alpha_U] = \left[\tilde{\alpha}\left(\frac{z_{\nu/2}}{\sqrt{2n}} + 1\right)^{-1}, \tilde{\alpha}\left(\frac{z_{1-\nu/2}}{\sqrt{2n}} + 1\right)^{-1}\right],$$

and

$$CI(\beta)_{ME} = [\beta_L, \beta_U] = \left[\tilde{\beta}\left(\frac{z_{\nu/2}}{\sqrt{nh_2(\tilde{\alpha})}} + 1\right)^{-1}, \tilde{\beta}\left(\frac{z_{1-\nu/2}}{\sqrt{nh_2(\tilde{\alpha})}} + 1\right)^{-1}\right],$$

where $h_2(x) = \frac{1+0.75x^2}{(1+0.5x^2)^2}$.

3.4. Confidence Intervals Based on Modified Method of Moments Estimator (MME) Approach

Ng et al. [7] performed extensive Monte Carlo simulations to evaluate the performance of the method of moments estimator. They found that the method of moments estimator is highly biased if α is large and sample size is small. Ng et al. [12] proposed the almost unbiased method of moments estimators of α and β, denoted by $\tilde{\alpha}^*$ and $\tilde{\beta}^*$. These bias-corrected estimators are given by

$$\tilde{\alpha}^* = \left(\frac{n}{n-1}\right)\tilde{\alpha} \quad \text{and} \quad \tilde{\beta}^* = \left(1 + \frac{(\tilde{\alpha}^*)^2}{4n}\right)\tilde{\beta}.$$

The joint distribution of $\tilde{\alpha}^*$ and $\tilde{\beta}^*$ is bivariate normal

$$\begin{pmatrix}\tilde{\alpha}^*\\ \tilde{\beta}^*\end{pmatrix} \sim N\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\frac{n\alpha^2}{2(n-1)^2} & 0\\ 0 & \frac{16n(\alpha\beta)^2\left(1+0.75\alpha^2\right)}{(4n+\alpha^2)^2(1+0.25\alpha^2)^2}\end{pmatrix}\right],$$

They proposed the $(1-\nu)\,100\%$ two-sided confidence intervals for α and β based on the almost unbiased method of moments estimators are:

$$CI(\alpha)_{MME} = [\alpha_L, \alpha_U] = \left[\tilde{\alpha}^*\left(\sqrt{\frac{n}{2}}\frac{z_{\nu/2}}{n-1} + 1\right)^{-1}, \tilde{\alpha}^*\left(\sqrt{\frac{n}{2}}\frac{z_{1-\nu/2}}{n-1} + 1\right)^{-1}\right],$$

and

$$CI(\beta)_{MME} = [\beta_L, \beta_U] = \left[\tilde{\beta}^*\left(\sqrt{\frac{n}{h_2\left(\tilde{\alpha}^*\right)}}\frac{4z_{\nu/2}}{4n+\tilde{\alpha^*}^2} + 1\right)^{-1}, \tilde{\beta}^*\left(\sqrt{\frac{n}{h_2(\tilde{\alpha}^*)}}\frac{4z_{1-\nu/2}}{4n+\tilde{\alpha^*}^2} + 1\right)^{-1}\right].$$

3.5. Confidence Interval Based on the Generalized Pivotal (GP) Approach

Weerahandi [15] defined a *generalized pivotal* as a statistic that has a distribution free of unknown parameters and an observed value of generalized pivotal does not depend on nuisance parameters. For the general theory of this method, it has been shown that the generalized pivotal quantities are allowed to be a function of nuisance parameters, whereas conventional pivotal quantities can only be function of the sample and the parameter of interest. In this section, we present the GP approach for parameters of the Birnbaum−Saunders distribution.

First let α be the parameter of interest, and β be a nuisance parameter. The procedure of the confidence interval construction for the parameter α is as follows: the first is construct a generalized pivotal quantity, R, based on the two sufficient statistics sample mean $\bar{T}$ and harmonic mean H. Remind that the observed versions of these statistics are denoted as s and r, respectively.

Let $\mathbf{T} = \{T_1, \dots, T_n\}$ be a random sample of size n from $BS(\alpha, \beta)$ distribution. As we already mentioned (see (2)),

$$Y_i = \sqrt{\frac{T_i}{\beta}} - \sqrt{\frac{\beta}{T_i}}, i = 1, 2, \dots, n,$$

are independent normal $N(0, \alpha^2)$ random variables. The sample mean and sample variance of Ys

$$\bar{Y} = \bar{Y}(\beta, \mathbf{T}) = \frac{1}{n}\sum_{i=1}^{n} Y_i \quad \text{and} \quad S_Y^2 = S_Y(\beta, \mathbf{T}) = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2,$$

are independent with

$$\bar{Y} \sim N\left(0, \frac{\alpha^2}{n}\right), \quad \frac{(n-1)S_Y^2}{\alpha^2} \sim \chi^2(n-1).$$

Note that we write that $\bar{Y} = \bar{Y}(\beta, \mathbf{T})$ and $S_Y = S_Y(\beta, \mathbf{T})$ because we used only the values of β and $\mathbf{T}$ to calculate them. This point is important for our further discussion of generalized pivotal quantity for α.

Therefore $Q(\beta) = \sqrt{n}\bar{Y}/S_Y$ has Student's $t(n-1)$ distribution with $n-1$ degrees of freedom. Again, we should write $Q(\beta, \mathbf{T})$ instead of $Q(\beta)$ only because the formula for $Q(\beta)$ calculation is based on β and $\mathbf{T}$. But what is important to know that this formula is based on β and $\mathbf{T}$ only, it does not depend on α.

With the help of some monotonicity properties of the function $Q(\beta) = Q(\beta, \mathbf{T})$ established in Sun [16], Wang [17] derived the pivotal quantity of β as follows

$$g(Q, \mathbf{T}) = \begin{cases} \max\{\beta_1, \beta_2\}, & \text{if} \quad Q \le 0, \\ \min\{\beta_1, \beta_2\}, & \text{if} \quad Q > 0, \end{cases}$$

where the equation $Q(\beta) = Q$ and β_1, β_2 are two solutions of the following equation:

$$[(n-1)B^2 - DQ^2]\beta^2 - 2[(n-1)AB - (1-AB)Q^2]\beta + (n-1)A^2 - CQ^2 = 0.$$

In this equation we denote

$$A = \frac{1}{n}\sum_{i=1}^{n}\sqrt{T_i}, B = \frac{1}{n}\sum_{i=1}^{n} 1/\sqrt{T_i}, C = \frac{1}{n}\sum_{i=1}^{n}(\sqrt{T_i} - A)^2,$$

$$D = \frac{1}{n}\sum_{i=1}^{n}(1/\sqrt{T_i} - B)^2 \quad \text{and} \quad Q = Q(\beta).$$

Since $Y_i \sim N(0, \alpha^2)$, we can get

$$V = \sum_{i=1}^{n}\left(\frac{Y_i - 0}{\alpha}\right)^2 = \frac{\sum_{i=1}^{n} Y_i^2}{\alpha^2} \sim \chi^2(n).$$

Then $Q(\beta)$ and V are independent. Hence

$$\alpha = \sqrt{\frac{\sum_{i=1}^{n} Y_i(\beta)^2}{V}} = \sqrt{\frac{\sum_{i=1}^{n}\left(\sqrt{\frac{T_i}{\beta}} - \sqrt{\frac{\beta}{T_i}}\right)^2}{V}}$$
$$= \sqrt{\frac{\sum_{i=1}^{n} T_i - 2n\beta + \beta^2 \sum_{i=1}^{n} T_i^{-1}}{\beta V}} = \sqrt{\frac{\bar{T} - 2n\beta + \beta^2 H^{-1}}{\beta V}}. \quad (5)$$

A generalized pivotal quantity $R = h(\mathbf{T}; \mathbf{t}, \alpha, \beta)$ is some function of $\mathbf{T}$, possibly $\mathbf{t}, \alpha$ and β as well, because a generalized pivotal can be a function of all unknown parameters.

Using the expression of α presented in (5), the generalized pivotal for the shape parameter can be defined as follows

$$R_\alpha = h(\mathbf{T}; \mathbf{t}, \alpha, \beta) = \sqrt{\frac{s - 2ng(Q, \mathbf{t}) + g(Q, \mathbf{t})^2 r^{-1}}{g(Q, \mathbf{t})V}}, \quad (6)$$

where $V \sim \chi^2(n)$ and $Q \sim t(n-1)$. Note that here we consider s and r as functions of the *observed* values $\mathbf{t}$.

The generalized pivotal $R_\alpha = h(\mathbf{T}; \mathbf{t}, \alpha, \beta)$, for interval estimation has the following two properties, which are in the line with the required properties of a generalized pivotal outlined above.

(i) $R_\alpha = h(\mathbf{T}; \mathbf{t}, \alpha, \beta)$ has a probability distribution free of unknown parameters (the observed values s and r being treated as constants),

(ii) The observed pivotal, that is defined as $r_{obs} = h(\mathbf{t}; \mathbf{t}, \alpha, \beta)$ does not depend on the nuisance parameter. This proper is imposed to guarantee that such probability statements based on a generalized pivotal quantity will lead to confidence regions involving observed data t only.

Confidence intervals for α based on the GP approach can be constructed with the help of R_α. If $R_\alpha(1-\nu)$ is the $100(1-\nu)$th percentile of the R_α distribution, then $R_\alpha(1-\nu)$ is the $(1-\nu)100\%$ upper confidence limit for α. Thus,

$$CI\,(\alpha)_{GP} = [\alpha_L, \alpha_U] = [R_\alpha(\nu/2), R_\alpha(1-\nu/2)]$$

is a $(1-\nu)\,100\%$ two-sided GP confidence interval for the shape parameter α of the Birnbaum–Saunders distribution.

In the next section we conduct a simulation study to evaluate the accuracy properties of this method of confidence intervals construction. In order to understand the performance of the GP confidence interval, we estimate its coverage probability by Algorithm 1.

Algorithm 1. Fix values of n (sample size), α, β (parameters to be estimated), m_1 (the number of replications for the generalized pivotal computations, in our calculations we assume $m_1 = 5.000$) and m_2 (the number of replications for computation the GP confidence interval, in our calculations we assume $m_2 = 10.000$). The GP confidence interval can be computed by the following steps.

0. Let $j = 1$.

1. Simulate values $t_1, t_2, ..., t_n$ from $BS\,(\alpha, \beta)$ distribution.

2. Compute sample mean $s = \frac{1}{n}\sum_{i=1}^{n} t_i$ and harmonic mean $r = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{-1}\right]^{-1}$.

3. Generate $T \sim t(n-1)$ and $V \sim \chi^2(n)$, independently, for computing $g(Q, t)$ (if $g(Q, t) \leq 0$ then regenerate $T \sim t(n-1)$).

4. Compute R_α following (6).

5. Repeat Steps 3–4 a total of m_1 times and obtain an array of R_α's.

6. Rank this array of R_α's from smallest to largest.

7. If the $100(1-\nu/2)$th percentile of R_α's is greater than α and the $100(\nu/2)$th percentile R_α's is smaller than α, set $K_j = 1$, otherwise set $K_j = 0$ for $j = 1, 2, \ldots, m_2$.

8. Let j be $j+1$. Repeat Steps 1–7 a total of m_2 times and obtain an array of K_j's.

The value $\frac{1}{m_2}\sum_{i=1}^{m_2} K_j$ is an estimate of the coverage probability of two-sided GP $(1-\nu)\,100\%$ confidence interval for the shape parameter α of the Birnbaum–Saunders distribution $BS(\alpha, \beta)$.

The generalized pivotal for the scale parameter β can be defined as follows $R_\beta = g(Q,t)$. Confidence intervals for β based on the GP approach can be constructed with the help of R_β. If $R_\beta(1-\nu)$ is the $(1-\nu)$100th percentile of the R_β distribution, then $R_\beta(1-\nu)$ is the $(1-\nu)$ 100% upper confidence limit for β. Thus,

$$CI(\beta)_{GP} = [\beta_L, \beta_U] = [R_\beta(\nu/2), R_\beta(1-\nu/2)]$$

is a $(1-\nu)$ 100% two-sided GP for the scale parameters of the two-parameter Birnbaum—Saunders distribution.

3.6. Confidence Interval Based on the Percentile Bootstrap (PB) Approach

The last, we consider the percentile bootstrap (PB) approach for constructing confidence interval for parameters of the Birnbaum—Saunders distribution. Bootstrap methods are one type of re-sampling approaches that can be used to reduce the bias of the MLE. The basic theory of bootstrapping was presented by Efron and Tibshirani [18]. The bootstrap estimators for the shape and scale parameters are calculated by using the method of Lemonte et al. [19].

The method of Lemonte et al. [19] is a strategy of bias correction of the maximum-likelihood estimators for the parameters that index the distribution via bootstrap and can be described as follows.

Let $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ be a random sample of size n, where each element is a random draw from the random variable Y with the distribution function $F = F_\theta(y)$ and $\hat{\theta}$ be an estimator of θ based on y; we write $\hat{\theta} = s(y)$. Next, let $\mathbf{B}$ be bootstrap samples $(\mathbf{y}^{*1}, \mathbf{y}^{*2}, \ldots, \mathbf{y}^{*B})$ which are generated independently from the original sample y. The respective bootstrap replications are denoted as $(\hat{\theta}^{*1}, \hat{\theta}^{*2}, \ldots, \hat{\theta}^{*B})$, where $\hat{\theta}^{*b} = s(\mathbf{y}^{*b})$ and $b = 1, 2, \ldots, B$. The approximate bootstrap estimator is calculated by the mean $\hat{\theta}^{*(.)} = 1/B \sum_{b=1}^{B} \hat{\theta}^{*b}$. Therefore, the bootstrap bias estimates based on B replications of $\hat{\theta}$ are $\hat{B}_{F_{\hat{\theta}}}(\hat{\theta}, \theta) = \hat{\theta}^{*(.)} - s(\mathbf{y})$ for the parametric versions.

By using the idea of constant-bias-correcting (CBC) estimates by MacKinnon and Smith [20], we arrive at the following bias-corrected estimator

$$\bar{\theta}^* = s(\mathbf{y}) - \hat{B}_{F_{\hat{\theta}}}(\hat{\theta}, \theta) = 2\hat{\theta} - \hat{\theta}^{*(.)}.$$

We have evaluated, through Monte-Carlo simulations, the performance of the maximum likelihood estimators $\hat{\theta} = (\hat{\alpha}, \hat{\beta})^T$ of the vector of parameters $\theta = (\alpha, \beta)^T$ of the two-parameter Birnbaum—Saunders distribution and its bootstrap estimator $\bar{\theta}^* = (\bar{\alpha}^*, \bar{\beta}^*)^T$.

Suppose we want to use the m bootstrap samples to form a 95% confidence interval. We shall need to calculate the $\bar{\theta}^*$ number of m values to ordered value in the list of B standardized bootstrap estimates of θ. Therefore, the percentile $(1-\nu)$ 100% bootstrap confidence interval for α are given by

$$CI(\alpha)_{PB} = [\alpha_L, \alpha_U] = \left[\bar{\alpha}^*_{(\nu/2)}, \bar{\alpha}^*_{(1-\nu/2)}\right],$$

where $\bar{\alpha}^*_{(\nu/2)}$ is the $B\,(\nu/2)$th ordered value in the list of B standardized bootstrap estimates of α.

The Percentile $(1-\nu)$ 100% bootstrap confidence interval for β are given by

$$CI(\beta)_{PB} = [\beta_L, \beta_U] = \left[\bar{\beta}^*_{(\nu/2)}, \bar{\beta}^*_{(1-\nu/2)}\right],$$

where $\bar{\beta}^*_{(\nu/2)}$ is the $B\,(\nu/2)$th ordered value in the list of B standardized bootstrap estimates of β.

Table 1. Comparison of coverage probabilities and average lengths of α for the Two-parameter Birnbaum–Saunders distribution at the 0.95 nominal level ($\beta = 1$)

n	α	ML		MML		ME		MME		GP		PB	
		AL[a]	CP[b]	AL[a]	CP[b]	AL[a]	CP[b]	AL[a]	CP[b]	AL[a]	CP[b]	AL[a]	CP[b]
5	0.1	0.039	0.912	0.044	0.957	0.035	0.943	0.039	0.961	0.032	0.964	0.044	0.945
	0.5	0.196	0.922	0.224	0.955	0.071	0.942	0.190	0.958	0.186	0.955	0.217	0.943
	1.0	0.462	0.925	0.526	0.954	0.143	0.940	0.365	0.948	0.749	0.954	0.416	0.943
	1.5	1.178	0.930	1.330	0.950	0.236	0.943	0.523	0.946	1.112	0.953	0.597	0.944
	2.0	1.745	0.937	1.988	0.943	0.345	0.921	0.670	0.945	1.486	0.953	0.764	0.954
15	0.1	0.052	0.927	0.055	0.959	0.463	0.949	0.052	0.958	0.058	0.963	0.055	0.953
	0.5	0.266	0.922	0.280	0.954	0.041	0.949	0.257	0.958	0.273	0.961	0.270	0.951
	1.0	0.569	0.925	0.599	0.947	0.082	0.946	0.495	0.950	0.525	0.961	0.521	0.949
	1.5	1.702	0.952	1.781	0.940	0.163	0.949	0.716	0.957	0.784	0.959	0.753	0.949
	2.0	3.234	0.937	3.422	0.936	0.267	0.928	0.924	0.930	1.062	0.955	0.973	0.951
30	0.1	0.058	0.926	0.060	0.955	0.402	0.946	0.058	0.967	0.041	0.952	0.060	0.955
	0.5	0.296	0.937	0.304	0.952	0.533	0.937	0.286	0.959	0.194	0.951	0.294	0.951
	1.0	0.622	0.942	0.638	0.953	0.043	0.935	0.552	0.951	0.373	0.951	0.567	0.951
	1.5	1.751	0.950	1.814	0.950	0.086	0.931	0.800	0.945	0.558	0.951	0.822	0.952
	2.0	4.390	0.902	4.469	0.943	0.171	0.921	1.036	0.942	0.761	0.958	1.065	0.947
50	0.1	0.062	0.932	0.063	0.962	0.280	0.943	0.062	0.962	0.033	0.955	0.063	0.954
	0.5	0.313	0.924	0.318	0.956	0.424	0.944	0.303	0.955	0.159	0.954	0.308	0.955
	1.0	0.652	0.921	0.663	0.946	0.045	0.941	0.585	0.950	0.305	0.952	0.595	0.952
	1.5	1.730	0.963	1.757	0.945	0.089	0.929	0.849	0.944	0.457	0.950	0.863	0.946
	2.0	5.282	0.935	5.360	0.943	0.179	0.916	1.101	0.936	0.624	0.956	1.119	0.950
100	0.1	0.065	0.930	0.066	0.963	0.294	0.944	0.065	0.976	0.026	0.954	0.066	0.950
	0.5	0.331	0.926	0.334	0.957	0.442	0.947	0.321	0.969	0.123	0.953	0.323	0.953
	1.0	0.687	0.937	0.693	0.951	0.589	0.945	0.620	0.954	0.237	0.952	0.625	0.952
	1.5	1.718	0.937	1.736	0.947	0.048	0.946	0.899	0.948	0.355	0.950	0.907	0.954
	2.0	6.700	0.915	6.707	0.940	0.095	0.947	1.168	0.934	0.486	0.959	1.178	0.949
500	0.1	0.072	0.930	0.072	0.958	0.190	0.949	0.072	0.959	0.018	0.958	0.072	0.952
	0.5	0.363	0.933	0.364	0.958	0.313	0.947	0.352	0.951	0.087	0.953	0.353	0.950
	1.0	0.751	0.929	0.752	0.951	0.473	0.948	0.681	0.949	0.168	0.952	0.682	0.958
	1.5	1.839	0.920	1.840	0.949	0.632	0.942	0.989	0.947	0.252	0.950	0.990	0.950
	2.0	11.905	0.931	11.956	0.946	0.748	0.933	1.285	0.936	0.345	0.950	1.286	0.953

4. SIMULATION STUDIES AND RESULTS

The simulation studies are carried out to evaluate coverage probabilities and average widths of each confidence interval. The sample sizes are chosen to be $n = 5$, 15, 30, 50, 100, 500, the values of the shape parameter are $\alpha = 0.1$, 0.5, 1.0, 1.5, 2.0, the scale parameter β is kept fixed at 1.0, the nominal values are 0.95 and 0.99, the number of replications is 10.000. For the generalized pivotal

Table 2. Comparison of coverage probabilities and average lengths of α for the Two-parameter Birnbaum–Saunders distribution at the 0.99 nominal level ($\beta = 1$)

n	α	ML		MML		ME		MME		GP		PB	
		AL^a	CP^b	AL^a	CP^b	AL^a	CP^b	AL^a	CP^b	AL^a	CP^b	AL^a	CP^b
5	0.1	0.165	0.972	0.348	0.993	0.165	0.878	0.126	0.984	0.347	0.988	0.304	0.988
	0.5	0.835	0.974	1.762	0.991	0.810	0.893	0.620	0.940	1.708	0.982	1.491	0.972
	1.0	1.968	0.978	4.145	0.986	1.557	0.892	1.196	0.996	3.280	0.993	2.878	0.915
	1.5	5.017	0.983	10.478	0.985	2.228	0.892	1.702	0.880	4.707	0.993	4.097	0.980
	2.0	7.435	0.983	15.661	0.981	2.856	0.873	2.184	0.940	6.023	0.985	5.256	0.919
15	0.1	0.110	0.945	0.123	0.996	0.110	0.985	0.075	0.989	0.123	0.988	0.098	0.991
	0.5	0.562	0.953	0.627	0.995	0.543	0.978	0.365	0.991	0.605	0.982	0.477	0.980
	1.0	1.203	0.959	1.344	0.992	1.047	0.966	0.708	0.953	1.169	0.980	0.926	0.995
	1.5	3.599	0.962	3.996	0.992	1.513	0.906	1.025	0.918	1.690	0.984	1.341	0.987
	2.0	6.838	0.989	7.676	0.988	1.954	0.890	1.307	0.979	2.182	0.994	1.708	0.976
30	0.1	0.098	0.947	0.102	0.995	0.097	0.982	0.049	0.984	0.102	0.995	0.055	0.991
	0.5	0.496	0.950	0.520	0.991	0.480	0.976	0.242	0.969	0.502	0.982	0.272	0.986
	1.0	1.043	0.951	1.090	0.984	0.927	0.983	0.469	0.997	0.970	0.992	0.525	0.989
	1.5	2.938	0.964	3.100	0.987	1.343	0.985	0.677	0.978	1.405	0.992	0.759	0.989
	2.0	7.364	0.978	7.639	0.984	1.738	0.986	0.874	0.962	1.820	0.988	0.980	0.997
50	0.1	0.092	0.971	0.094	0.995	0.092	0.981	0.039	0.985	0.094	0.986	0.042	0.990
	0.5	0.465	0.974	0.477	0.991	0.450	0.989	0.194	0.987	0.462	0.994	0.209	0.994
	1.0	0.970	0.977	0.995	0.995	0.871	0.966	0.375	0.967	0.893	0.985	0.403	0.991
	1.5	2.573	0.974	2.635	0.991	1.262	0.969	0.542	0.999	1.294	0.989	0.583	0.984
	2.0	7.857	0.984	8.040	0.989	1.637	0.995	0.705	0.979	1.679	0.985	0.758	0.979
100	0.1	0.086	0.970	0.087	0.994	0.086	0.981	0.030	0.984	0.087	0.994	0.031	0.989
	0.5	0.438	0.972	0.442	0.990	0.424	0.978	0.148	0.980	0.429	0.986	0.154	0.989
	1.0	0.908	0.972	0.918	0.990	0.819	0.983	0.285	0.975	0.829	0.985	0.297	0.991
	1.5	2.271	0.977	2.301	0.993	1.189	0.997	0.413	0.991	1.203	0.985	0.431	0.995
	2.0	8.857	0.981	8.891	0.993	1.543	0.996	0.536	0.966	1.561	0.984	0.559	0.999
500	0.1	0.078	0.960	0.078	0.991	0.078	0.980	0.021	0.985	0.078	0.996	0.021	0.990
	0.5	0.397	0.969	0.397	0.985	0.384	0.981	0.103	0.986	0.385	0.991	0.105	0.991
	1.0	0.820	0.978	0.821	0.985	0.744	0.981	0.199	0.989	0.744	0.978	0.203	0.992
	1.5	2.008	0.978	2.009	0.986	1.079	0.980	0.290	0.987	1.081	0.997	0.296	0.990
	2.0	12.996	0.980	13.053	0.985	1.403	0.971	0.375	0.988	1.404	0.997	0.383	0.995

computations 5.000 pivotal quantities are used. For each simulation, the number of bootstrap samples B is 5.000. All computer simulations are studied by using written functions in R statistical programming environment [21]. Generally speaking, we prefer a confidence interval with a coverage probability close to the nominal coverage level and a shorter width.

Table 3. Comparison of coverage probabilities and average lengths of $\beta = 1$ for the Two-parameter Birnbaum–Saunders distribution at the 0.95 nominal level

n	α	ML		MML		ME		MME		GP		PB	
		AL[a]	CP[b]	AL[a]	CP[b]	AL[a]	CP[b]	AL[a]	CP[b]	AL[a]	CP[b]	AL[a]	CP[b]
5	0.1	0.071	0.928	0.223	0.949	0.068	0.951	0.635	0.949	0.069	0.949	0.078	0.952
	0.5	0.400	0.897	1.140	0.958	0.350	0.896	0.671	0.941	0.333	0.937	0.380	0.958
	1.0	1.787	0.957	5.279	0.923	1.619	0.918	0.743	0.912	0.639	0.914	0.728	0.974
	1.5	7.052	0.962	20.201	0.923	6.196	0.899	0.836	0.945	0.930	0.937	1.063	0.981
	2.0	9.593	0.849	27.622	0.996	8.472	0.928	0.945	0.995	1.220	0.987	1.387	0.982
15	0.1	0.076	0.929	0.081	0.952	0.073	0.950	0.641	0.950	0.073	0.950	0.076	0.950
	0.5	0.402	0.914	0.427	0.949	0.383	0.966	0.682	0.945	0.354	0.948	0.373	0.951
	1.0	1.124	0.891	1.193	0.946	1.070	0.929	0.763	0.953	0.684	0.955	0.718	0.957
	1.5	7.980	0.897	8.713	0.986	7.818	0.939	0.867	0.948	0.997	0.925	1.047	0.959
	2.0	15.200	0.959	15.911	0.922	14.277	0.922	1.002	0.943	1.308	0.943	1.374	0.960
30	0.1	0.076	0.931	0.050	0.951	0.074	0.950	0.643	0.950	0.074	0.948	0.076	0.950
	0.5	0.401	0.922	0.264	0.949	0.390	0.952	0.686	0.948	0.361	0.949	0.370	0.950
	1.0	0.997	0.935	0.658	0.951	0.972	0.934	0.771	0.950	0.695	0.951	0.715	0.959
	1.5	6.603	0.935	4.386	0.952	6.474	0.938	0.880	0.949	1.014	0.958	1.042	0.955
	2.0	19.048	0.921	12.983	0.952	19.164	0.937	1.024	0.954	1.328	0.969	1.366	0.957
50	0.1	0.075	0.930	0.036	0.950	0.074	0.950	0.689	0.950	0.074	0.950	0.075	0.950
	0.5	0.394	0.933	0.189	0.953	0.387	0.955	0.779	0.952	0.363	0.949	0.369	0.950
	1.0	0.938	0.923	0.451	0.946	0.925	0.958	0.894	0.951	0.701	0.951	0.713	0.949
	1.5	4.618	0.927	2.281	0.947	4.679	0.946	1.042	0.947	1.022	0.957	1.039	0.947
	2.0	22.560	0.917	10.767	0.955	22.083	0.962	0.648	0.944	1.338	0.955	1.361	0.955
100	0.1	0.075	0.930	0.024	0.950	0.074	0.949	0.695	0.950	0.074	0.950	0.075	0.950
	0.5	0.387	0.929	0.124	0.949	0.384	0.949	0.790	0.949	0.365	0.949	0.368	0.948
	1.0	0.886	0.926	0.283	0.951	0.878	0.954	0.913	0.951	0.705	0.952	0.711	0.950
	1.5	2.990	0.924	0.961	0.957	2.986	0.952	1.073	0.952	1.028	0.948	1.036	0.952
	2.0	28.218	0.920	9.274	0.955	28.822	0.955	0.648	0.955	1.343	0.943	1.355	0.946
500	0.1	0.075	0.930	0.007	0.950	0.075	0.950	0.696	0.950	0.075	0.950	0.075	0.950
	0.5	0.381	0.930	0.035	0.950	0.381	0.950	0.793	0.951	0.367	0.949	0.368	0.950
	1.0	0.813	0.925	0.074	0.950	0.812	0.947	0.918	0.951	0.710	0.949	0.711	0.950
	1.5	2.128	0.934	0.195	0.950	2.126	0.947	1.080	0.950	1.032	0.952	1.033	0.951
	2.0	49.079	0.931	4.444	0.947	48.484	0.954	0.635	0.953	1.345	0.951	1.346	0.958

In the following, the notation "^{a}" is used when the average width of a confidence interval is reported and "^{b}" is used for the actual coverage probability.

In Tables 1 and 2 we presented the average width and coverage probabilities of 95% and 99% confidence intervals for α. From these tables, we readily observe that the coverage probabilities of the ML and ME are considerably smaller than the nominal levels, particularly when the shape parameter

Table 4. Comparison of coverage probabilities and average lengths of $\beta = 1$ for the Two-parameter Birnbaum–Saunders distribution at the 0.99 nominal level

n	α	ML		MML		ME		MME		GP		PB	
		AL[a]	CP[b]	AL[a]	CP[b]	AL[a]	CP[b]	AL[a]	CP[b]	AL[a]	CP[b]	AL[a]	CP[b]
5	0.1	0.616	0.986	0.537	0.990	0.292	0.988	0.679	0.989	0.292	0.991	0.615	0.992
	0.5	3.153	0.972	2.753	0.976	1.490	0.972	0.762	0.998	1.421	0.981	2.996	0.993
	1.0	14.082	0.934	12.295	0.996	6.899	0.915	0.931	0.994	2.721	0.991	5.733	0.993
	1.5	55.558	0.884	48.506	0.968	26.397	0.980	1.169	0.948	3.960	0.995	8.374	0.995
	2.0	75.578	0.834	65.985	0.976	36.094	0.919	1.500	0.914	5.198	0.936	10.927	0.996
15	0.1	0.172	0.978	0.095	0.991	0.154	0.991	0.664	0.990	0.153	0.992	0.171	0.990
	0.5	0.902	0.993	0.500	0.997	0.809	0.980	0.727	0.988	0.749	0.985	0.836	0.991
	1.0	2.522	0.996	1.398	0.991	2.263	0.995	0.855	0.989	1.446	0.977	1.611	0.993
	1.5	17.904	0.964	9.924	0.969	16.531	0.987	1.026	0.998	2.108	0.997	2.349	0.991
	2.0	34.102	0.972	18.902	0.999	30.188	0.976	1.260	0.943	2.766	0.996	3.083	0.998
30	0.1	0.129	0.980	0.054	0.990	0.124	0.991	0.660	0.990	0.123	0.989	0.129	0.990
	0.5	0.685	0.977	0.284	0.985	0.654	0.986	0.720	0.986	0.605	0.984	0.633	0.994
	1.0	1.704	0.980	0.707	0.995	1.630	0.989	0.841	0.978	1.166	0.984	1.221	0.986
	1.5	11.285	0.981	4.682	0.988	10.860	0.989	0.999	0.986	1.702	0.982	1.781	0.981
	2.0	32.555	0.989	13.507	0.983	32.147	0.997	1.219	0.974	2.228	0.992	2.334	0.998
50	0.1	0.113	0.980	0.038	0.990	0.110	0.990	0.713	0.990	0.110	0.990	0.113	0.991
	0.5	0.591	0.981	0.197	0.991	0.576	0.994	0.827	0.986	0.540	0.991	0.554	0.992
	1.0	1.407	0.975	0.469	0.984	1.375	0.991	0.977	0.985	1.043	0.986	1.069	0.988
	1.5	6.927	0.979	2.309	0.994	6.961	0.984	1.175	0.980	1.520	0.991	1.559	0.988
	2.0	33.840	0.994	11.280	0.988	32.850	0.979	0.653	0.997	1.990	0.998	2.041	0.985
100	0.1	0.400	0.980	0.024	0.990	0.498	0.989	0.705	0.990	0.098	0.990	0.099	0.990
	0.5	0.514	0.980	0.126	0.988	0.507	0.989	0.811	0.990	0.483	0.991	0.488	0.990
	1.0	1.175	0.984	0.289	0.987	1.161	0.991	0.949	0.991	0.932	0.991	0.943	0.990
	1.5	3.964	0.972	0.974	0.992	3.947	0.995	1.132	0.992	1.359	0.989	1.373	0.990
	2.0	37.405	0.959	9.187	0.984	38.097	0.999	0.652	0.997	1.775	0.987	1.796	0.996
500	0.1	0.382	0.980	0.007	0.990	0.382	0.990	0.704	0.990	0.081	0.990	0.082	0.990
	0.5	0.416	0.979	0.035	0.990	0.416	0.991	0.808	0.990	0.401	0.990	0.401	0.990
	1.0	0.888	0.979	0.075	0.991	0.887	0.992	0.943	0.988	0.775	0.992	0.776	0.990
	1.5	2.323	0.980	0.195	0.991	2.321	0.990	1.121	0.992	1.127	0.989	1.128	0.993
	2.0	53.583	0.978	4.504	0.991	52.928	0.995	0.679	0.986	1.468	0.993	1.470	0.991

α is small. However, the average width of the ME is smaller than that of the ML, especially when the shape parameter α is large. The widths of each confidence interval seem to be related to the values of α. The performance of the MML and MME are both better than ML and ME in terms of coverage probabilities. The average widths of the MML and MME have similar values and change in the same direction when the parameters and sample sizes change. Moreover, the MME yields the length width

Table 5. Interval estimations of α and β at the 0.95 nominal level

Methods	Parameters	
	α	β
ML	(0.138, 0.203)	(131.781, 131.856)
MML	(0.139, 0.205)	(131.772, 131.846)
ME	(0.023, 0.317)	(131.782, 131.856)
MME	(0.140, 0.205)	(131.462, 132.157)
GP	(0.159, 0.185)	(131.761, 131.835)
PB	(0.139, 0.205)	(131.766, 131.831)

shorter than MML. In respect to coverage probabilities, the GP approach appears to be the clear winner. The coverage probability of the GP approach is generally closer to the nominal level than that of the PB approach. The average width of the GP approach is smaller than that of the PB approach, especially when the sample sizes are large.

The average widths and coverage probabilities of 95% and 99% confidence intervals for β are reported in Tables 3 and 4, respectively. From the simulation results, it is clear that the average widths of the ML and the ME are both short if n and α are small. The performance of the MML and MME are almost identical for different sample sizes, if the shape parameter α is not too large. The average width of the GP approach is as smaller than that of the PB approach, especially when the sample sizes are large. The PB approach works very well in all case for both the parameters even for small samples. The performance of the GP approach and PB approach are almost identical for different shape parameters and the large sample sizes. The PB approach performs better than all other methods in terms of coverage probabilities.

5. ILLUSTRATIVE EXAMPLES

To illustrate the computation of proposed confidence intervals in this paper, we use the data by Birnbaum and Saunders [14] on the fatigue life of 6061-T6 aluminium coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set consists of 101 observations with maximum stress per cycle 31000 psi.

By this information on the fatigue lifetimes, we compute the 95% confidence interval for α and β using all 6 methods. The results are presented in Table 5. The confidence interval for α using the GP approach provides the shortest width among all methods. For the scale parameter β, the PB confidence interval is the winner with the shortest width of interval as 0.065.

6. CONCLUSION

The aim of this article is to propose the GP and the PB approaches for confidence interval estimation of the shape and scale parameters of the Birnbaum–Saunders distribution. These methods are compared with the ML, the MML, the ME, and the MME approaches, which were presented by Ng et al. [7] and [12]. The performances of these confidence intervals were assessed in terms of coverage probabilities and average widths through simulation studies. The simulation study indicates that the proposed confidence intervals perform well in terms of coverage probabilities and average widths. For most cases, coverage probabilities of the proposed methods are equal or above the nominal values and these confidence intervals are also short confidence intervals in comparison with all other methods. The GP approach provides the shorter average width, especially, when the sample size is small. In this case, we suggest the GP approach for the confidence interval construction for the shape parameter α and the PB approach for the scale parameter β confidence interval.

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