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On complete convergence for arrays

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Abstract

In this note we present new results on a complete convergence for arrays of rowwise independent random variables that generalize the results of Hu et al. [2003. Complete convergence for arrays of rowwise independent random variables. Comm. Korean Math. Soc. 18, 375–383], Kuczmaszewska [2004. On some conditions for complete convergence for arrays. Statist. Probab. Lett. 66, 399–405], and Sung et al. [2005. More on complete convergence for arrays. Statist. Probab. Lett. 71, 303–311]. Additional results that deal with complete convergence for rowwise dependent arrays are given.

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1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins (1947) as follows. A sequence $\{U_n, n\geqslant 1\}$ of random variables converges completely to the constant θ if

$$\sum_{n=1}^{\infty} P\{|U_n-\theta|>\varepsilon\}<\infty \quad \text{for all } \varepsilon>0.$$

The first results concerning the complete convergence were due to Hsu and Robbins (1947) and Erdös (1949). The paper Hu et al. (1998) unifies and extends the ideas of previously obtained results on complete convergence. In the main results of Hu et al. (1998), no assumptions are made concerning the existence of

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expected values or absolute moments of the random variables. The proof of Hu et al. (1998) is mistakenly based on the fact that the assumptions of their theorem (cf. Corollary 1) imply convergence in probability of the corresponding partial sums. Counterexamples to this proof were presented in Hu and Volodin (2000) and Hu et al. (2003).

Many attempts to solve this problem led to weaker variants of this theorem. Hu et al. (2003) gave a first partial solution to this question (cf. Corollary 2). Next partial solution was given by Kuczmaszewska (2004) (cf. Corollary 4) and the question was solved completely in Sung et al. (2005) (cf. Corollary 1), where the result is proved as stated in Hu et al. (1998). The proof of Sung et al. is different from those of Hu et al. (1998) in the sense that it does not use a symmetrization procedure.

The initial objective of our investigation that lead to the present paper was to find a proof of the main result of Hu et al. (1998) that is based on the symmetrization procedure. But it appears that a more general result can be proved, cf. Theorem 1. As it is shown in corollaries, the main results of Hu et al. (1998, 2003), Kuczmaszewska (2004), and Sung et al. (2005) can be derived simply from the main result of this paper.

2. Main results

With the preliminaries accounted for we can now state and prove our main results. In the following we let $\{X_{nk}, 1 \leqslant k \leqslant m_n, n \geqslant 1\}$ be an array of rowwise independent random variables defined on a probability space $(\Omega, \mathscr{F}, P)$, $\{m_n, n \geqslant 1\}$ be a sequence of positive integers such that $\lim_{n\to\infty} m_n = \infty$, and $\{c_n, n \geqslant 1\}$ be a sequence of positive constants. For an event $A \in \mathscr{F}$ we denote $I[A]$ the indicator function. We should note that all the results of this paper remain true in the case $m_n = \infty$ for some/all $n \geqslant 1$, provided the series $\sum_{k=1}^{\infty} X_{nk}$ converges almost surely. Certainly, we should consider sup instead of max in the case of infinite sums.

Theorem 1. *Let* $\{X_{nk}, 1 \leqslant k \leqslant m_n, n \geqslant 1\}$ *and* $\{c_n, n \geqslant 1\}$ *satisfy the following conditions*:

(i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{m_n} P\{|X_{nk}| > \varepsilon\} < \infty$ *for all* $\varepsilon > 0$,
(ii) *there exist* $j > 0$, $\delta > 0$ *and* $p \geqslant 1$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(E \left| \sum_{k=1}^{m_n} [X_{nk} I[|X_{nk}| \leqslant \delta] - E(X_{nk} I[|X_{nk}| \leqslant \delta])] \right|^p \right)^j < \infty.$$

Then

$$\sum_{n=1}^{\infty} c_n P\left\{ \max_{1 \leqslant m \leqslant m_n} \left| \sum_{k=1}^{m} [X_{nk} - E(X_{nk} I[|X_{nk}| \leqslant \delta])] \right| > \varepsilon \right\} < \infty \tag{2.1}$$

for any $\varepsilon > 0$.

Proof. The conclusion of the theorem is obvious if $\sum_{n=1}^{\infty} c_n < \infty$. For $n \geqslant 1$ and $1 \leqslant k,\ m \leqslant m_n$ let

$$Y_{nk} = X_{nk} I[|X_{nk}| \leqslant \delta], \quad T_m = \sum_{k=1}^{m} Y_{nk}, \quad S_m = \sum_{k=1}^{m} X_{nk},$$

$$A = \bigcap_{k=1}^{m_n} \{X_{nk} = Y_{nk}\}, \quad B = \bigcup_{k=1}^{m_n} \{X_{nk} \neq Y_{nk}\}.$$

Note that

$$
\begin{aligned}
P\left\{\max_{1\leqslant m\leqslant m_n}|S_m-ET_m|>\varepsilon\right\} &= P\left\{\left\{\max_{1\leqslant m\leqslant m_n}|T_m-ET_m|>\varepsilon\right\}\cap A\right\} \\
&\quad + P\left\{\left\{\max_{1\leqslant m\leqslant m_n}|S_m-ET_m|>\varepsilon\right\}\cap B\right\} \\
&\leqslant P\left\{\max_{1\leqslant m\leqslant m_n}|T_m-ET_m|>\varepsilon\right\}+\sum_{k=1}^{m_n}P\{|X_{nk}|>\delta\}.
\end{aligned}
$$

By (i) it is enough to prove that for all $\varepsilon>0$

$$\sum_{n=1}^{\infty}c_nP\left\{\max_{1\leqslant m\leqslant m_n}|T_m-ET_m|>\varepsilon\right\}<\infty.$$

For any $n\geqslant 1$, the random variables $V_m=|T_m-ET_m|^p$, $1\leqslant m\leqslant m_n$, form a submartingale with filtration $\mathscr{F}_m=\sigma(Y_{n1},\ldots,Y_{nm})$, $m=1,\ldots,m_n$, and hence $EV_1\leqslant\cdots\leqslant EV_{m_n}$. Here $\sigma(Y_{n1},\ldots,Y_{nm})$ is the sigma-algebra generated by $Y_{n1},\ldots,Y_{nm}$. Consider a partition of the set of natural numbers $\mathbb{N}$ into two parts

$$\mathbb{N}'=\{n:EV_{m_n}\leqslant\varepsilon^p/2^{p+1}\}\quad\text{and}\quad\mathbb{N}''=\{n:EV_{m_n}>\varepsilon^p/2^{p+1}\}.$$

Applying (ii) we obtain

$$\sum_{n\in\mathbb{N}''}c_n\leqslant\sum_{n\in\mathbb{N}''}c_n(EV_{m_n}/(\varepsilon^p/2^{p+1}))^j=\frac{2^{(p+1)j}}{\varepsilon^{pj}}\sum_{n\in\mathbb{N}''}c_n(EV_{m_n})^j<\infty.$$

Hence it is enough to show that for all $\varepsilon>0$

$$\sum_{n\in\mathbb{N}'}c_nP\left\{\max_{1\leqslant m\leqslant m_n}|T_m-ET_m|>\varepsilon\right\}<\infty. \tag{2.2}$$

By Markov inequality

$$P\{|T_m-ET_m|>(2EV_m)^{1/p}\}=P\{V_m>2EV_m\}\leqslant\tfrac{1}{2}$$

and hence $|\mathrm{med}\,T_m-ET_m|\leqslant(2EV_m)^{1/p}$. Therefore

$$\max_{1\leqslant m\leqslant m_n}|\mathrm{med}\,T_m-ET_m|\leqslant\max_{1\leqslant m\leqslant m_n}(2EV_m)^{1/p}=(2EV_{m_n})^{1/p}\leqslant\varepsilon/2$$

for all $n\in\mathbb{N}'$. From here and (2.2) it follows that it is enough to prove

$$\sum_{n\in\mathbb{N}'}c_nP\left\{\max_{1\leqslant m\leqslant m_n}|T_m-\mathrm{med}\,T_m|>\varepsilon/2\right\}<\infty. \tag{2.3}$$

Denote $\{Y'_{nk},1\leqslant k\leqslant m_n,n\geqslant 1\}$ rowwise independent random variables which are independent copies of $\{Y_{nk},1\leqslant k\leqslant m_n,n\geqslant 1\}$; that is Y'_{nk} and Y_{nk} have the same distribution and independent for all $1\leqslant k\leqslant m_n$, $n\geqslant 1$. Then random variables $\{Y_{nk}-Y'_{nk},1\leqslant k\leqslant m_n,n\geqslant 1\}$ are rowwise independent and symmetrically distributed. By the symmetrization inequality and by Lévy inequality (Loève, 1977 18.1B and C) we obtain

$$
\begin{aligned}
P\left\{\max_{1\leqslant m\leqslant m_n}|T_m-\mathrm{med}\,T_m|>\varepsilon/2\right\} &\leqslant 2P\left\{\max_{1\leqslant m\leqslant m_n}\left|\sum_{k=1}^{m}(Y_{nk}-Y'_{nk})\right|>\varepsilon/2\right\} \\
&\leqslant 4P\left\{\left|\sum_{k=1}^{m_n}(Y_{nk}-Y'_{nk})\right|>\varepsilon/2\right\}.
\end{aligned} \tag{2.4}
$$

Choose an integer $l \geqslant 1$ such that $j \leqslant 2^l$. Then by the iterated Hoffmann-Jøregensen inequality (Jain, 1975) there exist positive constants C_l and D_l such that

$$P\left\{\left|\sum_{k=1}^{m_n}(Y_{nk}-Y'_{nk})\right|>\varepsilon/2\right\} \leqslant C_l P\left\{\max_{1\leqslant k\leqslant m_n}|Y_{nk}-Y'_{nk}|>\varepsilon/(2\cdot 3^l)\right\} + D_l\left(P\left\{\left|\sum_{k=1}^{m_n}(Y_{nk}-Y'_{nk})\right|>\varepsilon/(2\cdot 3^l)\right\}\right)^j. \tag{2.5}$$

By Markov inequality and using the fact that $EY_{nk}=EY'_{nk}$ we have

$$\begin{aligned} P\left\{\left|\sum_{k=1}^{m_n}(Y_{nk}-Y'_{nk})\right|>\varepsilon/(2\cdot 3^l)\right\} &\leqslant \frac{2^p 3^{lp}}{\varepsilon^p} E\left|\sum_{k=1}^{m_n}(Y_{nk}-Y'_{nk})\right|^p \\ &= \frac{2^p 3^{lp}}{\varepsilon^p} E\left|\sum_{k=1}^{m_n}(Y_{nk}-EY_{nk}+EY'_{nk}-Y'_{nk})\right|^p \\ &\leqslant \frac{2^{2p} 3^{lp}}{\varepsilon^p}\left[E\left|\sum_{k=1}^{m_n}(Y_{nk}-EY_{nk})\right|^p + E\left|\sum_{k=1}^{m_n}(Y'_{nk}-EY'_{nk})\right|^p\right] \\ &= \frac{2^{2p+1} 3^{lp}}{\varepsilon^p} E\left|\sum_{k=1}^{m_n}(Y_{nk}-EY_{nk})\right|^p \\ &= \frac{2^{2p+1} 3^{lp}}{\varepsilon^p} EV_{m_n}. \end{aligned} \tag{2.6}$$

By (2.4), (2.5), (2.6) and the assumptions of the theorem we see that (2.3) holds. $\square$

In the next theorem we simplify condition (ii) of Theorem 1 when absolute moments of some order $p \geqslant 1$ exist for the row sums of mean zero random variables comprising the array.

Theorem 2. *Let assumption* (i) *is fulfilled and* $EX_{nk}=0$ *for all* $1\leqslant k\leqslant m_n$, $n\geqslant 1$. *Assume that for some* $p\geqslant 1$ *and some* $j>0$

$$\sum_{n=1}^{\infty} c_n\left(E\left|\sum_{k=1}^{m_n} X_{nk}\right|^p\right)^j < \infty. \tag{2.7}$$

Then

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\leqslant m\leqslant m_n}\left|\sum_{k=1}^{m} X_{nk}\right|>\varepsilon\right\}<\infty \tag{2.8}$$

for all $\varepsilon>0$.

Proof. The proof is actually the same as that of Theorem 1. We are using the notations from the proof of Theorem 1. For any $n\geqslant 1$, the random variables $|S_m|^p$, $1\leqslant m\leqslant m_n$ where $S_m=X_{n1}+\ldots+X_{nm}$, form a submartingale with filtration $\mathscr{F}_m=\sigma(X_{n1},\ldots,X_{nm})$, $m=1,\ldots,m_n$, and hence $E|S_1|^p\leqslant\cdots\leqslant E|S_{m_n}|^p$. Partition the set $\mathbb{N}$ of all natural numbers into two parts

$$\mathbb{K}'=\{n: E|S_{m_n}|^p\leqslant \varepsilon^p/2^{p+1}\} \quad \text{and} \quad \mathbb{K}''=\{n: E|S_{m_n}|^p> \varepsilon^p/2^{p+1}\}.$$

By (2.7) we obtain

$$\sum_{n\in\mathbb{K}''} c_n \leqslant \sum_{n\in\mathbb{K}''} c_n(E|S_{m_n}|^p/(\varepsilon^p/2^{p+1}))^j = \frac{2^{(p+1)j}}{\varepsilon^{pj}}\sum_{n\in\mathbb{K}''} c_n(E|S_{m_n}|^p)^j<\infty.$$

Hence it is sufficient to prove that for all $\varepsilon > 0$

$$\sum_{n\in\mathbb{K}'} c_n P\left\{\max_{1\leqslant m\leqslant m_n}\left|\sum_{k=1}^{m} X_{nk}\right| > \varepsilon\right\} < \infty. \tag{2.9}$$

By the Markov inequality $P\{|S_m| > (2E|S_m|^p)^{1/p}\} \leqslant \frac{1}{2}$ and hence

$$|\mathrm{med}\, S_m| \leqslant (2E|S_m|^p)^{1/p} \leqslant (2E|S_{m_n}|^p)^{1/p}.$$

Therefore $\max_{1\leqslant m\leqslant m_n}|\mathrm{med}\, S_m| \leqslant \varepsilon/2$ for all $n \in \mathbb{K}'$. From here and (2.9) it follows that it is enough to prove that

$$\sum_{n\in\mathbb{K}'} c_n P\left\{\max_{1\leqslant m\leqslant m_n}|S_m - \mathrm{med}\, S_m| > \varepsilon/2\right\} < \infty. \tag{2.10}$$

Statement (2.10) is an analog of statement (2.3) and the proof of (2.3) is applicable to a proof of (2.10). All necessary changes can be obtained by the substitution T_m by S_m. As the result, we obtain

$$\begin{aligned}\sum_{n\in\mathbb{K}'} c_n P\left\{\max_{1\leqslant m\leqslant m_n}|S_m - \mathrm{med}\, S_m| > \varepsilon/2\right\} &\leqslant 8C_l \sum_{n\in\mathbb{K}'} c_n \sum_{k=1}^{m_n} P\{|X_{nk}| > \varepsilon/(4\cdot 3^l)\} \\ &\quad + \frac{4\cdot 2^{(2p+1)j}3^{lpj}}{\varepsilon^{pj}} D_l \sum_{n\in\mathbb{K}'} c_n (E|S_{m_n}|^p)^j < \infty\end{aligned}$$

by the assumptions of the theorem. $\square$

Remark 1. Assumptions (ii) and (2.7) can be strengthen for $p \in [1,2]$ as follows:

(ii)′ There exist $j > 0$, $\delta > 0$ and $p \in [1,2]$ such that

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{m_n} E|X_{nk} I[|X_{nk}| \leqslant \delta] - E(X_{nk} I[|X_{nk}| \leqslant \delta])|^p\right)^j < \infty$$

and

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{m_n} E|X_{nk}|^p\right)^j < \infty. \tag{2.7′}$$

Proof. By von Bahr–Esseen inequality for the pth absolute moment of a sum of random variables (cf. for example Hoffmann-Jørgensen, 1994, 4.32 where this inequality is called Khinchine's inequality), there exists a positive constant $d = d_p \geqslant 1$ such that

$$E|T_m - ET_m|^p \leqslant d \sum_{k=1}^{m} E|Y_{nk} - EY_{nk}|^p \quad \text{and} \quad E|\sum_{k=1}^{m} X_{nk}|^p \leqslant d \sum_{k=1}^{m} E|X_{nk}|^p$$

for any $1 \leqslant m \leqslant m_n$, $n \geqslant 1$. It is obvious that $d = d_p = 1$ for $p = 1$ and $p = 2$. For $p = 2$ assumptions (ii) and (ii)′ as well as (2.7) and (2.7)′ coincide. $\square$

Remark 2. Condition (i) in Theorem 2 can be omitted if condition (2.7) is fulfilled with $j \in (0,1]$. Really, by the maximal inequalities (cf. Chow and Teicher, 1997, Theorem 7.4.8)

$$\begin{aligned}\sum_{n=1}^{\infty} c_n P\left\{\max_{1\leqslant m\leqslant m_n}\left|\sum_{k=1}^{m} X_{nk}\right| > \varepsilon\right\} &\leqslant \sum_{n=1}^{\infty} c_n \left(P\left\{\max_{1\leqslant m\leqslant m_n}\left|\sum_{k=1}^{m} X_{nk}\right| > \varepsilon\right\}\right)^j \\ &\leqslant \frac{\kappa_p^j}{\varepsilon^{pj}} \sum_{n=1}^{\infty} c_n \left(E\left|\sum_{k=1}^{m_n} X_{nk}\right|^p\right)^j < \infty,\end{aligned}$$

where $\kappa_p = 1$ if $p = 1$, and $\kappa_p = (p/(p-1))^p$ if $p > 1$.

3. Corollaries

Now we present some corollaries from the main results of the paper. Corollaries 1, 2, and 4 contain as particular cases the main results of Hu et al. (1998, 2003), Kuczmaszewska (2004) and Sung et al. (2005). Corollaries 8 and 9 show how some well known results on complete convergence can be easily derived from Theorem 2.

In the proofs of all corollaries we use the notation from the proof of Theorem 1.

The first corollary contains the main result of Hu et al. (1998) and Sung et al. (2005). Their result corresponds to the case $p=2$ and $j\geqslant 2$ in condition (ii).

Corollary 1. *Let* $\{X_{nk}, 1\leqslant k\leqslant m_n, n\geqslant 1\}$ *and* $\{c_n, n\geqslant 1\}$ *satisfy assumptions* (i), (ii) *and* (iii) $\sum_{k=1}^{m_n} EX_{nk}I[|X_{nk}|\leqslant\delta]\to 0$ *as* $n\to\infty$.

Then $\sum_{n=1}^{\infty} c_n P\{|\sum_{k=1}^{m_n} X_{nk}|>\varepsilon\}<\infty$ *for all* $\varepsilon>0$.

Remark 3. Note that the original assumption (ii) in Hu et al. (1998) and Sung et al. (2005) was in the form: There exists $j\geqslant 2$ such that

$$\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{m_n} EX_{nk}^2 I[|X_{nk}|\leqslant\delta]\right)^j<\infty,$$

which is stronger than assumption (ii) presented here.

The second corollary contains the main result of Hu et al. (2003). Their result corresponds to the case $p=2$ and $j\geqslant 2$ in condition (ii).

Corollary 2. *Let* $\{X_{nk}, 1\leqslant k\leqslant m_n, n\geqslant 1\}$ *and* $\{c_n, n\geqslant 1\}$ *satisfy assumptions* (i), (ii) *and* (iv) $\max_{1\leqslant m\leqslant m_n}|\sum_{i=1}^{m} EX_{ni}I[|X_{ni}|\leqslant\delta]|\to 0$ *as* $n\to\infty$.

Then $\sum_{n=1}^{\infty} c_n P\{\max_{1\leqslant m\leqslant m_n}|\sum_{i=1}^{m} X_{ni}|>\varepsilon\}<\infty$ *for all* $\varepsilon>0$.

Remark 4. We should mention that the original conclusion of Hu et al. (2003) was that

$$\sum_{n=1}^{\infty} c_n P\left\{\left|\sum_{k=1}^{m_n} X_{nk}\right|>\varepsilon\right\}<\infty.$$

Hence Corollary 2 gives actually a stronger statement than the main result of Hu et al. (2003).

Before we start with the proof of the main result of Kuczmaszewska (2004) (Corollary 4 with $p=2$ and $j\geqslant 2$ in condition (ii)) we present the following corollary which is of independent interest.

Corollary 3. *If assumptions* (i) *and* (ii) *are fulfilled, then*

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\leqslant m\leqslant m_n}\left|\sum_{k=1}^{m} X_{nk} - \operatorname{med}\left(\sum_{k=1}^{m} X_{nk}I[|X_{nk}|\leqslant\delta]\right)\right|>\varepsilon\right\}<\infty \tag{3.1}$$

for all $\varepsilon>0$.

Proof. According to (i), (2.2) and by the symmetrization inequality (Loève, 1977, 18.1B) we obtain

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\leqslant m\leqslant m_n}|S_m-\operatorname{med} T_m|>\varepsilon\right\}\leqslant 4\sum_{n=1}^{\infty} c_n P\left\{\max_{1\leqslant m\leqslant m_n}|T_m-ET_m|>\varepsilon/2\right\} + \sum_{n=1}^{\infty} c_n\sum_{k=1}^{m_n} P\{|X_{nk}|>\delta\}<\infty. \qquad \square$$

Corollary 4. *Let* $\{X_{nk}, 1\leqslant k\leqslant m_n, n\geqslant 1\}$ *and* $\{c_n, n\geqslant 1\}$ *satisfy assumptions* (i), (ii) *and*

$$\lim_{n\to\infty}\operatorname{med}\left(\sum_{k=1}^{m_n} X_{nk}I[|X_{nk}|\leqslant\delta]\right)=0.$$

Then $\sum_{n=1}^{\infty} c_n P\{|\sum_{k=1}^{m_n} X_{nk}|>\varepsilon\}<\infty$ *for all* $\varepsilon>0$.

Moreover, we can strengthen the conclusion of Corollary 4 in the following way.

Corollary 5. *If assumptions* (i) *and* (ii) *are fulfilled and*

$$\lim_{n\to\infty} \max_{1\leqslant m\leqslant m_n} \left| \operatorname{med}\left(\sum_{k=1}^{m} X_{nk} I[|X_{nk}| \leqslant \delta] \right) \right| = 0,$$

then (2.8) *holds.*

The next two corollaries deal with medians of not truncated random variables.

Corollary 6. *If assumptions* (i) *and* (ii) *are fulfilled, then*

$$\sum_{n=1}^{\infty} c_n P\left\{ \max_{1\leqslant m\leqslant m_n} \left| \sum_{k=1}^{m} X_{nk} - \operatorname{med}\left(\sum_{k=1}^{m} X_{nk} \right) \right| > \varepsilon \right\} < \infty \tag{3.2}$$

for all $\varepsilon > 0$.

Corollary 7. *If assumptions* (i) *and* (ii) *are fulfilled and*

$$\lim_{n\to\infty} \operatorname{med}\left(\sum_{k=1}^{m_n} X_{nk} \right) = 0,$$

then

$$\sum_{n=1}^{\infty} c_n P\left\{ \left| \sum_{k=1}^{m_n} X_{nk} \right| > \varepsilon \right\} < \infty$$

for all $\varepsilon > 0$.

Before we start with corollaries to Theorem 2 we need to recall some well known notions.
Recall that the array $\{(X_{nk}, 1\leqslant k\leqslant m_n), n\geqslant 1\}$ of random variables is said to be:

(1) *stochastically dominated in the Cesàro sense* by a random variable X if $m_n < \infty$ and there exists a constant $D > 0$ such that $\sum_{k=1}^{m_n} P\{|X_{nk}| > x\} \leqslant D m_n P\{|X| > x\}$ for all $x > 0$ and all n.
(2) *stochastically dominated* by a random variable X if there exists a constant $D > 0$ such that

$$P\{|X_{nk}| > x\} \leqslant DP\{|X| > x\}$$

for all $x > 0$ and for all k and n.

First we obtain a generalization of the main result of Hu et al. (1989) (cf. also Gut, 1992, Theorem 2.1). Furthermore, we can obtain the rate of convergence as follows.

Corollary 8. *Let* $\{(X_{nk}, 1\leqslant k\leqslant n), n\geqslant 1\}$ *be an array of rowwise independent mean zero random variables which are stochastically dominated in the Cesàro sense by a random variable* $X, r > 1$, *and* $1\leqslant q < 2$. *If* $E|X|^{rq} < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{ \max_{1\leqslant m\leqslant n} \left| \sum_{k=1}^{m} X_{nk} \right| > \varepsilon n^{1/q} \right\} < \infty$$

for any $\varepsilon > 0$.

Proof. It is sufficient to check that the assumptions of Theorem 2 are fulfilled with $c_n = n^{r-2}$ and $X_{nk}/n^{1/q}$ for all X_{nk}.

In order to show assumption (i), we note that

$$\begin{aligned}\sum_{n=1}^{\infty} n^{r-2} \sum_{k=1}^{n} P\{|X_{nk}| > \varepsilon n^{1/q}\} &\leqslant D \sum_{n=1}^{\infty} n^{r-1} P\{|X| > \varepsilon n^{1/q}\} \\ &= D \sum_{n=1}^{\infty} n^{r-1} P\{|X|^{rq} > \varepsilon^{rq} n^{r}\} \\ &= D \sum_{n=1}^{\infty} n^{r-1} \sum_{k=n}^{\infty} P\{\varepsilon^{rq} k^{r} < |X|^{rq} \leqslant \varepsilon^{rq}(k+1)^{r}\} \\ &\leqslant D \sum_{k=1}^{\infty} k^{r} P\{\varepsilon^{rq} k^{r} < |X|^{rq} \leqslant \varepsilon^{rq}(k+1)^{r}\} \leqslant \frac{D}{\varepsilon^{rq}} E|X|^{rq} < \infty.\end{aligned}$$

For assumption (ii) let $p = \min\{2, rq\}$. By the integration by parts formula we obtain the inequality $\sum_{k=1}^{n} E|X_{nk}|^{p} \leqslant DnE|X|^{p}$. Next,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{r-2} \left(\sum_{k=1}^{n} E|X_{nk}/n^{1/q}|^{p} \right)^{j} &\leqslant \sum_{n=1}^{\infty} n^{r-2} \left(\frac{n}{n^{p/q}} DE|X|^{p} \right)^{j} \\ &= (DE|X|^{p})^{j} \sum_{n=1}^{\infty} \frac{1}{n^{(p/q-1)j-r+2}} < \infty\end{aligned}$$

for $j > (r-1)q/(p-q)$. From here and $(2.7)'$ follows that assumption (2.7) is fulfilled. $\square$

Secondly we present a generalization of a result of Rohatgi (1971). Recall that a double array $\{a_{nk}; k, n \geqslant 1\}$ of real numbers is said to be a *Toeplitz sequence* if $\lim_{n\to\infty} a_{nk} = 0$ for each k and $\sum_{k} |a_{nk}| \leqslant C$ for each n, where C is a positive constant.

Corollary 9. *Let $\{X_{nk}; k, n \geqslant 1\}$ be an array of rowwise independent mean zero random variables, stochastically dominated by a random variable X, and let $\{a_{nk}; k, n \geqslant 1\}$ be a Toeplitz sequence. If*

I. $\max_{k\geqslant 1} |a_{nk}| = O(n^{-r})$ *for some* $r > 0$,
II. $E|X|^{1+1/r} < \infty$, *then*

$$\sum_{n=1}^{\infty} P\left\{ \sup_{m \geqslant 1} \left| \sum_{k=1}^{m} a_{nk} X_{nk} \right| > \varepsilon \right\} < \infty.$$

Proof. By Lemma 1 of Rohatgi (1971)

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} P\{|a_{nk} X_{nk}| > \varepsilon\} \leqslant \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} P\{|a_{nk} X| > \varepsilon\} < \infty.$$

Next, let $p = \min\{2, 1+1/r\}$ and $q = \min\{1, 1/r\}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} |a_{nk}|^{p} E|X_{nk}|^{p} \right)^{j} &\leqslant \sum_{n=1}^{\infty} \left(D \sup_{k \geqslant 1} |a_{nk}|^{q} \sum_{k=1}^{\infty} |a_{nk}| E|X|^{p} \right)^{j} \\ &\leqslant \sum_{n=1}^{\infty} (O(n^{-rq}))^{j} < \infty \quad \text{if } rqj > 1.\end{aligned}$$

From here and $(2.7)'$ follows that condition (2.7) is fulfilled. The corollary follows from Theorem 2 with $c_n \equiv 1$. $\square$

4. Some additional results

In the conclusion we present two results that relax the condition of rowwise independence in an array of random variables under the consideration. The first proposition shows that the main result of Hu et al. (1998) and Sung et al. (2005) can be partially generalized on the case of rowwise pairwise independent array of random variables. The second proposition shows that Theorem 1 can be partially generalized on the case of arbitrary random variables if the centering by conditional expectations is considered.

Proposition 1. *Let* $\{X_{nk}, 1 \leqslant k \leqslant m_n, n \geqslant 1\}$ *be an array of rowwise pairwise independent random variables and* $\{c_n, n \geqslant 1\}$ *be a sequence of positive constants such that conditions* (i) *and* (ii) *with* $p = 2$ *and* $j = 1$ *are satisfied. Then*

$$\sum_{n=1}^{\infty} c_n P\left\{\left|\sum_{k=1}^{m_n}[X_{nk} - E(X_{nk}I[|X_{nk}| \leqslant \delta])]\right| > \varepsilon\right\} < \infty$$

for any $\varepsilon > 0$.

The proof of Proposition 1 is very easy and it can be derived from the proof of Theorem 1.

Proposition 2. *Let* $\{X_{nk}, 1 \leqslant k \leqslant m_n, n \geqslant 1\}$ *be an array of arbitrary random variables defined on a probability space* $(\Omega, \mathscr{F}, P)$ *and* $\{c_n, n \geqslant 1\}$ *be a sequence of positive constants. For* $n \geqslant 1$ *let sigma-algebras* $\mathscr{F}_{nk} = \sigma(X_{n1}, \ldots, X_{nk})$ *for* $1 \leqslant k \leqslant m_n$ *and* $\mathscr{F}_{n0} = \{\Omega, \emptyset\}$. *Assume that conditions* (i) *and* (ii)$''$ *for some* $\delta > 0$

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{m_n} E[X_{nk}I[|X_{nk}| \leqslant \delta] - E(X_{nk}I[|X_{nk}| \leqslant \delta]|\mathscr{F}_{nk-1})]^2 < \infty$$

are satisfied. Here $\mathscr{F}_{nk-1} = \sigma\mathscr{X}_{nj},\ 1 \leqslant j \leqslant k-1$.

Then

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1 \leqslant m \leqslant m_n}\left|\sum_{k=1}^{m}[X_{nk} - E(X_{nk}I[|X_{nk}| \leqslant \delta]|\mathscr{F}_{nk-1})]\right| > \varepsilon\right\} < \infty$$

for any $\varepsilon > 0$.

Proof. Note that for every $n \geqslant 1$ the random variables

$$\sum_{k=1}^{m}[Y_{nk} - E(Y_{nk}|\mathscr{F}_{nk-1})], \quad 1 \leqslant m \leqslant m_n,$$

form a martingale with filtration $\mathscr{F}_{nm},\ 1 \leqslant m \leqslant m_n$. By (i) and the maximal inequalities (cf. Chow and Teicher, 1997, Theorem 7.4.8)

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n P\left\{\max_{1 \leqslant m \leqslant m_n} |[X_{nk} - E(Y_{nk}|\mathscr{F}_{nk-1})]| > \varepsilon\right\} \\
&\quad \leqslant \sum_{n=1}^{\infty} c_n P\left\{\max_{1 \leqslant m \leqslant m_n} |[Y_{nk} - E(Y_{nk}|\mathscr{F}_{nk-1})]| > \varepsilon\right\} + \sum_{n=1}^{\infty} c_n \sum_{k=1}^{m_n} P\{|X_{nk}| > \delta\} \\
&\quad \leqslant \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty} c_n \sum_{k=1}^{m_n} E|[Y_{nk} - E(Y_{nk}|\mathscr{F}_{nk-1})]^2 + \sum_{n=1}^{\infty} c_n \sum_{k=1}^{m_n} P\{|X_{nk}| > \delta\} < \infty. \qquad \square
\end{aligned}$$

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