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A Remark on Complete Convergence for Arrays of Rowwise Negatively Associated Random Variables

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ABSTRACT We obtain complete convergence result for arrays of rowwise negatively associated random variables, which extend and generalize the results of Hu *et al*. (1998), Hu *et al*. (2003), and Sung *et al*. (2005). As applications, some well-known results on independent random variables can be easily extended to the case of negatively associated random variables.

Keywords Negatively associated random variables; Array of rowwise negatively associated random variables; Complete convergence.

1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins (1947) as follows. A sequence $\{U_n, n \geq 1\}$ of random variables converges completely to the constant θ if

$$\sum_{n=1}^{\infty} P\{|U_n - \theta| > \varepsilon\} < \infty \quad \text{for all } \varepsilon > 0.$$

The first results concerning the complete convergence were due to Hsu and Robbins (1947) and Erdös (1949). Since then there were many authors who devoted the study to complete convergence for sums and weighted sums of independent random variables. Hu *et al*. (1998) announced the general result on complete convergence for arrays of rowwise independent random variables, presented as Theorem A below.

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In the following we let $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, $\{k_n, n \ge 1\}$ be a sequence of positive integers such that $\lim_{n\to\infty} k_n = \infty$, and $\{c_n, n \ge 1\}$ be a sequence of positive constants. For an event $A \in \mathcal{F}$ we denote $I\{A\}$ the indicator function. We should note that all the results of this paper remain true in the case $k_n = \infty$ for some/all $n \ge 1$, provided the series $\sum_{k=1}^{\infty} X_{nk}$ converges almost surely. Certainly, we should consider sup instead of max in the case of infinite sums.

Theorem A *Let* $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ *be array of rowwise independent random variables. Suppose that for every* $\varepsilon > 0$ *and some* $\delta > 0$*:*
(i) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} P\{|X_{ni}| > \varepsilon\} < \infty$,
(ii) there exists $j \ge 2$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{k_n} EX_{ni}^2 I\{|X_{ni}| \le \delta\} \right)^j < \infty,$$

(iii) $\sum_{i=1}^{k_n} EX_{ni} I\{|X_{ni}| \le \delta\} \to 0$ *as* $n \to \infty$.
Then

$$\sum_{n=1}^{\infty} c_n P\left\{ |\sum_{k=1}^{k_n} X_{nk}| > \varepsilon \right\} < \infty \text{ for all } \varepsilon > 0.$$

The paper Hu *et al.* [6] unifies and extends the ideas of previously obtained results on complete convergence. In the main results of Hu *et al.* [6], no assumptions are made concerning the existence of expected values or absolute moments of the random variables. The *proof* of Hu *et al.* [6] is mistakenly based on the fact that the assumptions of their theorem imply convergence in probability of the corresponding partial sums. Counterexamples to this proof were presented in Hu and Volodin [7] and Hu *et al.* [5]. We would like to stress that the both examples are the counterexamples to the *proof* of Theorem A, but not to the result. At the same time, they mentioned that the problem whether the Theorem A is true for any positive constants $\{c_n,\ n \ge 1\}$ has remained open.

Many attempts to solve this problem led to weaker variants of this theorem. Hu *et al.* [5] gave a first partial solution to this question. Next partial solution was given by Kuczmaszewska [10] and the question was solved completely in Sung *et al.* [18], where the result is proved as stated in Hu *et al.* [6]. The approach of Sung *et al.* [18] is different from those of Hu *et al.* [6] in the sense that it does not use the symmetrization procedure.

The next paper which, to our best knowledge, presents the most general results on complete convergence for arrays of rowwise *independent* random variables is Kruglov *et al.* [9]. As it is shown in corollaries of that paper, the main results of Hu *et al.* [6], Hu *et al.* [5], Kuczmaszewska [10], and Sung *et al.* [18] can be derived easily from the first theorem of Kruglov *et al.* [9].

The proofs of the main results of Kruglov *et al.* [9], as well as of Hu *et al.* [6], Hu *et al.* [5], Kuczmaszewska [10], and Sung *et al.* [18], make use of the well-known Hoffmann-Jørgensen's inequality (cf. Hoffmann-Jørgensen [3]). Hoffmann-Jørgensen's maximal inequality is a powerful tool which has now become a standard technique in proving limit theorem for independent random variables.

The main purpose of this investigation is to extend Theorem A for the case of arrays of rowwise *negatively associated* random variables. But for associated random variables, especially for negative associated random variables, it is still an open question whether Hoffmann-Jørgensen's maximal inequality is true or not. In order to extend Theorem A to the case of negatively associated random variables, we need to find another way. In the paper, we use the exponential inequality for negatively associated random variables, which is established by Shao [15], cf. Lemma below.

Recall that a finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be *negatively associated* (abbreviated to NA in the following) if for any disjoint subsets A and B of $\{1, 2, \cdots, n\}$ and any real coordinate-wise nondecreasing functions f on $\mathbf{R}^A$ and g on $\mathbf{R}^B$,

$$\mathrm{Cov}\left(f(X_i, i \in A), g(X_j, j \in B)\right) \le 0$$

whenever the covariance exists. An infinite family of random variables $\{X_i, i \ge 1\}$ is NA if every finite subfamily is NA.

This concept was introduced by Joag-Dev and Proschan [8]. They also pointed out and proved in their paper that a number of well-known multivariate distributions possess the NA property. NA random variables have wide applications in reliability theory and multivariate statistical analysis. Recently Su *et al.* [17] showed that NA structure plays an important role in risk management. Because of these reasons the notions of NA random variables have received more and more attention in recent years. A great number of papers for NA random variables are now in literature. We refer to Joag-Dev and Proschan [8] for fundamental properties, Newman [14] for the central limit theorem, Matula [13] for the three series theorem, Shao and Su [16] for the law of the iterated logarithm, Shao [15] for moment inequalities, Liu *et al.* [12] for the Hàjek-Rènyi inequality and Barbour *et al.* [1] for the Poison approximation.

Shao [15] showed the following important exponent inequality of Kolmogorov's type.

Lemma *Let $\{X_i, 1 \le i \le n\}$ be a sequence of NA mean zero random variables with $E|X_i|^2 < \infty$ for every $1 \le i \le n$ and $B_n = \sum_{i=1}^n EX_i^2$. Then for all $\varepsilon > 0,\ a > 0$*

$$P\left\{\max_{1\le k\le n} |\sum_{i=1}^k X_i| \ge \varepsilon\right\} \le 2P\left\{\max_{1\le i\le n} |X_i| > a\right\} + 4\exp\left\{-\frac{\varepsilon^2}{4(a\varepsilon + B_n)} \times \left[1 + \frac{2}{3}\ln\left(1 + \frac{a\varepsilon}{B_n}\right)\right]\right\}.$$

Note also that if $\{X_i, i \geq 1\}$ is a sequence of NA random variables, then the truncation in the usual way, for example the sequence $\{X_i I\{|X_i| < \delta\}, i \geq 1\}$, where $\delta > 0$, is not necessary NA any more. We should use so-called *monotone truncation* (cf. the definition of random variables Y_{nk} in the proof of Theorem B) in order to preserve this property.

2. Main Results, Corollaries, and Remarks

Now we state the main results. The proofs will be detailed in next section.

Theorem B *Let* $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ *be an array of rowwise NA random variables such that conditions (i) and (ii) are satisfied. Then*

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m}[X_{nk} - E(X_{nk}I\{|X_{nk}| \leq \delta\})]\right| > \varepsilon\right\} < \infty$$

for any $\varepsilon > 0$.

The following two corollaries of Theorem B are immediate.

Corollary 1 *Let* $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ *be an array of rowwise NA random variables such that conditions (i), (ii), and (iii) are satisfied. Then*

$$\sum_{n=1}^{\infty} c_n P\left\{\left|\sum_{k=1}^{k_n} X_{nk}\right| > \varepsilon\right\} < \infty$$

for any $\varepsilon > 0$.

Corollary 2. *Let* $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ *be an array of rowwise NA random variables such that conditions (i), (ii), and*

(iii)′ $\max_{1\leq m\leq k_n} |\sum_{i=1}^{m} EX_{ni}I\{|X_{ni}| \leq \delta\}| \to 0$ *as* $n \to \infty$

are satisfied. Then

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m} X_{nk}\right| > \varepsilon\right\} < \infty$$

for any $\varepsilon > 0$.

Similar argument as in Remark 2 of Hu *et al.* [6], we have

Theorem C *Let* $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ *be an array of rowwise NA random variables with* $EX_{nk} = 0$ *for all* $n \geq 1$ *and* $1 \leq k \leq k_n$. *Let* $\phi(x)$ *be a real function such that for some* $\delta > 0$:

$$\sup_{x>\delta}\frac{x}{\phi(x)} < \infty \text{ and } \sup_{0\leq x\leq\delta}\frac{x^2}{\phi(x)} < \infty.$$

Suppose that for all $\varepsilon > 0$, *conditions (i) and* *(ii)′ there exist* $j \geq 2$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\phi(|X_{nk}|) \right)^j < \infty \textit{ and } \sum_{k=1}^{k_n} E\phi(|X_{nk}|) \to 0 \textit{ as } n \to \infty$$

are satisfied, then

$$\sum_{n=1}^{\infty} c_n P \left\{ \max_{1 \leq m \leq k_n} \left| \sum_{k=1}^{m} X_{nk} \right| > \varepsilon \right\} < \infty$$

for any $\varepsilon > 0$.

Remark 1. Suppose $\liminf_{n\to\infty} c_n > 0$, then condition (i) in Corollary 2 is also necessary. In fact,

$$\sum_{n=1}^{\infty} c_n P \left\{ \max_{1 \leq m \leq k_n} \left| \sum_{k=1}^{m} X_{nk} \right| > \varepsilon \right\} < \infty$$

implies that

$$\sum_{n=1}^{\infty} c_n P \left\{ \max_{1 \leq i \leq k_n} |X_{ni}| > \varepsilon \right\} < \infty, \text{ for all } \varepsilon > 0,$$

hence we have $P\{\max_{1\leq i\leq k_n} |X_{ni}| > \varepsilon\} \to 0$ as $n \to \infty$. By Lemma 2 of Liang [11], we have for sufficiently large n,

$$\sum_{i=1}^{k_n} P\{|X_{ni}| > \varepsilon\} \leq CP \left\{ \max_{1 \leq i \leq k_n} |X_{ni}| > \varepsilon \right\}$$

for some $C > 0$, therefore condition (i) holds.

Remark 2. By Theorems B and C we have that Corollaries 1 and 2 of Hu *et al.* [6] are also true if we assume negative association instead of independence and replace weighted sums by the maxima of weighted sums.

3. Proofs

In the following, C always stands for a positive constant which may differ from one place to another.

Proof of Theorem B. The conclusion of the theorem is obvious if $\sum_{n=1}^{\infty} c_n < \infty$. Let $\{Y_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be a *monotone truncation* of $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$, that is

$$Y_{nk} = X_{nk} I\{|X_{nk}| \leq \delta\} + \delta I\{X_{nk} > \delta\} - \delta I\{X_{nk} < -\delta\}$$

where $\delta > 0$ and $1 \leq k \leq k_n, n \geq 1$. Note that by Property 6 of Joag-Dev and Proschan [8] (applied twice) we can conclude that $\{Y_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ is an array of rowwise NA random variables.

For $n \geq 1$ and $1 \leq m \leq k_n$ let

$$
\begin{aligned}
T_m &= \sum_{k=1}^{m} Y_{nk}, \; S_m = \sum_{k=1}^{m} X_{nk}, \; S'_m = \sum_{k=1}^{m} X_{nk} I\{|X_{nk}| \leq \delta\}, \\
A &= \bigcap_{k=1}^{k_n} \{X_{nk} = X_{nk} I\{|X_{nk}| \leq \delta\}\}, \; A^c = \bigcup_{k=1}^{k_n} \{X_{nk} \neq X_{nk} I\{|X_{nk}| \leq \delta\}\}.
\end{aligned}
$$

Note that for any $n \geq 1$

$$
\begin{aligned}
& P\{\max_{1\leq m\leq k_n} |S_m - ES'_m| > \varepsilon\} \\
= \; & P\left\{\{\max_{1\leq m\leq k_n} |S_m - ES'_m| > \varepsilon\} \cap A^c\right\} + P\left\{\{\max_{1\leq m\leq k_n} |S'_m - ES'_m| > \varepsilon\} \cap A\right\} \\
\leq \; & \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} + P\{\max_{1\leq m\leq k_n} |S'_m - ES'_m| > \varepsilon\} \\
\leq \; & \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} + P\{\max_{1\leq m\leq k_n} |T_m - ET_m| > \varepsilon/2\} \\
& + P\{\max_{1\leq m\leq k_n} |(S'_m - T_m) - E(S'_m - T_m)| > \varepsilon/2\} \\
\leq \; & \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} + P\{\max_{1\leq m\leq k_n} |T_m - ET_m| > \varepsilon/2\} \\
& + P\{\max_{1\leq m\leq k_n} \delta|\sum_{k=1}^{m}[I\{X_{nk} > \delta\} - I\{X_{nk} < -\delta\} \\
& - E(I\{X_{nk} > \delta\} - I\{X_{nk} < -\delta\})]| > \varepsilon/2\} \\
\leq \; & \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} + P\{\max_{1\leq m\leq k_n} |T_m - ET_m| > \varepsilon/2\} \\
& + CE \max_{1\leq m\leq k_n} |\sum_{k=1}^{m}[I\{X_{nk} > \delta\} - I\{X_{nk} < -\delta\} \\
& - E(I\{X_{nk} > \delta\} - I\{X_{nk} < -\delta\})]| \text{ (by Markov's inequality)} \\
\leq \; & \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} + P\{\max_{1\leq m\leq k_n} |T_m - ET_m| > \varepsilon/2\} \\
& + CE \max_{1\leq m\leq k_n} \sum_{k=1}^{m}[I\{X_{nk} > \delta\} + I\{X_{nk} < -\delta\} + P\{X_{nk} > \delta\} + P\{X_{nk} < -\delta\}] \\
\leq \; & \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} + P\{\max_{1\leq m\leq k_n} |T_m - ET_m| > \varepsilon/2\} \\
& + CE \sum_{k=1}^{k_n}[I\{X_{nk} > \delta\} + I\{X_{nk} < -\delta\} + P\{|X_{nk}| > \delta\}]
\end{aligned}
$$

$$\leq \quad P\{\max_{1\leq m\leq k_n} |T_m - ET_m| > \varepsilon/2\} + (1+2C)\sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\}.$$

By (i) it is enough to prove that for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} c_n P\{\max_{1\leq m\leq k_n} |T_m - ET_m| > \varepsilon\} < \infty.$$

Let $B_n = \sum_{k=1}^{k_n} \text{ Var}(Y_{nk})$. For any $\varepsilon > 0$ and $a > 0$ set

$$\begin{aligned}
\mathbf{N}_1 &= \{n : B_n > a\varepsilon\}, \\
\mathbf{N}_2 &= \left\{ n : \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} > \min\{1, a\varepsilon/(2\delta^2), a/(4\delta)\} \right\}, \\
\mathbf{N}_3 &= \left\{ n : \sum_{k=1}^{k_n} EX_{nk}^2 I\{|X_{nk}| \leq \delta\} > \min\{a\varepsilon/2, a^2/16\} \right\}, \\
\mathbf{N}_4 &= \mathbf{N} - (\mathbf{N}_2 \cup \mathbf{N}_3).
\end{aligned}$$

Since

$$\begin{aligned}
B_n &= \sum_{k=1}^{k_n} (EY_{nk}^2 - (EY_{nk})^2) \leq \sum_{k=1}^{k_n} EY_{nk}^2 \\
&= \delta^2 \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} + \sum_{k=1}^{k_n} EX_{nk}^2 I\{|X_{nk}| \leq \delta\},
\end{aligned}$$

we have $\mathbf{N}_1 \subseteq \mathbf{N}_2 \cup \mathbf{N}_3$. Note that

$$\begin{aligned}
&\sum_{n\in\mathbf{N}_2\cup\mathbf{N}_3} c_n P\{\max_{1\leq m\leq k_n} |T_m - ET_m| > \varepsilon\} \leq \sum_{n\in\mathbf{N}_2\cup\mathbf{N}_3} c_n \\
\leq\ & \left(\min\{1/2, a\varepsilon/(4\delta^2), a/(8\delta)\}\right)^{-1} \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} \\
&+ \left(\min\{a\varepsilon/4, a^2/32\}\right)^{-j} \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EX_{nk}^2 I\{|X_{nk}| \leq \delta\} \right)^j < \infty.
\end{aligned}$$

Hence it is sufficient to prove that

$$\sum_{n\in\mathbf{N}_4} c_n P\left\{ \max_{1\leq m\leq k_n} |\sum_{k=1}^{m} (Y_{nk} - EY_{nk})| > \varepsilon \right\} < \infty.$$

By Lemma we have that

$$\sum_{n\in\mathbf{N}_4} c_n P\left\{ \max_{1\leq m\leq k_n} \left| \sum_{k=1}^{m} (Y_{nk} - EY_{nk}) \right| > \varepsilon \right\}$$

$$\leq \sum_{n\in\mathbf{N}_4} c_n \left[2P\left\{\max_{1\leq k\leq k_n} |Y_{nk} - EY_{nk}| > a\right\} + 4\exp\left\{-\frac{\varepsilon^2}{4(a\varepsilon + B_n)}\left[1 + \frac{2}{3}\ln\left(1 + \frac{a\varepsilon}{B_n}\right)\right]\right\}\right]$$

Note that for any $n \in \mathbf{N}_4$

$$\begin{aligned}
&\max_{1\leq k\leq k_n} |EY_{nk}| \leq \max_{1\leq k\leq k_n} E|Y_{nk}| \\
&\leq \max_{1\leq k\leq k_n} (\delta P\{|X_{nk}| > \delta\} + E|X_{nk}|I\{|X_{nk}| \leq \delta\}) \\
&\leq \delta \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} + \left(\sum_{k=1}^{k_n} EX_{nk}^2 I\{|X_{nk}| \leq \delta\}\right)^{1/2} \\
&\leq \delta \min\{1, a\varepsilon/(2\delta^2), a/(4\delta)\} + \left(\min\{a\varepsilon/2, a^2/16\}\right)^{1/2} \\
&\qquad (\text{since } n \notin \mathbf{N}_2 \text{ and } n \notin \mathbf{N}_3) \\
&\leq a/4 + a/4 = a/2.
\end{aligned}$$

This implies that for any $n \in \mathbf{N}_4$, $\max_{1\leq k\leq k_n} |EY_{nk}| \leq a/2$ and

$$\begin{aligned}
&\sum_{n\in\mathbf{N}_4} c_n P\left\{\max_{1\leq k\leq k_n} |Y_{nk} - EY_{nk}| > a\right\} \\
\leq\ & \sum_{n=1}^{\infty} c_n P\left\{\max_{1\leq k\leq k_n} |Y_{nk}| > a/2\right\} \\
\leq\ & \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\left\{|X_{nk}| > \min\{\delta, a/2\}\right\} < \infty \text{ (by (i))}.
\end{aligned}$$

Therefore it is sufficient to prove that

$$\sum_{n\in\mathbf{N}_4} c_n \exp\left\{-\frac{\varepsilon^2}{4(a\varepsilon + B_n)}\left[1 + \frac{2}{3}\ln\left(1 + \frac{a\varepsilon}{B_n}\right)\right]\right\} < \infty.$$

When $n \in \mathbf{N}_4$, $B_n \leq a\varepsilon$ and $\sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} \leq 1$. Let $a = \varepsilon/(12j)$.

$$\begin{aligned}
&\sum_{n\in\mathbf{N}_4} c_n \exp\left\{-\frac{\varepsilon^2}{4(a\varepsilon + B_n)}\left[1 + \frac{2}{3}\ln\left(1 + \frac{a\varepsilon}{B_n}\right)\right]\right\} \\
\leq\ & \sum_{n\in\mathbf{N}_4} c_n \exp\left\{-\frac{\varepsilon^2}{8a\varepsilon}\left[1 + \frac{2}{3}\ln\left(1 + \frac{a\varepsilon}{B_n}\right)\right]\right\} \\
\leq\ & \exp\{-\frac{3}{2}j\} \sum_{n\in\mathbf{N}_4} c_n \exp\left\{-j\ln\left(\frac{B_n + a\varepsilon}{B_n}\right)\right\} \\
=\ & C\sum_{n\in\mathbf{N}_4} c_n \left(\frac{B_n}{B_n + a\varepsilon}\right)^j \leq C\sum_{n\in\mathbf{N}_4} c_n \left(\frac{B_n}{a\varepsilon}\right)^j = C\sum_{n\in\mathbf{N}_4} c_n (B_n)^j
\end{aligned}$$

$$
\begin{aligned}
&\le C \sum_{n \in \mathbf{N}_4} c_n \left[\delta^2 \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} + \sum_{k=1}^{k_n} EX_{nk}^2 I\{|X_{nk}| \le \delta\} \right]^j \\
&\le C \sum_{n \in \mathbf{N}_4} c_n \left\{ \delta^{2j} \left[\sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} \right]^j + \left[\sum_{k=1}^{k_n} EX_{nk}^2 I\{|X_{nk}| \le \delta\} \right]^j \right\} \\
&\le C \sum_{n \in \mathbf{N}_4} c_n \left\{ \delta^{2j} \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} + \left[\sum_{k=1}^{k_n} EX_{nk}^2 I\{|X_{nk}| \le \delta\} \right]^J \right\} \\
&\qquad \left(\text{since } \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} \le 1\right) \\
&\le C \sum_{n=1}^{\infty} c_n \left\{ \delta^{2j} \sum_{k=1}^{k_n} P\{|X_{nk}| > \delta\} + \left[\sum_{k=1}^{k_n} EX_{nk}^2 I\{|X_{nk}| \le \delta\} \right]^j \right\} < \infty
\end{aligned}
$$

by the assumptions. The proof is completed. □

Proof of Theorem C. In view of Corollary 2 it is enough to show that conditions (ii) and (iii)′ are satisfied.

For (ii) note that

$$
\sum_{k=1}^{k_n} EX_{nk}^2 I\{|X_{nk}| \le \delta\} \le \sup_{0 \le x \le \delta} \frac{x^2}{\phi(x)} \sum_{k=1}^{k_n} E\phi(|X_{nk}|).
$$

For (iii)′ since $EX_{nk} = 0$ it follows that

$$
\begin{aligned}
&\max_{1 \le m \le k_n} \left| \sum_{k=1}^{m} EX_{nk} I\{|X_{nk}| \le \delta\} \right| = \max_{1 \le m \le k_n} \left| \sum_{k=1}^{m} EX_{nk} I\{|X_{nk}| > \delta\} \right| \\
&\le \sup_{x > \delta} \frac{x}{\phi(x)} \sum_{k=1}^{k_n} E\phi(|X_{nk}|) \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

□

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