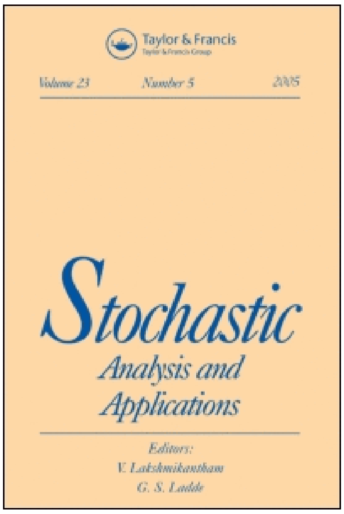

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On the Strong Rates of Convergence for Arrays of Rowwise Negatively Dependent Random Variables

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The strong convergence rate and complete convergence results for arrays of rowwise negatively dependent random variables are established. The results presented generalize the results of Chen et al. [1] and Sung et al. [2]. As applications, some well-known results on independent random variables can be easily extended to the case of negatively dependent random variables.

Keywords Arrays of rowwise negatively dependent random variables; Complete convergence; Negatively dependent random variables; Strong convergence rate.

Mathematics Subject Classification 60F15.

1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [3] as follows. A sequence $\{U_n, n \geq 1\}$ of random variables *converges completely* to the constant θ if

$$\sum_{n=1}^{\infty} P\{|U_n - \theta| > \epsilon\} < \infty \quad \text{for all } \epsilon > 0.$$

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The first results concerning the complete convergence were due to Hsu and Robbins [3] and Erdös [4, 5]. Since then there were many authors who are devoted to studying complete convergence for sums and weighted sums of independent random variables. We refer the reader to the expository article by Gut [6], where a detailed survey of results on complete convergence is given (including his own valuable contributions). After, Hu et al. [7] announced the general result on complete convergence for arrays of rowwise *independent* random variables, presented as Theorem A below.

In the following, we let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$, be an array of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, $\{k_n, n \geq 1\}$, be a sequence of positive integers such that $\lim_{n\to\infty} k_n = \infty$, and $\{a_n, n \geq 1\}$, be a sequence of positive constants. For an event $A \in \mathcal{F}$, we denote by $I(A)$ the indicator function. We should note that all the results of this article remain true in the case $k_n = \infty$ for some/all $n \geq 1$, provided the series $\sum_{k=1}^{\infty} X_{nk}$ converges almost surely. Of course, we should consider sup instead of max in the case of infinite sums.

Theorem A. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be array of rowwise independent random variables. Suppose that for every $\varepsilon > 0$ and some $\delta > 0$:*

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P\{|X_{ni}| > \varepsilon\} < \infty$,

(ii) *there exists $J \geq 1$ such that*

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} EX_{ni}^2 I\{|X_{ni}| \leq \delta\} \right)^J < \infty,$$

(iii) $\sum_{i=1}^{k_n} EX_{ni} I\{|X_{ni}| \leq \delta\} \to 0$ *as* $n \to \infty$.

Then

$$\sum_{n=1}^{\infty} a_n P\left\{ \left|\sum_{k=1}^{k_n} X_{nk}\right| > \varepsilon \right\} < \infty \quad \text{for all } \varepsilon > 0.$$

Hu et al. [7] unifies and extends the ideas of previously obtained results on complete convergence. In the main results of Hu et al. [7], no assumptions are made concerning the existence of expected values or absolute moments of the random variables. The *proof* of Hu et al. [7] is mistakenly based on the fact that the assumptions of their theorem imply convergence in probability of the corresponding partial sums. Counterexamples to this proof were presented in Hu and Volodin [9] and Hu et al. [10]. We would like to stress that the both examples are the counterexamples to the *proof* of Theorem A, but not to the result. At the same time, they mentioned that the problem as to whether Theorem A is true for any positive constants $\{a_n, n \geq 1\}$ has remained open.

Many attempts to solve this problem led to weaker variants of this theorem. Hu et al. [10] gave a first partial solution to this question. Next partial solution was given by Kuczmaszewska [11] and the question was solved completely in Sung et al. [2], where the result is proved as stated in Hu et al. [7]. The approach of Sung et al. [2] is different from those of Hu et al. [7] in the sense that it does not use the symmetrization procedure.

The next article, which, to our best knowledge, presents the most general results on complete convergence for arrays of rowwise *independent* random variables is

Kruglov et al. [8]. As it is shown in corollaries of that article, the main results of Hu et al. [7], Hu et al. [10], Kuczmaszewska [11], and Sung et al. [2] can be derived easily from the first theorem of Kruglov et al. [8].

The proofs of the main results of Kruglov et al. [8], as well as of Hu et al. [7], Hu et al. [10], Kuczmaszewska [11], and Sung et al. [2], make use of the well-known Hoffmann–Jørgensen's inequality (cf. [12]). Hoffmann–Jørgensen's maximal inequality is a powerful tool which has now become a standard technique in proving limit theorems for independent random variables.

The main purpose of Chen et al. [1] was to extend Theorem A for the case of arrays of rowwise *negatively associated* random variables. But for associated random variables, especially for negative associated random variables, it is still an open question as to whether Hoffmann-Jørgensen's maximal inequality is true or not. In order to extend Theorem A to the case of negatively associated random variables, we need to find another approach. In the article Chen et al. [1], the exponential inequality for negatively associated random variables of Kolmogorov's type, which is established by Shao [13] was used.

Negatively associated sequences have many good properties and extensive applications in multivariate statistical analysis and reliability theory. The notion of negative association has received considerable attention, there are many articles about negatively associated random variables, while articles about negatively dependent random variables are a few (see, e.g., [14–20]).

The main purpose of the current investigation is to extend Theorem A for the case of arrays of rowwise *negatively dependent* random variables. Joag-Dev and Proschan [14] pointed out that negative association property implies negatively dependency, but negative dependency does not imply negative association. They gave an example of a collection of random variables that are negatively dependent, but not negatively associated. Negative association is a much more restrictive and stronger property than negative dependence.

Hence, it not only the case that we cannot use Hoffmann-Jørgensen's maximal inequality to prove the results presented in the article, the exponential inequality by Shao [13] cannot be applied either (it is valid for negative associated random variables only). Because of that, we prove a new exponential inequality presented in Lemma 2 below.

The concept of negatively dependent random variables was introduced by Lehmann [21] as follows.

Definition 1. Random variables $Y_1, Y_2, \ldots$ are said to be *negatively dependent* if for each $n \geq 2$, the following two inequalities hold:

$$P\{Y_1 \leq y_1, \ldots, Y_n \leq y_n\} \leq \prod_{i=1}^{n} P\{Y_i \leq y_i\}$$

and

$$P\{Y_1 > y_1, \ldots, Y_n > y_n\} \leq \prod_{i=1}^{n} P\{Y_i > y_i\}$$

for every sequence $\{y_1, \ldots, y_n\}$ of real numbers.

Random variables $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ are said to be an *array of rowwise negatively dependent random variables* if for each $n \geq 1$, $\{X_{ni}, 1 \leq i \leq k_n\}$ is negatively dependent.

Note that if $\{X_i, i \geq 1\}$ is a sequence of negatively dependent random variables, then the truncation in the usual way, for example, the sequence $\{X_i I\{|X_i| < \delta\}, i \geq 1\}$, where $\delta > 0$, is not necessary negatively dependent any more. We should use so-called *monotone truncation* (see the definition of random variables Y_j in the proof of Lemma 2 below) in order to preserve this property.

The main purpose of this article is to discuss the strong convergence rate for arrays of rowwise negatively dependent random variables. Some new complete convergence results for arrays of rowwise negatively dependent random variables are obtained. The results partially extend the results of Chen et al. [1] (where the negatively associated case was considered) and Sung et al. [2] (where the independent case was considered).

Throughout this article, C will represent positive constants whose value may change from one place to another.

2. Lemmata

In order to prove our main result, we need the following lemmas. The first lemma is well known and trivial, so we omit the proof (see, e.g., [18, Lemma 1] and [15]).

Lemma 1. *Let $\{Y_n, n \geq 1\}$ be a sequence of negatively dependent random variables.*

1) *If $\{f_n, n \geq 1\}$ is a sequence of real measurable functions all of which are monotone increasing (or all monotone decreasing), then $\{f_n(Y_n), n \geq 1\}$ is a sequence of negatively dependent random variables.*
2) *For any $n \geq 1$: $E(\prod_{j=1}^n Y_j) \leq \prod_{j=1}^n E(Y_j)$ provided the expectations are finite.*

The following lemma presents an exponential inequality of Kolmogorov's type for negatively dependent random variables and plays a crucial role in the proof of the main result of the article. For a different exponential inequality of Kolmogorov's type we refer to Volodin [17].

Lemma 2. *Let $\{X_n, n \geq 1\}$ be a sequence of negatively dependent random variables with $EX_n = 0$ and $EX_n^2 < \infty, n \geq 1$. Let $S_n = \sum_{i=1}^n X_i, B_n = \sum_{i=1}^n EX_i^2$. Then for all $x > 0, y > 0$*

$$P(|S_n| > x) \leq 2P\left(\max_{1\leq k\leq n} |X_k| > y\right) + 2\exp\left\{\frac{x}{y} - \frac{x}{y}\ln\left(1 + \frac{xy}{B_n}\right)\right\}.$$

Proof. Let $Y_j = \min\{X_j, y\} = X_j I(X_j < y) + yI(X_j \geq y)$ and $T_n = \sum_{j=1}^n Y_j$. Obviously $EY_j \leq 0$ and $EY_j^2 \leq EX_j^2$. By Lemma 1(1) for $h > 0, \{e^{hY_j}, j \geq 1\}$ is nonnegative negatively dependent. Thus, by Lemma 1(2), we have

$$Ee^{hT_n} \leq \prod_{j=1}^n Ee^{hY_j}.$$

For fixed $h > 0$, the function $g(x) = (e^{hx} - 1 - hx)/x^2$ is increasing for all x, therefore

$$\begin{aligned} Ee^{hY_j} &= 1 + hEY_j + E\left[(e^{hX_j} - hX_j - 1)I(X_j < y)\right] + (e^{hy} - 1 - hy)P(X_j \geq y) \\ &\leq 1 + \frac{e^{hy} - hy - 1}{y^2}\left\{EX_j^2 I(X_j < y) + y^2 P(X_j \geq y)\right\} \\ &\leq 1 + \frac{e^{hy} - hy - 1}{y^2} EX_j^2. \end{aligned}$$

Since $e^x \geq 1 + x$ for all x and by Markov's inequality, we have

$$P(T_n > x) \leq e^{-hx} Ee^{hT_n} \leq \exp\left(-hx + \frac{e^{hy} - hy - 1}{y^2} B_n\right).$$

Chose $h = \frac{1}{y}\ln\left(1 + \frac{xy}{B_n}\right)$, then

$$\frac{e^{hy} - hy - 1}{y^2} B_n = \frac{x}{y} - \frac{B_n}{y^2}\ln\left(1 + \frac{xy}{B_n}\right) \leq \frac{x}{y},$$

and, hence,

$$P(T_n > x) \leq \exp\left\{\frac{x}{y} - \frac{x}{y}\ln\left(1 + \frac{xy}{B_n}\right)\right\}.$$

Clearly, the events $\{S_n > x\} \subset \left\{\max_{1\leq k\leq n} X_k > y\right\} \cup \{T_n > x\}$. Then

$$\begin{aligned} P(S_n > x) &\leq P\left(\max_{1\leq k\leq n} X_k > y\right) + P(T_n > x) \\ &\leq P\left(\max_{1\leq k\leq n} |X_k| > y\right) + \exp\left\{\frac{x}{y} - \frac{x}{y}\ln\left(1 + \frac{xy}{B_n}\right)\right\}. \end{aligned}$$

If we consider $-X_n$ instead of X_n in the arguments above, in a similar manner we obtain

$$P(-S_n > x) \leq P\left(\max_{1\leq k\leq n} |X_k| > y\right) + \exp\left\{\frac{x}{y} - \frac{x}{y}\ln\left(1 + \frac{xy}{B_n}\right)\right\}.$$

Therefore,

$$P(|S_n| > x) \leq 2P\left(\max_{1\leq k\leq n} |X_k| > y\right) + 2\exp\left\{\frac{x}{y} - \frac{x}{y}\ln\left(1 + \frac{xy}{B_n}\right)\right\} \qquad \square$$

3. Main Results

The first theorem is a slightly different result than a direct generalization of Theorem A for the case of negatively dependent random variables. The exact extension of Theorem A is presented in Corollary 1.

Theorem 1. *Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise negatively dependent random variables and $\{a_n, n \ge 1\}$ be a sequence of positive constants. Suppose that for every $\varepsilon > 0$ and some $\delta > 0$*

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|X_{ni}| > \epsilon) < \infty$,
(ii) *there exists $J \ge 1$ such that*

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} \mathrm{Var}(X_{ni} I(|X_{ni}| \le \delta)) \right)^J < \infty.$$

Then

$$\sum_{n=1}^{\infty} a_n P\left(\left| \sum_{i=1}^{k_n} (X_{ni} - EX_{ni} I(|X_{ni}| \le \delta)) \right| > \varepsilon \right) < \infty \quad \textit{for all } \varepsilon > 0.$$

Proof. Note that for any fixed $\varepsilon > 0$ and $n \ge 1$

$$\begin{aligned} & P\left(\left| \sum_{i=1}^{k_n} (X_{ni} - EX_{ni} I(|X_{ni}| \le \delta)) \right| > \varepsilon \right) \\ & \le P\left(\left| \sum_{i=1}^{k_n} (X_{ni} - EX_{ni} I(|X_{ni}| \le \delta)) \right| > \varepsilon, |X_{ni}| > \delta, \text{ for some } i, 1 \le i \le k_n \right) \\ & \quad + P\left(\left| \sum_{i=1}^{k_n} (X_{ni} - EX_{ni} I(|X_{ni}| \le \delta)) \right| > \varepsilon, |X_{ni}| \le \delta, \text{ for all } i, 1 \le i \le k_n \right) \\ & \le \sum_{i=1}^{k_n} P(|X_{ni}| > \delta) + P\left(\left| \sum_{i=1}^{k_n} (X_{ni} I(|X_{ni}| \le \delta) - EX_{ni} I(|X_{ni}| \le \delta)) \right| > \varepsilon \right). \end{aligned}$$

By condition (i), it is enough to prove that

$$\sum_{n=1}^{\infty} a_n P\left(\left| \sum_{i=1}^{k_n} (X_{ni} I(|X_{ni}| \le \delta) - EX_{ni} I(|X_{ni}| \le \delta)) \right| > \varepsilon \right) < \infty.$$

Set

$$\begin{aligned} U_{ni} &= \delta I(X_{ni} > \delta) + X_{ni} I(|X_{ni}| \le \delta) - \delta I(X_{ni} < -\delta) \quad \text{and} \\ U'_{ni} &= \delta I(X_{ni} > \delta) - \delta I(X_{ni} < -\delta), \quad \forall 1 \le i \le k_n, \;\; n \ge 1. \end{aligned}$$

We have

$$\begin{aligned} & \sum_{n=1}^{\infty} a_n P\left(\left| \sum_{i=1}^{k_n} (X_{ni} I(|X_{ni}| \le \delta) - EX_{ni} I(|X_{ni}| \le \delta)) \right| > \varepsilon \right) \\ & = \sum_{n=1}^{\infty} a_n P\left(\left| \sum_{i=1}^{k_n} (U_{ni} - EU_{ni} - U'_{ni} + EU'_{ni}) \right| > \varepsilon \right) \\ & \le \sum_{n=1}^{\infty} a_n P\left(\left| \sum_{i=1}^{k_n} (U'_{ni} - EU'_{ni}) \right| > \varepsilon/2 \right) + \sum_{n=1}^{\infty} a_n P\left(\left| \sum_{i=1}^{k_n} (U_{ni} - EU_{ni}) \right| > \varepsilon/2 \right) \\ & = I + II, \quad \text{say}. \end{aligned}$$

For I, by condition (i) and Markov's inequality, we have

$$I \leq C\sum_{n=1}^{\infty} a_n E\left|\sum_{i=1}^{k_n} U'_{ni}\right| \leq C\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|X_{ni}| > \delta) < \infty.$$

For II, let $B_n = \sum_{i=1}^{k_n} \mathrm{Var}(U_{ni})$, then

$$B_n \leq 3\sum_{i=1}^{k_n} \mathrm{Var}\,(X_{ni}I(|X_{ni}| \leq \delta)) + 6\delta^2 \sum_{i=1}^{k_n} P(|X_{ni}| > \delta).$$

For any $y > 0$, set

$$d = \min\left\{1, \frac{y}{6\delta}\right\}, \mathbf{N_1} = \left\{n : \sum_{i=1}^{k_n} P\left(|X_{ni}| > \min\left\{\delta, \frac{y}{6}\right\}\right) > d\right\}, \quad \text{and} \quad \mathbf{N_2} = \mathbf{N}/\mathbf{N_1}.$$

Note that

$$\begin{aligned}\sum_{n\in\mathbf{N_1}} a_n P\left(\left|\sum_{i=1}^{k_n}(U_{ni} - EU_{ni})\right| > \varepsilon/2\right) &\leq \sum_{n\in\mathbf{N_1}} a_n \\ &\leq \frac{1}{d}\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P\left(|X_{ni}| > \min\left\{\delta, \frac{y}{6}\right\}\right) < \infty.\end{aligned}$$

Hence, it is sufficient to prove that $\sum_{n\in\mathbf{N_2}} a_n P(|\sum_{i=1}^{k_n}(U_{ni} - EU_{ni})| > \varepsilon/2) < \infty$.

Since $\{U_{ni} - EU_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ is an array of rowwise negatively dependent random variables, by Lemma 2 we have

$$\begin{aligned}\sum_{n\in\mathbf{N_2}} a_n P\left(\left|\sum_{i=1}^{k_n}(U_{ni} - EU_{ni})\right| > \varepsilon/2\right) &\leq 2\sum_{n\in N_2} a_n P\left(\max_{1\leq i\leq k_n} |U_{ni} - EU_{ni}| > y\right) \\ &\quad + 2\sum_{n\in N_2} a_n \exp\left\{\frac{\varepsilon}{2y} - \frac{\varepsilon}{2y}\ln\left(1 + \frac{\varepsilon y}{2B_n}\right)\right\}.\end{aligned}$$

Note that

$$\begin{aligned}P\left(\max_{1\leq i\leq k_n} |U_{ni} - EU_{ni}| > y\right) &\leq P\left(\max_{1\leq i\leq k_n} |X_{ni}I(|X_{ni}| \leq \delta) - EX_{ni}I(|X_{ni}| \leq \delta)| > y/2\right) \\ &\quad + P\left(\max_{1\leq i\leq k_n} |U'_{ni} - EU'_{ni}| > y/2\right).\end{aligned}$$

Next, for any $n \in \mathbf{N_2}$

$$\begin{aligned}\max_{1\leq i\leq k_n} |EX_{ni}I(|X_{ni}| \leq \delta)| &\leq \max_{1\leq i\leq k_n} E|X_{ni}|I(|X_{ni}| \leq \delta) \\ &\leq \max_{1\leq i\leq k_n} (E|X_{ni}|I(|X_{ni}| \leq y/6) + E|X_{ni}|I(y/6 < |X_{ni}| \leq \delta)) \\ &\leq y/6 + \delta\sum_{i=1}^{k_n} P\,(|X_{ni}| > \min\{y/6, \delta\}) \leq y/6 + \delta d \leq y/3.\end{aligned}$$

Therefore,

$$\sum_{n\in\mathbf{N_2}} a_n P\left(\max_{1\le i\le k_n} |U_{ni}-EU_{ni}|>y\right)$$
$$\le \sum_{n=1}^{\infty} a_n P\left(\max_{1\le i\le k_n} |X_{ni}|I(|X_{ni}|\le\delta)>y/6\right)+\sum_{n=1}^{\infty} a_n P\left(\max_{1\le i\le k_n} |U'_{ni}-EU'_{ni}|>y/2\right)$$
$$\le \sum_{n=1}^{\infty} a_n \sum_{1=1}^{k_n} P(|X_{ni}|>y/6)+C\sum_{n=1}^{\infty} a_n \sum_{1=1}^{k_n} P(|X_{ni}|>\delta)<\infty.$$

When $n\in\mathbf{N_2}$, we have that $\sum_{i=1}^{k_n} P(|X_{ni}|>\delta)\le 1$. Let $y=\varepsilon/(2J)$, by conditions (i) and (ii) we obtain

$$\sum_{n\in\mathbf{N_2}} a_n \exp\left\{\frac{\varepsilon}{2y}-\frac{\varepsilon}{2y}\ln\left(1+\frac{\varepsilon y}{2B_n}\right)\right\}$$
$$\le \exp\left(\frac{\varepsilon}{2y}\right)\sum_{n\in\mathbf{N_2}} a_n\left(1+\frac{\varepsilon y}{2B_n}\right)^{-\varepsilon/(2y)} \le \exp\left(\frac{\varepsilon}{2y}\right)\sum_{n\in\mathbf{N_2}} a_n\left(\frac{2B_n}{\varepsilon y}\right)^{J}$$
$$\le C\sum_{n=1}^{\infty} a_n B_n^J \le C\sum_{n=1}^{\infty} a_n\left(\sum_{i=1}^{k_n} P(|X_{ni}|>\delta)\right)^J + C\sum_{n=1}^{\infty} a_n\left(\sum_{i=1}^{k_n} \mathrm{Var}(X_{ni}I(|X_{ni}|\le\delta))\right)^J$$
$$\le C\sum_{n=1}^{\infty} a_n\sum_{i=1}^{k_n} P(|X_{ni}|>\delta) + C\sum_{n=1}^{\infty} a_n\left(\sum_{i=1}^{k_n} \mathrm{Var}(X_{ni}I(|X_{ni}|\le\delta))\right)^J<\infty.$$

□

The following are two corollaries to Theorem 1.

Corollary 1. *Let $\{X_{ni}, 1\le i\le k_n, n\ge 1\}$ be an array of rowwise negatively dependent random variables. If conditions* (i) *and* (ii) *of Theorem* 1 *and*

(iii) $\lim_{n\to\infty}\sum_{i=1}^{k_n} EX_{ni}I(|X_{ni}|\le\delta)=0$

are satisfied, then

$$\sum_{n=1}^{\infty} a_n P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right|>\varepsilon\right)<\infty \quad \textit{for all } \varepsilon>0.$$

With Theorem 1 in hand, the proof of Corollary 1 is obvious and, hence, is omitted.

Corollary 2. *Let $\{X_{ni}, 1\le i\le k_n, n\ge 1\}$ be an array of rowwise negatively dependent random variables with $EX_{ni}=0$ for all $1\le i\le k_n,\ n\ge 1$. Let $\phi(x)$ be a real function such that for some $\delta>0$*

$$\sup_{x>\delta}\frac{x}{\phi(x)}<\infty \quad \textit{and} \quad \sup_{0\le x\le\delta}\frac{x^2}{\phi(x)}<\infty.$$

Suppose that for all $\varepsilon>0$, condition (i) is satisfied and there exists $J\ge 1$ such that $\sum_{n=1}^{\infty} a_n\big(\sum_{i=1}^{k_n} E\phi(|X_{ni}|)\big)^J<\infty$ and $\sum_{i=1}^{k_n} E\phi(|X_{ni}|)\to 0$ as $n\to\infty$.

Then

$$\sum_{n=1}^{\infty} a_n P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon\right) < \infty \quad \textit{for all } \varepsilon > 0.$$

Proof. We show that the conditions of Corollary 1 are satisfied.

Note that

$$\sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}| \leq \delta) \leq \sup_{0 \leq x \leq \delta} \frac{x^2}{\phi(x)} \sum_{i=1}^{k_n} E\phi(|X_{ni}|).$$

Since $EX_{ni} = 0$ it follows that

$$\left|\sum_{i=1}^{k_n} EX_{ni} I(|X_{ni}| \leq \delta)\right| \leq \left|\sum_{i=1}^{k_n} EX_{ni} I(|X_{ni}| > \delta)\right|$$
$$\leq \sup_{x > \delta} \frac{x}{\phi(x)} \sum_{i=1}^{k_n} E\phi(|X_{ni}|) \to 0 \quad \text{as } n \to \infty. \qquad \square$$

The second theorem refines Theorem 1 for the case when all random variables comprising the array are centered and have a finite moment.

Theorem 2. *Let* $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ *be an array of rowwise negatively dependent random variables with* $EX_{ni} = 0$ *for all* $1 \leq i \leq k_n, n \geq 1$. *Suppose that for all* $\varepsilon > 0$, *condition* (*i*) *is satisfied and there exists* $J \geq 1$ *and* $0 < p \leq 2$ *such that* $\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} E|X_{ni}|^p\right)^J < \infty$.

Then

$$\sum_{n=1}^{\infty} a_n P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0.$$

Proof. Note that for any fixed $\varepsilon > 0$ and $n \geq 1$ the events

$$\left\{\left|\sum_{i=1}^{k_n}(X_{ni} I(|X_{ni}| > \delta) - EX_{ni} I(|X_{ni}| > \delta))\right| > \varepsilon/2\right\}$$
$$\subset \{|X_{ni}| > \delta \text{ for at least one value of } i, 1 \leq i \leq k_n\} \subset \cup_{i=1}^{k_n}\{|X_{ni}| > \delta\}.$$

Therefore, by $EX_{ni} = 0$, we have

$$P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon\right)$$
$$\leq P\left(\left|\sum_{i=1}^{k_n}(X_{ni} I(|X_{ni}| > \delta) - EX_{ni} I(|X_{ni}| > \delta))\right| > \varepsilon/2\right)$$
$$+ P\left(\left|\sum_{i=1}^{k_n}(X_{ni} I(|X_{ni}| \leq \delta) - EX_{ni} I(|X_{ni}| \leq \delta))\right| > \varepsilon/2\right)$$
$$\leq \sum_{i=1}^{k_n} P(|X_{ni}| > \delta) + P\left(\left|\sum_{i=1}^{k_n}(X_{ni} I(|X_{ni}| \leq \delta) - EX_{ni} I(|X_{ni}| \leq \delta))\right| > \varepsilon/2\right),$$

By the same argument as in the proof of Theorem 1, it suffices to prove that condition (ii) is satisfied. Since $0 < p \leq 2$, we have

$$\sum_{i=1}^{k_n} \mathrm{Var}\,(X_{ni}I(|X_{ni}| \leq \delta)) \leq \sum_{i=1}^{k_n} E\,(X_{ni}I(|X_{ni}| \leq \delta))^2 = \delta^2 \sum_{i=1}^{k_n} E\left(\frac{X_{ni}I(|X_{ni}| \leq \delta)}{\delta}\right)^2$$
$$\leq \delta^2 \sum_{i=1}^{k_n} E\left(\frac{|X_{ni}|I(|X_{ni}| \leq \delta)}{\delta}\right)^p \leq \delta^{2-p} \sum_{i=1}^{k_n} E|X_{ni}|^p. \qquad \square$$

An open problem. Note that the results of Kruglov et al. [8] and Chen et al. [1] provide a stronger conclusion on the rate of convergence for *maximums* of partial sums than the results presented in Theorems 1 and 2 above; that is, they obtained results of the form

$$\sum_{n=1}^{\infty} a_n P\left(\max_{1\leq m\leq k_n}\left|\sum_{k=1}^{m} X_{nk}\right| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0.$$

It is still an open problem to obtain results of this type for arrays of rowwise negatively dependent random variables. The authors suggest that a solution can be obtained if a better exponential inequality than that which is presented above in Lemma 2 could be established.

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