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Almost Sure lim sup Behavior of Dependent Bootstrap Means

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Abstract: In this article, the upper bound for the exact convergence rate (i.e., the law of the logarithm type result) is obtained for dependent bootstrap means.

Keywords: Bootstrap means; Dependent bootstrap; Kolmogorov exponential inequality; Law of the logarithm; Pairwise i.i.d. sequences; Stationary ergodic sequences.

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I. INTRODUCTION

The main focus of the present investigation is to obtain the upper bound of the exact convergence rate (i.e., a law of the logarithm type result) for dependent bootstrap means from a sequence of random variables. The work on the consistency of bootstrap estimators has

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received much attention in recent years due to a growing demand for the procedure, both theoretically and practically. It is important to note that exponential inequalities are of practical use in establishing the strong asymptotic validity of bootstrap mean.

We begin with a brief discussion of results in the literature pertaining to a sequence of independent and identically distributed (i.i.d.) random variables and the classical (independent) bootstrap of the mean. Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. For $\omega \in \Omega$ and $n \geq 1$, let $P_n(\omega) = n^{-1} \sum_{i=1}^{n} \delta_{X_i(\omega)}$ denote the empirical measure, and let $\{\widehat{X}_{n,j}^{(\omega)}, 1 \leq j \leq m(n)\}$ be i.i.d. random variables with law $P_n(\omega)$ where $\{m(n), n \geq 1\}$ is a sequence of positive integers. In other words, the random variables $\{\widehat{X}_{n,j}^{(\omega)}, 1 \leq j \leq m(n)\}$ result by sampling $m(n)$ times with replacement from the n observations

$$X_1(\omega), \ldots, X_n(\omega)$$

such that for each of the $m(n)$ selections, each $X_j(\omega)$ has probability n^{-1} of being chosen. For each $n \geq 1$, $\{\widehat{X}_{n,j}^{(\omega)}, 1 \leq j \leq m(n)\}$ is the so-called Efron [1] bootstrap sample from $X_1, \ldots, X_n$ with bootstrap sample size $m(n)$. Let $\overline{X}_n(\omega) = \frac{1}{n} \sum_{j=1}^{n} X_j(\omega)$ denote the sample mean of $\{X_j(\omega), 1 \leq j \leq n\}$, $n \geq 1$.

When X is nondegenerate and $EX^2 < \infty$, Bickel and Freedman [16] showed that for almost every $\omega \in \Omega$ the central limit theorem (CLT)

$$n^{1/2}\left(\frac{1}{n}\sum_{j=1}^{n} \widehat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega)\right) \xrightarrow{d} N(0, \sigma^2)$$

obtains. Here and below and $\sigma^2 = \operatorname{Var} X$. Note that by the Glivenko–Cantelli theorem $P_n(\omega)$ is close to $\mathscr{L}(X)$ for almost every $\omega \in \Omega$ and all large n, and by the classical Lévy CLT

$$n^{1/2}\left(\frac{1}{n}\sum_{j=1}^{n} X_j - EX\right) \xrightarrow{d} N(0, \sigma^2).$$

It follows that for almost every $\omega \in \Omega$, the bootstrap statistic

$$n^{1/2}\left(\frac{1}{n}\sum_{j=1}^{n} \widehat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega)\right)$$

is close in distribution to that of

$$n^{1/2}\left(\frac{1}{n}\sum_{j=1}^{n} X_j - EX\right)$$

for all large n: This is the basic idea behind the bootstrap. See the pioneering work of Efron [1] where this nice idea is made explicit and where it is substantiated with several important examples.

Moreover, strong laws of large numbers were proved by Athreya [2] and Csörgö [3] for bootstrap means. Arenal-Gutiérrez et al. [4] analyzed the results of Athreya [2] and Csörgő [3]. Then, by taking into account the different growth rates for the resampling size $m(n)$, they gave new and simple proofs of those results. They also provided examples that show that the sizes of resampling required by their results to ensure almost sure (a.s.) convergence are not far from optimal.

Another article which is important for this article is the work carried out by Mikosch [5]. He established a series of useful exponential inequalities that are an important tool for deriving results on the consistency of the bootstrap mean. Moreover, the pioneering work establishing a law of the logarithm type result for bootstrap means for the sequence of i.i.d. random variables was carried out in Mikosch. The law of logarithm type result for the bootstrapped means from the arbitrary sequence (not necessary independent or identically distributed) random variables was established in Ahmed et al. [6].

The main goal of the present article is to extend and generalize the results of Ahmed et al. [6]. On the law of the logarithm to the case of dependent bootstrap procedure. The notion of the dependent bootstrap procedure will be presented in the next section.

II. DEPENDENT BOOTSTRAP

The results from this section are modifications, generalizations and extensions of the results of Smith and Taylor [7] for the dependent bootstrap from the sequence of arbitrary (not necessarily i.i.d.) random variables. We mention that Smith and Taylor [7] consider only i.i.d. case. We present this section as a simple reference since it plays a role in what follows.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables (which are not necessarily independent or identically distributed) defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\{m(n), n \geq 1\}$ and $\{k(n), n \geq 1\}$ be two sequences of positive integers such that for all $n \geq 1$:

$$m(n) \leq nk(n).$$

For $\omega \in \Omega$ and $n \geq 1$, the dependent bootstrap is defined as the sample of size $m(n)$, denoted $\{X_{n,j}^{(\omega)}, 1 \leq j \leq m(n)\}$, drawn without replacement from the collection of $nk(n)$ items made up of $k(n)$ copies each of the sample observations $X_1(\omega), \ldots, X_n(\omega)$.

The dependent bootstrap procedure is proposed as a procedure to reduce variation of estimators and to obtain better confidence intervals. We refer to the paper Smith and Taylor [8] where this fact is proven and simulated confidence intervals are used to examine possible gains in coverage probabilities and interval lengths.

The first proposition gives us the joint distribution of the dependent bootstrap random variables. We need the following notations.

For $\omega \in \Omega$, $n \geq 1$, and a real number x, denote

$$\tau(x) = \sum_{j=1}^{n} I\{X_j(\omega) \leq x\},$$

where $I(\cdot)$ is the indicator function. Hence, $\tau(x)$ is the random variable that counts the number of observations less than or equal to x.

For a finite set $\{x_1, x_2, \ldots, x_m\}$ of real numbers, let $\{x_{(1)}, x_{(2)}, \ldots, x_{(m)}\}$ denote its nondecreasing rearrangement, that is $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(m)}$ and for any $1 \leq j \leq m$ there exists $1 \leq i \leq m$ such that $x_i = x_{(j)}$.

Proposition 1. *For* $\omega \in \Omega$, $n \geq 1$, *and a set* $\{x_1, x_2, \ldots, x_m\}$ *of real numbers*:

1) *If* $k(n)\tau(x_{(j)}) \geq j$ *for all* $1 \leq j \leq m(n)$, *then*

$$P\{\widehat{X}_{n,1}^{(\omega)} \leq x_1, \ldots, \widehat{X}_{n,m(n)}^{(\omega)} \leq x_{m(n)}\} = \prod_{j=1}^{m(n)} \frac{k(n)\tau(x_{(j)}) - (j-1)}{k(n)n - (j-1)}.$$

2) *If* $k(n)\tau(x_{(j)}) < j$ *for at least one* $1 \leq j \leq m(n)$, *then the above probability is zero.*

Proof. Let π be the reordering of $\{1, 2, \ldots, m(n)\}$ such that $\pi(j) = i$ for $x_i = x_{(j)}$. Then

$$\begin{aligned}
&P\{\widehat{X}_{n,1}^{(\omega)} \leq x_1, \ldots, \widehat{X}_{n,m(n)}^{(\omega)} \leq x_{m(n)}\} \\
&= P\{\widehat{X}_{n,\pi(1)}^{(\omega)} \leq x_{(1)}, \ldots \widehat{X}_{n,\pi(m(n))}^{(\omega)} \leq x_{(m(n))}\} \\
&= P\{\widehat{X}_{n,\pi(1)}^{(\omega)} \leq x_{(1)}\} \times P\{\widehat{X}_{n,\pi(2)}^{(\omega)} \leq x_{(2)} | \widehat{X}_{n,\pi(1)}^{(\omega)} \leq x_{((1))}\} \\
&\quad \times \cdots \times P\{\widehat{X}_{n,\pi(m(n))}^{(\omega)} \leq x_{(m(n))} | \widehat{X}_{n,\pi(1)}^{(\omega)} \leq x_{(1)}, \ldots, \widehat{X}_{n,\pi(m(n)-1)}^{(\omega)} \leq x_{(m(n)-1)}\} \\
&= \prod_{j=1}^{m(n)} \frac{k(n)\tau(x_{(j)}) - (j-1)}{k(n)n - (j-1)}
\end{aligned}$$

if $k(n)\tau(x_{(j)}) \geq j$ for all $1 \leq j \leq m(n)$.

The second part of the proposition is obvious.

Of course, the dependent bootstrap random variables $\{\widehat{X}_{n,j}^{(\omega)}, 1 \leq j \leq m(n)\}$ are dependent. They obey the so-called negatively dependent property; this property will be established in Proposition 2. The concept of negatively dependent random variables was introduced by Lehmann [9] as follows.

The random variables $Y_1, Y_2, \ldots$ are said to be negatively dependent if for each $n \geq 2$ the following two inequalities hold:

$$P\{Y_1 \leq y_1, \ldots, Y_n \leq y_n\} \leq \prod_{i=1}^{n} P\{Y_i \leq y_i\}$$

and

$$P\{Y_1 > y_1, \ldots, Y_n > y_n\} \leq \prod_{i=1}^{n} P\{Y_i > y_i\}$$

for any set $\{y_1, \ldots, y_n\}$ of real numbers.

Proposition 2. *For $\omega \in \Omega$ and $n \geq 1$ the dependent bootstrap random variables $\{X_{n,j}^{(\omega)}, 1 \leq j \leq m(n)\}$ are negatively dependent and finitely exchangeable.*

Proof. For the negative dependence property we will prove only the first inequality. The proof of the second one is similar.

Let $\{x_1, x_2, \ldots, x_{m(n)}\}$ be a sequence of real numbers. We need to consider only the case $k(n)\tau(x_{(j)}) \geq j$ for all $1 \leq j \leq m(n)$. By Proposition 1

$$\begin{aligned} P\{\widehat{X}_{n,1}^{(\omega)} \leq x_1, \ldots, \widehat{X}_{n,m(n)}^{(\omega)} \leq x_{m(n)}\} &= \prod_{j=1}^{m(n)} \frac{k(n)\tau(x_{(j)}) - (j-1)}{k(n)n - (j-1)} \\ &\leq \prod_{j=1}^{m(n)} \frac{k(n)\tau(x_{(j)})}{k(n)n} = \prod_{j=1}^{m(n)} P\{\widehat{X}_{n,j}^{(\omega)} \leq x_j\}. \end{aligned}$$

The exchangeability is obvious by Proposition 1.

Remark 1. Note that the dependent bootstrap random variables are only *finitely* exchangeable, they cannot be embedded into *infinite* sequence of exchangeable ranom variables. This follows from the simple observation that an infinite sequence of exchangeable random variables cannot be negatively correlated (cf. Taylor et al. [10] inequality (1.1.4), p. 8), while in view of Lemma 1(2) given below, negatively dependent random variables are negatively correlated, provided the expecatations exist.

III. A FEW TECHNICAL LEMMAS

In this section we present a few technical results that we will use in proofs of the main results of the article. Some of the lemmas are only generalizations and extensions of well-known results. For expository purposes we outline their proofs.

For the simplicity, by the log-function in this section we mean the natural logarithm function. The results can be easily generalized to any other logarithm function with base greater than one.

The first lemma is well known and trivial (cf. for example Bozorgnia et al. [11]).

Lemma 1. *Let $\{Y_n, n \geq 1\}$ be a sequence of negatively dependent random variables.*

1) *If $\{f_n, n \geq 1\}$ is a sequence of measurable real functions all of which are monotone increasing (or all monotone decreasing), then $\{f_n(Y_n), n \geq 1\}$ is a sequence of negatively dependent random variables.*
2) *For any $n \geq 1 : E(\prod_{j=1}^n Y_j) \leq \prod_{j=1}^n E(Y_j)$ provided the expectations are finite.*

Next two lemmas deal with convergence of maximums of random variables. No assumption of independence is made.

Lemma 2. *Let $\{X_n, n \geq 1\}$ be a sequence of positive random variables and $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive constants such that $b_n \to \infty$. Then the assertions $X_n/b_n \to 0$ a.s. and $\frac{1}{b^n} \max_{1 \leq j \leq n} X_j \to 0$ a.s. are equivalent.*

Proof. Let $X_n/b_n \to 0$ a.s. For arbitrary $n \geq k \geq 2$,

$$\begin{aligned}
\frac{1}{b_n} \max_{1 \leq j \leq n} X_j &\leq \frac{1}{b_n} \max_{1 \leq j \leq k-1} X_j + \frac{1}{b_n} \max_{k \leq j \leq n} X_j \\
&\leq \frac{1}{b_n} \max_{1 \leq j \leq k-1} X_j + \max_{k \leq j \leq n} X_j/b_j \quad (\text{since } \{b_n, n \geq 1\} \text{ is nondecreasing}) \\
&\leq \frac{1}{b_n} \max_{1 \leq j \leq k-1} X_j + \sup_{j \geq k} X_j/b_j \to 0
\end{aligned}$$

as first $n \to \infty$ and then $k \to \infty$. The reverse implication is obvious.

The following lemma in this section is a generalization of the Corollary to Theorem 3 of Barnes and Tucker [12].

Lemma 3. *Let $\psi(t), t \geq 0$ be a strictly increasing function and $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive numbers such that*

$$\psi(b_n) \geq Cn, \quad n \geq 1,$$

where constant C does not depend on n. Let moreover $\{X_n, n \geq 1\}$ be a sequence of positive identically distributed random variables such that $E\psi(X_1/\epsilon) < \infty$ for all $\epsilon > 0$. Then

$$\frac{1}{b^n} \max_{1 \leq j \leq n} X_j \to 0 \quad a.s.$$

Proof. For arbitrary $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\{X_n > \epsilon b_n\} \leq \sum_{n=1}^{\infty} P\left\{\frac{1}{C}\psi\left(\frac{X_1}{\epsilon}\right) > n\right\} \leq \frac{1}{C}E\psi\left(\frac{X_1}{\epsilon}\right) < \infty.$$

Then by the Borel–Cantelli lemma $X_n/b_n \to 0$ a.s. By Lemma 2 we obtain that

$$\frac{1}{b_n}\max_{1\leq j\leq n} X_j \to 0 \quad \text{a.s.}$$

The last lemma in this section is only a technical result that will help us to improve a constant in the Kolmogorov exponential in equality.

Lemma 4. *Let* $a > 0$ *and* $0 < \alpha \leq \frac{a^3}{2(e^a-1-a-a^2/2)}$. *Then*

$$e^x - 1 - x - \frac{x^2}{2} \leq \frac{x^3}{2\alpha}$$

for all $0 \leq x \leq a$.

Proof. Consider the function

$$f(x, \alpha) = \ln\left(1 + x + \frac{x^2}{2} + \frac{x^3}{2\alpha}\right) - x.$$

We need to prove that $f(x, \alpha) \geq 0$ for all $0 < \alpha \leq \frac{a^3}{2(e^a-1-a-a^2/2)}$ and $0 \leq x \leq a$.

Take the derivative

$$\frac{\partial f}{\partial x} = -\frac{x^2(x-(3-\alpha))}{2\alpha(1+x+x^2/2+x^3/(2\alpha))}.$$

Hence, f is increasing in x on the interval $(0, 3-\alpha)$ and decreasing on the interval $(3-\alpha, a)$.

Note that $f(0, \alpha) = 0$ and $f(a, \alpha) \geq 0$ since $\alpha \leq \frac{a^3}{2(e^a-1-a-a^2/2)}$.

The next lemma is a generalization of the famous Komogorov exponential inequality (cf. for example Stout [13], Theorem 5.2.2(i)). The result was announced in Volodin [14].

Lemma 5. *Let* $\{X_n, n \geq 1\}$ *be a sequence of negatively dependent random variables with zero means and finite variances. Let*

$$s_n^2 = \sum_{k=1}^{n} EX_k^2, \quad n \geq 1,$$

and assume that $|X_k| \le Cs_n$ *a.s. for each* $1 \le k \le n$ *and* $n \ge 1$. *For each* $a > 0$ *and* $n \ge 1$, *if* $\epsilon C \le a$ *and* $0 < \alpha \le \frac{a^3}{2(e^a - 1 - a - a^2/2)}$, *then*

$$P\left\{S_n/s_n > \epsilon\right\} \le \exp\left\{-\frac{\epsilon^2}{2}\left(1 - \frac{\epsilon C}{\alpha}\right)\right\},$$

where $S_n = \sum_{k=1}^n X_k$, $n \ge 1$.

Proof. Fix $n \ge 1$ and $a > 0$. Suppose $x = \epsilon C \le a$. For each $1 \le k \le n$,

$$\begin{aligned} E\exp\left\{\epsilon X_k/s_n\right\} &= 1 + \frac{\epsilon^2 EX_k^2}{2!s_n^2} + \frac{\epsilon^3 X_k^3}{3!s_n^3} + \cdots \\ &\le 1 + \frac{\epsilon^2 EX_k^2}{2s_n^2}\left(1 + \frac{\epsilon C}{3} + \frac{\epsilon^2 C^2}{3\cdot 4} + \cdots\right) \\ &= 1 + \frac{\epsilon^2 EX_k^2}{2s_n^2}\left(1 + \frac{x}{3} + \frac{x^2}{3\cdot 4} + \cdots\right). \end{aligned}$$

By Lemma 4,

$$e^x - x - \frac{x^2}{2} = 1 + \frac{x^2}{2}\left(\frac{x}{3} + \frac{x^2}{3\cdot 4} + \cdots\right) \le 1 + \frac{x^3}{2\alpha} = 1 + \frac{x^2}{2}\cdot\frac{x}{\alpha},$$

Hence,

$$\frac{x}{3} + \frac{x^2}{3\cdot 4} + \cdots \le \frac{x}{\alpha} \quad \text{or} \quad 1 + \frac{x}{3} + \frac{x^2}{3\cdot 4} + \cdots \le 1 + \frac{x}{\alpha}.$$

Therefore,

$$E\exp\{\epsilon X_k/s_n\} \le 1 + \frac{\epsilon^2 EX_k^2}{2s_n^2}\left(1 + \frac{x}{\alpha}\right) \le \exp\left\{\frac{\epsilon^2 EX_k^2}{2s_n^2}\left(1 + \frac{x}{\alpha}\right)\right\}$$

since $1 + t \le e^t$ for all t. By Lemma 1,

$$E\exp\{\epsilon S_n/s_n\} \le \exp\left\{\frac{\epsilon^2}{2}\left(1 + \frac{\epsilon C}{\alpha}\right)\right\}.$$

Thus,

$$P\{S_n/s_n > \epsilon\} \le \exp\{-\epsilon^2\}E\exp\{\epsilon S_n/s_n\} \le \exp\left\{-\frac{\epsilon^2}{2}\left(1 - \frac{\epsilon C}{\alpha}\right)\right\}.$$

Remarks. 2. Even for $a = 1$, our lemma gives better constant

$$\alpha = \frac{1}{2e - 5} = 2.2906\cdots > 2,$$

while in the Kolmogorov original inequality we have $\alpha = 2$ (cf. Stout [13], p. 263).

3. If $a \to 0$ then $\alpha \to 3$. We need $a \to 0$ for the proof of the law of the logarithm.

4. Another interesting advantage of the lemma is that we can consider any positive a, while in the Kolmogorov inequality $a = 1$. In our inequality the upper bound involves a fixed (given) C and fixed ϵ and a variable α. Now, α is a function of a, for $a \geq \epsilon C$. Note that the left-hand side of the inequality does not involve a anywhere, whereas the right-hand side is, in effect, a function of a. So the best possible inequality occurs when a is chosen so that $\alpha(a)$ is maximized on the interval $[\epsilon C, \infty]$. However, α is a decreasing function of a, so the maximum value of the upper bound occurs when a is as small as possible; that is, when $a = \epsilon C$.

In short, we technically then have a family of inequalities – one for each value of $a \geq \epsilon C$. However, the special case where $\alpha = \alpha(\epsilon C)$ implies the validity of the inequality for all larger values of α. So there is really only one inequality, for one specific value of α.

The next lemma is a general result for arrays of rowwise negatively dependent random variables and is the key lemma used in the proof of the upper bound for the law of the logarithm type result for the dependent bootstrap of the mean presented in the theorem. Lemma 6 is given in a form somewhat more general than what is required for the proof of the theorem and maybe of independent interest.

Lemma 6. *Let* $\{Y_{n,j}, 1 \leq j \leq m(n) < \infty, n \geq 1\}$ *be an array of rowwise negatively dependent random variables such that* $EY_{n,j} = 0$ *and* $|Y_{n,j}| \leq c_n$, $1 \leq j \leq m(n)$, $n \geq 1$ *where* $\{c_n, n \geq 1\}$ *is a sequence of constants in* $(0, \infty)$. *Set* $s_n^2 = \sum_{j=1}^{m(n)} EY_{n,j}^2$, $n \geq 1$ *and suppose that* $s_n^2 > 0$, $n \geq 1$. *Let* $\{a_n, n \geq 1\}$ *be a sequence of positive constants such that*

$$\sum_{n=1}^{\infty} \exp\{-\beta^2 a_n^2\} < \infty$$

for some $0 < \beta < \infty$. *Then*:

(i) *If* $c_n a_n = o(s_n)$,

$$\limsup_{n\to\infty} \frac{\left|\sum_{j=1}^{m(n)} Y_{n,j}\right|}{s_n a_n} \leq \sqrt{2} B_0 \ a.s.,$$

where

$$B_0 = \inf\left\{B \in (0, \beta] : \sum_{n=1}^{\infty} \exp\left\{-B^2 a_n^2\right\} < \infty\right\}.$$

(ii) *If* $c_n(\log n)^{1/2} = o(s_n)$,

$$\limsup_{n\to\infty} \frac{\left|\sum_{j=1}^{m(n)} Y_{n,j}\right|}{s_n(2\log n)^{1/2}} \le 1 \ a.s.$$

Proof. Part (ii) follows immediately from (i) by taking $a_1 = (\log 2)^{1/2}$, $a_n = (\log n)^{1/2}$, $n \ge 2$. To prove (i), note that for arbitrary $\alpha > 0$, there exists a positive integer $N(\alpha)$ such that for all $n \ge N(\alpha)$

$$(B_0+\alpha)^2\left(1 - \frac{\sqrt{2}(B_0+\alpha)a_n c_n}{2s_n}\right) \ge \left(B_0 + \frac{\alpha}{2}\right)^2.$$

Employing Lemma 5 we have for $n \ge N(\alpha)$

$$\begin{aligned} &P\left\{\frac{\left|\sum_{j=1}^{m(n)} Y_{n,j}\right|}{s_n a_n} \ge \sqrt{2}(B_0+\alpha)\right\} \\ &\le 2\exp\left\{-(B_0+\alpha)^2 a_n^2\left(1 - \frac{\sqrt{2}(B_0+\alpha)a_n c_n}{2s_n}\right)\right\} \\ &\le 2\ \exp\left\{-\left(B_0+\frac{\alpha}{2}\right)^2 a_n^2\right\}. \end{aligned}$$

By definition of B_0, it follows that

$$\sum_{n=1}^{\infty} P\left\{\frac{\left|\sum_{j=1}^{m(n)} Y_{n,j}\right|}{s_n a_n} \ge \sqrt{2}\left(B_0+\frac{\alpha}{2}\right)\right\} < \infty.$$

Then, by the Borel–Cantelli lemma,

$$P\left\{\frac{\left|\sum_{j=1}^{m(n)} Y_{n,j}\right|}{s_n a_n} \ge \sqrt{2}\left(B_0+\frac{\alpha}{2}\right) \text{ i.o. } (n)\right\} = 0$$

and, hence,

$$\limsup_{n\to\infty} \frac{\left|\sum_{j=1}^{m(n)} Y_{n,j}\right|}{s_n a_n} \le \sqrt{2}\left(B_0+\frac{\alpha}{2}\right) \quad \text{a.s.}$$

Since α is arbitrary,

$$\limsup_{n\to\infty} \frac{\left|\sum_{j=1}^{m(n)} Y_{n,j}\right|}{s_n a_n} \le \sqrt{2}B_0 \quad \text{a.s.}$$

IV. UPPER BOUND FOR THE LAW OF THE LOGARITHM FOR THE DEPENDENT BOOTSTRAP OF THE MEAN

The next theorem can be considered as a upper bound for the law of the logarithm for the dependent bootstrap of the mean. The theorem is an analog of Theorem 1 of Ahmed et al. [6] for the case of the dependent bootstrap.

$$\tilde{\sigma}_n = \left(\frac{\sum_{i=1}^n (X_i - \overline{X}_n)^2}{n}\right)^{1/2} = \left(\frac{\sum_{i=1}^n X_i^2}{n} - \left(\frac{\sum_{i=1}^n X_i}{n}\right)^2\right)^{1/2}, \quad n \geq 1.$$

Theorem. *Let* $\{X_n, n \geq 1\}$ *be a sequence of random vaiables (which are not necessarily independent or identically distributed) and let* $\{m(n), n \geq 1\}$ *be a sequence of positive integers. Suppose that*

(i) $$\lim_{n\to\infty} \frac{(\log n)\max_{1\leq i\leq n}(X_i - \overline{X}_n)^2}{m(n)} = 0 \quad a.s.$$

and

(ii) *for almost every* $\omega \in \Omega$ *the limit* $\lim_{n\to\infty} \tilde{\sigma}_n(\omega) \equiv \tilde{\sigma}(\omega) > 0$ *exists.*
Then for almost every $\omega \in \Omega$

$$\limsup_{n\to\infty} \left(\frac{m(n)}{2\log n}\right)^{1/2} \left|\frac{\sum_{j=1}^{m(n)} \widehat{X}_{n,j}^{(\omega)}}{m(n)} - \overline{X}_n(\omega)\right| \leq \tilde{\sigma}(\omega) \quad \text{a.s.}$$

Proof. The conclusion of the theorem is equivalent to: for almost every $\omega \in \Omega$

$$\limsup_{n\to\infty} \frac{\left|\sum_{j=1}^{m(n)} (\widehat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega))\right|}{(2m(n)\log n)^{1/2}} \leq \tilde{\sigma}(\omega) \quad \text{a.s.}$$

To prove this, set

$$Y_{n,j}^{(\omega)} = \widehat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega), \quad 1 \leq j \leq m(n), \ n \geq 1.$$

Note that

$$EY_{n,j}^{(\omega)} = 0, \quad |Y_{n,j}^{(\omega)}| \leq \max_{1\leq i\leq n} |X_i(\omega) - \overline{X}_n(\omega)|, \quad 1 \leq j \leq m(n), \quad n \geq 1$$

and

$$\sum_{j=1}^{m(n)} E(Y_{n,j}^{(\omega)})^2 = m(n)E(Y_{n,1}^{(\omega)})^2 = m(n)\tilde{\sigma}_n^2(\omega).$$

Now by (i)

$$\frac{(\max_{1\leq i\leq n} |X_i(\omega) - \overline{X}_n(\omega)|)(\log n)^{1/2}}{\left(\sum_{j=1}^{m(n)} E(Y_{n,j}^{(\omega)})^2\right)^{1/2}} = \frac{(\max_{1\leq i\leq n} |X_i(\omega) - \overline{X}_n(\omega)|)(\log n)^{1/2}}{(m(n))^{1/2}\tilde{\sigma}_n(\omega)} \to 0$$

and so by Lemma 6(ii)

$$\limsup_{n\to\infty} \frac{\left|\sum_{j=1}^{m(n)} Y_{n,j}^{(\omega)}\right|}{\left((2\log n)\sum_{j=1}^{m(n)} E(Y_{n,j}^{(\omega)})^2\right)^{1/2}} \le 1 \quad \text{a.s.}$$

Then

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\left|\sum_{j=1}^{m(n)} (\widehat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega))\right|}{(2m(n)\log n)^{1/2}} \\
&= \limsup_{n\to\infty} \left(\frac{\left|\sum_{j=1}^{m(n)} Y_{n,j}^{(\omega)}\right|}{\left((2\log n)\sum_{j=1}^{m(n)} E(Y_{n,j}^{(\omega)})^2\right)^{1/2}} \cdot \left(\frac{\sum_{j=1}^{m(n)} E(Y_{n,j}^{(\omega)})^2}{m(n)}\right)^{1/2} \right) \\
&= \limsup_{n\to\infty} \frac{\left|\sum_{j=1}^{m(n)} Y_{n,j}^{(\omega)}\right|}{\left((2\log n)\sum_{j=1}^{m(n)} E(Y_{n,j}^{(\omega)})^2\right)^{1/2}} \tilde{\sigma}_n(\omega) \\
&\le \tilde{\sigma}(\omega) \quad a.s
\end{aligned}$$

Corollary. *Let $\{X_n, n \ge 1\}$ be a sequence of nondegenerate random variables such that either*

(i) $\{X_n, n \ge 1\}$ *is a sequence of pairwise i.i.d. random variables or*
(ii) $\{X_n, n \ge 1\}$ *is a stationary ergodic sequence of random variables.*

Let $\{m(n), n \ge 1\}$ be a sequence of positive integers such that

$$\frac{m(n)}{\log n} \uparrow .$$

Suppose that there exists a constant $\theta \ge 1$ such that

$$n^{1/\theta} \log n = \mathscr{O}(m(n))$$

and

$$E|X_1|^{2\theta} < \infty.$$

Set $\sigma^2 = \operatorname{Var} X_1$. Then for almost every $\omega \in \Omega$

$$\limsup_{n\to\infty} \left(\frac{m(n)}{2\log n}\right)^{1/2} \left| \frac{\sum_{j=1}^{m(n)} \widehat{X}_{n,j}^{(\omega)}}{m(n)} - \overline{X}_n(\omega) \right| \le \sigma \quad a.s.$$

Proof. Under case (i), $\{X_n^2, n \ge 1\}$ is also a sequence of pairwise i.i.d. random variables. Now by a double application of the Etemadi [15] strong law of large numbers

$$\tilde{\sigma}_n = \left(\frac{\sum_{i=1}^n X_i^2}{n} - \left(\frac{\sum_{i=1}^n X_i}{n}\right)^2 \right)^{1/2} \to (EX_1^2 - (EX_1)^2)^{1/2} = \sigma > 0 \quad \text{a.s.}$$

Under case (ii), $\{X_n^2, n \geq 1\}$ is also a stationary ergodic sequence by Theorem 3.5.8 of Stout, ([6] p. 182). By a double application of the pointwise ergodic theorem for stationary sequences (see, e.g., Stout [6], p. 181), we again have $\tilde{\sigma}_n \to \sigma > 0$ a.s.

Next, there exists a constant $M < \infty$ such that $n^{1/\theta} \log n \leq Mm(n)$, $n \geq 1$. Then for arbitrary $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\left\{\frac{(\log n)X_n^2}{m(n)} > \epsilon\right\} \leq \sum_{n=1}^{\infty} P\left\{|X_1| > \left(\frac{\epsilon}{M}\right)^{\frac{1}{2}} n^{\frac{1}{2\theta}}\right\} < CE|X_1|^{2\theta} < \infty$$

Thus by the Borel–Cantelli lemma

$$\lim_{n\to\infty} \frac{(\log n)X_n^2}{m(n)} = 0 \quad \text{a.s.}$$

Since $\frac{m(n)}{\log n} \uparrow$, condition (i) of the theorem then follows from Lemmas 2 and 3. The conclusion results directly from the theorem.

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