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# Improved Statistical Inference for Three-Parameter Crack Lifetime Distribution

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**ABSTRACT** In this article, we develop the maximum likelihood estimation for the three-parameter of the Crack lifetime distribution and also consider the bias-reduction of the estimators obtained from the classical estimation. Moreover, we consider the Bayesian estimation which we provide by assuming an informative priors. The Bayes estimators are obtained from the Gibbs sampling procedure to generate samples from the posterior distribution and also from the Lindley's approximation method. A simulation study carried out to estimate and compare the various point estimation methods considered.

***Keywords*** Bayesian estimation; Bias reduction; Bootstrap resampling, Composition method; Crack lifetime distribution; Gibbs sampling; Inverse transform method; Jackknife; Lindley's approximation; Markov chain Monte Carlo; Maximum likelihood estimation.

## 1. Introduction

The engineering interpretation of Crack random variable as the time after a crack started to develop in a machine element because of a cyclic or non-cyclic loading until the crack achieves the critical value. At the beginning, it may be a small crack in the machine, but the element could still work. When it achieves the critical point, tolerance exceeds and the element does not work anymore. The three-parameter Crack lifetime distribution had been introduced by Volodin and Dzhungurova [12] as a distribution that is performed by adding weighted parameter and combining the Inverse Gaussian distribution and Length Biased Inverse Gaussian distribution. Thus, the Crack lifetime distribution contains as special cases three known distributions, i.e., the Birnbaum-Saunders distribution, the Inverse Gaussian distribution and the Length Biased Inverse Gaussian distribution. Bowonrattanaset and Budsaba [1] established properties of this distribution. The probability density function of the Crack lifetime distribution is given by:

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$$fnCR(x;\lambda,\theta,p)=\begin{cases}\dfrac{1}{\theta\sqrt{2\pi}}\left[p\lambda\left(\dfrac{\theta}{x}\right)^{\frac{3}{2}}+(1-p)\left(\dfrac{\theta}{x}\right)^{\frac{1}{2}}\right]\exp\left[-\dfrac{1}{2}\left(\sqrt{\dfrac{x}{\theta}}-\lambda\sqrt{\dfrac{\theta}{x}}\right)^{2}\right], & x>0\\ 0 & \text{, otherwise}\end{cases} \quad (1)$$

which we will denote by $CR(\lambda,\theta,p)$, where $\lambda>0,\theta>0$ and $0\le p\le 1$ are parameters correspond to the thickness of a machine element, the nominal treatment pressure on a machine element and weighted parameter accordingly.

Bowonrattanaset [2] compared the Acceptance-Rejection method and the buit-in command in Wolfram Mathematica8 for generating random number of the Crack lifetime distribution. Next, in Bowonrattanaset [2] two classical estimations (the maximum likelihood and the method of moment estimations) have been proposed and compared numerically. Nevertheless, one of the most unsatisfactory facts is that the estimation of the parameter $p$ performs poorly. The performance of the maximum likelihood estimators and the method of moment estimators are asymptotically unbiased and highly biased in case of small sample sizes.

The Bayesian estimation has been used widely for parameter estimation. In many cases of the Bayesian estimation, the researchers found that the Bayes estimators cannot be obtained in closed form. Hence, some new the methods for evaluating the Bayes estimators have been suggested. Robert and Odong [11] suggested the Bayesian estimation of the parameters be obtained from the Birnbaum-Saunders distribution by using the Lindley's approximation technique under assumption informative priors and compared with the maximum likelihood estimates. Kundu and Gupta [6] considered the Bayesian estimation of the two-parameter exponential distribution. They used the idea of Lindley to compute the approximated Bayes estimators and also proposed the Gibbs sampling procedure in order to approximate the Bayes estimators under assumption of non-informative priors. Moreover, they compared the Bayes estimators with the maximum likelihood estimators by Monte Carlo simulations. Pradhan and Kundu [10] compared the performances of the estimators which are the classical moment estimators, the maximum likelihood estimators and the Bayes estimators of the gamma distribution. They approximated the Bayes estimators by using the Lindley's approximation and the Gibbs sampling procedure. Pandey and Bandyopadhyay [9] proposed the Bayesian estimation of parameters for the Inverse-Gaussian distribution under assumption of informative priors. They discussed two different methods which are the Lindley's approximation method and the Gibbs sampling method. They compared the performance of the Bayes estimators, the maximum likelihood estimators and the uniformly minimum variance unbiased estimators. Kohansal and Rezakhah [5] provided the Bayes inference for unknown parameters for Type-II hybrid censored weighted exponential distribution. They provided two approximations, namely the Lindley's approximation and the Gibbs sampling procedure under assumption of non-informative priors. They compared performances of the Bayes estimators from the two methods and the maximum likelihood estimators.

In this paper, we estimate the three-parameter Crack lifetime distribution by the maximum likelihood estimation and the Bayesian estimation based on the Gibbs sampling procedure. Moreover, we estimate and compare the performances of the procedures with fixed $p$ of the Crack lifetime distribution by the method of moment estimation, the maximum likelihood estimation, the Bayesian estimation based on the Lindley's approximation and the Gibbs sampling procedure and the bias-reduction of the method of moment estimators and the maximum likelihood estimators which we estimate based on the bootstrap resampling and the Jackknife technique.

The rest of the paper is arranged as follows. In Section 2, we propose the algorithm of the maximum likelihood estimation. We give the prior and posterior distribution in Section 3. The approximate Bayes estimators are also suggested in Section 4. In Section 5, we consider the bias reduction methods. Numerical results from simulation study are presented in Section 6. Finally we conclude the paper in Section 7.

## 2. Maximum Likelihood Estimation

The log-likelihood function for a random sample $(X_1, X_2, ..., X_n)$ where $X_i \sim CR(\lambda, \theta, p)$, is

$$L(\lambda,\theta,p) = C + L_1(\lambda,\theta) + L_2(\lambda,\theta,p) \tag{2}$$

when $C$ is constant and

$$L_1(\lambda,\theta) = \sum_{i=1}^{n} \log \phi\left(\sqrt{\frac{x_i}{\theta}} - \lambda\sqrt{\frac{\theta}{x_i}}\right) - \frac{n}{2}\log\theta, \quad L_2(\lambda,\theta,p) = \sum_{i=1}^{n} \log\left[1 + \left(\frac{\lambda\theta}{x_i} - 1\right)p\right].$$

Nothing that the weight parameter $p$ appears only at $L_2(\lambda,\theta,p)$, one can use the following iterative algorithm to obtain the maximum likelihood estimators.

ALGORITHM:

<u>Step 1</u> Set the initial value of the weight parameter $p^{(0)}$.

<u>Step 2</u> Given the stage $(m-1)$ parameter estimate $p^{(m-1)}$, find the zeros of the following first derivatives of $L(\lambda,\theta,p)$ with respect $\lambda$ and $\theta$ respectively:

$$\frac{\partial}{\partial\lambda} L(\lambda,\theta,\hat{p}^{(m-1)}) = 0 \quad \text{and} \quad \frac{\partial}{\partial\theta} L(\lambda,\theta,\hat{p}^{(m-1)}) = 0. \tag{3 and 4}$$

In case that analytical expressions are available, one can use the Newton-Raphson method with the first derivatives

$$\frac{\partial}{\partial\lambda} L(\lambda,\theta,\hat{p}^{(m-1)}) \quad \text{and} \quad \frac{\partial}{\partial\theta} L(\lambda,\theta,\hat{p}^{(m-1)})$$

and the second derivatives

$$\frac{\partial^2}{\partial\lambda^2} L(\lambda,\theta,\hat{p}^{(m-1)}), \ \frac{\partial^2}{\partial\lambda\partial\theta} L(\lambda,\theta,\hat{p}^{(m-1)}), \ \frac{\partial^2}{\partial\theta\partial\lambda} L(\lambda,\theta,\hat{p}^{(m-1)}), \text{ and } \frac{\partial^2}{\partial\theta^2} L(\lambda,\theta,\hat{p}^{(m-1)}).$$

Let the solutions of Equations (3) and (4) be $(\hat{\lambda}^{(m)}, \hat{\theta}^{(m)})$.

Step 3 Given the stage $(m)$ parameter estimate $(\hat{\lambda}^{(m)},\ \hat{\theta}^{(m)})$, find the zeros of the following first derivative of $L_2(\lambda,\theta,p)$ with respect $p$ :

$$\frac{d}{dp}L_2(\hat{\lambda}^{(m)},\ \hat{\theta}^{(m)},\ p)=0. \tag{5}$$

One may use the Newton-Raphson method with the first derivative

$$\frac{d}{dp}L_2(\hat{\lambda}^{(m)},\ \hat{\theta}^{(m)},\ p)$$

and the second derivative

$$\frac{d^2}{dp^2}L_2(\hat{\lambda}^{(m)},\ \hat{\theta}^{(m)},\ p).$$

Let the solution of Equation (5) be $\hat{p}^{(m)}$. Repeat steps 2 and 3 until convergence is obtained.

## 3. Prior and Posterior Distribution

In this section we provide the prior and posterior distributions. Let $x=(x_1,x_2,\dots,x_n)$ be a random sample of size $n$ from $CR(\lambda,\theta,p)$. The likelihood function of the observed sample can be written as:

$$l(\lambda,\theta,p\,|\,\underset{\sim}{x})=(2\pi)^{-\frac{n}{2}}\theta^{-\frac{n}{2}}e^{n\lambda-\frac{1}{2\theta}\sum_{i=1}^{n}x_i-\frac{\lambda^2\theta}{2}\sum_{i=1}^{n}\frac{1}{x_i}}\prod_{i=1}^{n}\frac{1}{x_i^{\frac{1}{2}}}\left[1-p+\frac{\lambda\theta p}{x_i}\right]. \tag{6}$$

Under the assumption of independence of $\lambda,\theta$ and $p$, the joint prior of $(\lambda,\ \theta,\ p)$ is

$$\pi(\lambda,\ \theta,\ p)=\pi_1(\lambda)\pi_1(\ \theta)\pi_1(p)\ .$$

The prior distributions of $\lambda$ and $\theta$ are gamma distribution and that of $p$ is a beta distribution for which the probability density functions are

$$\pi_1(\lambda)=\frac{a^{\gamma}}{\Gamma(\gamma)}\lambda^{\gamma-1}e^{-a\lambda}\ ,\gamma>0\ \text{and}\ a>0, \tag{7}$$

$$\pi_2(\theta)=\frac{b^{\eta}}{\Gamma(\eta)}\theta^{\eta-1}e^{-b\theta}\ ,\eta>0\ \text{and}\ b>0, \tag{8}$$

and

$$\pi_3(p)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}\ ,\alpha>0\ \text{and}\ \beta>0. \tag{9}$$

Thus, the joint posterior of $(\lambda,\ \theta,\ p)$ can be written as

$$p(\lambda,\theta,p\,|\,\underset{\sim}{x})=C\theta^{-\frac{n}{2}+\eta-1}\lambda^{\gamma-1}e^{n\lambda-\frac{1}{2\theta}\sum_{i=1}^{n}x_i-\frac{\lambda^2\theta}{2}\sum_{i=1}^{n}\frac{1}{x_i}-a\lambda-b\theta}p^{\alpha-1}(1-p)^{\beta-1}\prod_{i=1}^{n}\left[1-p+\frac{\lambda\theta p}{x_i}\right] \tag{10}$$

where $C$ is the normalizing constant. From Equation (10) it is clear that the Bayes estimate of

$h(\lambda,\theta,p)$ of $\lambda,\theta$ and $p$ under squared error loss function is the posterior expectation

$$\hat{h}_B(\lambda,\theta,p)=\int_0^\infty\int_0^\infty\int_0^1 h(\lambda,\theta,p)\,p(\lambda,\theta,p|\underset{\sim}{x})\,dp\,d\theta\,d\lambda\,. \tag{11}$$

Unfortunately, this integral cannot be computed in an explicit form. Thus, we will consider the Lindley's approximation and Gibbs sampling MCMC technique so as to approximate the Bayes estimators.

## 4. Bayesian Point Estimation

In this section we discuss the approximate Bayes estimates of the parameters based on the prior assumption mentioned under squared error loss function in previous section.

### 4.1 Lindley's Approximation

Lindley's [7] proposed the approximation to compute the ratio of two integral. In this case we specify the priors on $\lambda,\theta$ and $p$ which mentioned in section 2. The approximate Bayes estimates of $\lambda,\theta$ and $p$ under the squared error loss function which base on Lindley's approximation are:

$$\begin{aligned}\hat{\lambda}_L=\hat{\lambda}&+(\hat{\rho}_\lambda\hat{\sigma}_{\lambda\lambda}+\hat{\rho}_\theta\hat{\sigma}_{\lambda\theta}+\hat{\rho}_p\hat{\sigma}_{\lambda p})\\
&+\frac{1}{2}[(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\lambda\theta}\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\theta\lambda}+\hat{L}_{\lambda\lambda p}\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{p\lambda}+\hat{L}_{\lambda\theta\lambda}\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\theta\theta}\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta\lambda}\\
&\quad+\hat{L}_{\lambda\theta p}\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{p\lambda}+\hat{L}_{\lambda p\lambda}\hat{\sigma}_{\lambda p}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda p\theta}\hat{\sigma}_{\lambda p}\hat{\sigma}_{\theta\lambda}+\hat{L}_{\lambda pp}\hat{\sigma}_{\lambda p}\hat{\sigma}_{p\lambda})\\
&\quad+(\hat{L}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\theta\lambda\theta}\hat{\sigma}_{\theta\lambda}\hat{\sigma}_{\theta\lambda}+\hat{L}_{\theta\lambda p}\hat{\sigma}_{\theta\lambda}\hat{\sigma}_{p\lambda}+\hat{L}_{\theta\theta\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\theta\lambda}\\
&\quad+\hat{L}_{\theta\theta p}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{p\lambda}+\hat{L}_{\theta p\lambda}\hat{\sigma}_{\theta p}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\theta p\theta}\hat{\sigma}_{\theta p}\hat{\sigma}_{\theta\lambda}+\hat{L}_{\theta pp}\hat{\sigma}_{\theta p}\hat{\sigma}_{p\lambda})\\
&\quad+(\hat{L}_{p\lambda\lambda}\hat{\sigma}_{p\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{p\lambda\theta}\hat{\sigma}_{p\lambda}\hat{\sigma}_{\theta\lambda}+\hat{L}_{p\lambda p}\hat{\sigma}_{p\lambda}\hat{\sigma}_{p\lambda}+\hat{L}_{p\theta\lambda}\hat{\sigma}_{p\theta}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{p\theta\theta}\hat{\sigma}_{p\theta}\hat{\sigma}_{\theta\lambda}\\
&\quad+\hat{L}_{p\theta p}\hat{\sigma}_{p\theta}\hat{\sigma}_{p\lambda}+\hat{L}_{pp\lambda}\hat{\sigma}_{pp}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{pp\theta}\hat{\sigma}_{pp}\hat{\sigma}_{\theta\lambda}+\hat{L}_{ppp}\hat{\sigma}_{pp}\hat{\sigma}_{p\lambda})],\end{aligned} \tag{12}$$

$$\begin{aligned}\hat{\theta}_L=\hat{\theta}&+(\hat{\rho}_\lambda\hat{\sigma}_{\theta\lambda}+\hat{\rho}_\theta\hat{\sigma}_{\theta\theta}+\hat{\rho}_p\hat{\sigma}_{\theta p})\\
&+\frac{1}{2}[(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\lambda\theta}+\hat{L}_{\lambda\lambda\theta}\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\theta\theta}+\hat{L}_{\lambda\lambda p}\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{p\theta}+\hat{L}_{\lambda\theta\lambda}\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\lambda\theta}+\hat{L}_{\lambda\theta\theta}\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta\theta}\\
&\quad+\hat{L}_{\lambda\theta p}\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{p\theta}+\hat{L}_{\lambda p\lambda}\hat{\sigma}_{\lambda p}\hat{\sigma}_{\lambda\theta}+\hat{L}_{\lambda p\theta}\hat{\sigma}_{\lambda p}\hat{\sigma}_{\theta\theta}+\hat{L}_{\lambda pp}\hat{\sigma}_{\lambda p}\hat{\sigma}_{p\theta})\\
&\quad+(\hat{L}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\lambda}\hat{\sigma}_{\lambda\theta}+\hat{L}_{\theta\lambda\theta}\hat{\sigma}_{\theta\lambda}\hat{\sigma}_{\theta\theta}+\hat{L}_{\theta\lambda p}\hat{\sigma}_{\theta\lambda}\hat{\sigma}_{p\theta}+\hat{L}_{\theta\theta\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\theta}+\hat{L}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\theta\theta}\\
&\quad+\hat{L}_{\theta\theta p}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{p\theta}+\hat{L}_{\theta p\lambda}\hat{\sigma}_{\theta p}\hat{\sigma}_{\lambda\theta}+\hat{L}_{\theta p\theta}\hat{\sigma}_{\theta p}\hat{\sigma}_{\theta\theta}+\hat{L}_{\theta pp}\hat{\sigma}_{\theta p}\hat{\sigma}_{p\theta})\\
&\quad+(\hat{L}_{p\lambda\lambda}\hat{\sigma}_{p\lambda}\hat{\sigma}_{\lambda\theta}+\hat{L}_{p\lambda\theta}\hat{\sigma}_{p\lambda}\hat{\sigma}_{\theta\theta}+\hat{L}_{p\lambda p}\hat{\sigma}_{p\lambda}\hat{\sigma}_{p\theta}+\hat{L}_{p\theta\lambda}\hat{\sigma}_{p\theta}\hat{\sigma}_{\lambda\theta}+\hat{L}_{p\theta\theta}\hat{\sigma}_{p\theta}\hat{\sigma}_{\theta\theta}\\
&\quad+\hat{L}_{p\theta p}\hat{\sigma}_{p\theta}\hat{\sigma}_{p\theta}+\hat{L}_{pp\lambda}\hat{\sigma}_{pp}\hat{\sigma}_{\lambda\theta}+\hat{L}_{pp\theta}\hat{\sigma}_{pp}\hat{\sigma}_{\theta\theta}+\hat{L}_{ppp}\hat{\sigma}_{pp}\hat{\sigma}_{p\theta})],\end{aligned} \tag{13}$$

and

$$\begin{aligned}\hat{p}_L=\hat{p}&+(\hat{\rho}_\lambda\hat{\sigma}_{p\lambda}+\hat{\rho}_\theta\hat{\sigma}_{p\theta}+\hat{\rho}_p\hat{\sigma}_{pp})\\
&+\frac{1}{2}[(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\lambda p}+\hat{L}_{\lambda\lambda\theta}\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\theta p}+\hat{L}_{\lambda\lambda p}\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{pp}+\hat{L}_{\lambda\theta\lambda}\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\lambda p}+\hat{L}_{\lambda\theta\theta}\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta p}\\
&\quad+\hat{L}_{\lambda\theta p}\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{pp}+\hat{L}_{\lambda p\lambda}\hat{\sigma}_{\lambda p}\hat{\sigma}_{\lambda p}+\hat{L}_{\lambda p\theta}\hat{\sigma}_{\lambda p}\hat{\sigma}_{\theta p}+\hat{L}_{\lambda pp}\hat{\sigma}_{\lambda p}\hat{\sigma}_{pp})\end{aligned}$$

$$+(\hat{L}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\lambda}\hat{\sigma}_{\lambda p}+\hat{L}_{\theta\lambda\theta}\hat{\sigma}_{\theta\lambda}\hat{\sigma}_{\theta p}+\hat{L}_{\theta\lambda p}\hat{\sigma}_{\theta\lambda}\hat{\sigma}_{pp}+\hat{L}_{\theta\theta\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda p}+\hat{L}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\theta p}$$
$$+\hat{L}_{\theta\theta p}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{pp}+\hat{L}_{\theta p\lambda}\hat{\sigma}_{\theta p}\hat{\sigma}_{\lambda p}+\hat{L}_{\theta p\theta}\hat{\sigma}_{\theta p}\hat{\sigma}_{\theta p}+\hat{L}_{\theta pp}\hat{\sigma}_{\theta p}\hat{\sigma}_{pp})$$
$$+(\hat{L}_{p\lambda\lambda}\hat{\sigma}_{p\lambda}\hat{\sigma}_{\lambda p}+\hat{L}_{p\lambda\theta}\hat{\sigma}_{p\lambda}\hat{\sigma}_{\theta p}+\hat{L}_{p\lambda p}\hat{\sigma}_{p\lambda}\hat{\sigma}_{pp}+\hat{L}_{p\theta\lambda}\hat{\sigma}_{p\theta}\hat{\sigma}_{\lambda p}+\hat{L}_{p\theta\theta}\hat{\sigma}_{p\theta}\hat{\sigma}_{\theta p}$$
$$+\hat{L}_{p\theta p}\hat{\sigma}_{p\theta}\hat{\sigma}_{pp}+\hat{L}_{pp\lambda}\hat{\sigma}_{pp}\hat{\sigma}_{\lambda p}+\hat{L}_{pp\theta}\hat{\sigma}_{pp}\hat{\sigma}_{\theta p}+\hat{L}_{ppp}\hat{\sigma}_{pp}\hat{\sigma}_{pp})] \quad (14)$$

where $\hat{\lambda}$, $\hat{\theta}$, and $\hat{p}$ are the maximum likelihood estimators of $\lambda, \theta$, and $p$ respectively. For the proofs of Equations (12)-(14), they can be obtained as given in the Appendix.

In case we shall assume that the parameter $p$ is fixed, we will consider the parameters of the Crack lifetime distribution which are $\lambda$ and $\theta$. The approximate Bayes estimates of $\lambda$ and $\theta$ under the squared error loss function which base on Lindley's approximation are

$$\hat{\lambda}_L=\hat{\lambda}+(\hat{\rho}_{\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{\rho}_{\theta}\hat{\sigma}_{\lambda\theta})+\frac{1}{2}[(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\lambda\theta}\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\theta\lambda}+\hat{L}_{\lambda\theta\lambda}\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\theta\theta}\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta\lambda})$$
$$+(\hat{L}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\theta\lambda\theta}\hat{\sigma}_{\theta\lambda}\hat{\sigma}_{\theta\lambda}+\hat{L}_{\theta\theta\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\theta\lambda})] \quad (15)$$

and

$$\hat{\theta}_L=\hat{\theta}+(\hat{\rho}_{\lambda}\hat{\sigma}_{\theta\lambda}+\hat{\rho}_{\theta}\hat{\sigma}_{\theta\theta})+\frac{1}{2}[(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\lambda\theta}+\hat{L}_{\lambda\lambda\theta}\hat{\sigma}_{\lambda\lambda}\hat{\sigma}_{\theta\theta}+\hat{L}_{\lambda\theta\lambda}\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\lambda\theta}+\hat{L}_{\lambda\theta\theta}\hat{\sigma}_{\lambda\theta}\hat{\sigma}_{\theta\theta})$$
$$+(\hat{L}_{\theta\lambda\lambda}\hat{\sigma}_{\theta\lambda}\hat{\sigma}_{\lambda\theta}+\hat{L}_{\theta\lambda\theta}\hat{\sigma}_{\theta\lambda}\hat{\sigma}_{\theta\theta}+\hat{L}_{\theta\theta\lambda}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\lambda\theta}+\hat{L}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta}\hat{\sigma}_{\theta\theta})] \quad (16)$$

where $\hat{\lambda}$ and $\hat{\theta}$ are the maximum likelihood estimators of $\lambda$ and $\theta$ respectively. For the proofs of Equations (15)-(16), they can be obtained as given in the Appendix.

## 4.2 Gibbs Sampling

In this subsection, we consider Gibbs sampling MCMC technique to generate sample from posterior density function under the assumption of $\lambda,\theta$ and $p$ in Section 3 and the square error loss function.

From the posterior distribution of $(\lambda, \theta, p)$ in Equation (10), the full conditional distributions for $\lambda,\theta$ and $p$ can be written as

$$p(\lambda|\theta, p, \underset{\sim}{x}) \propto \lambda^{\gamma-1} e^{n\lambda-\frac{\lambda^2\theta}{2}\sum_{i=1}^{n}\frac{1}{x_i}-a\lambda}\prod_{i=1}^{n}\left[1-p+\frac{\lambda\theta p}{x_i}\right], \quad (17)$$

$$p(\theta|\lambda, p, \underset{\sim}{x}) \propto \theta^{-\frac{n}{2}+\eta-1} e^{-\frac{1}{2\theta}\sum_{i=1}^{n}x_i-\frac{\lambda^2\theta}{2}\sum_{i=1}^{n}\frac{1}{x_i}-b\theta}\prod_{i=1}^{n}\left[1-p+\frac{\lambda\theta p}{x_i}\right], \quad (18)$$

and

$$p(p|\lambda, \theta, \underset{\sim}{x}) \propto p^{\alpha-1}(1-p)^{\beta-1}\prod_{i=1}^{n}\left[1-p+\frac{\lambda\theta p}{x_i}\right]. \quad (19)$$

Now using the inverse transform method and following the idea of German and German [4], we propose the following scheme to generate $(\lambda, \theta, p)$ from Equations (17)-(19) respectively. We can easily simulate via a three-stage Gibbs sampler and also suggest the following algorithm.

ALGORITHM:

Step 1 Take some initial starting point $(\lambda_0, \theta_0)$.

Step 2 Generate $p_1$ from the full conditional function $p(p|\lambda_0, \theta_0, \underset{\sim}{x})$ by using the inverse transform method.

Step 3 Generate $\theta_1$ from the full conditional function $p(p|\lambda_0, p_1, \underset{\sim}{x})$ by using the inverse transform method.

Step 4 Generate $\lambda_1$ from the full conditional function $p(p|\theta_1, p_1, \underset{\sim}{x})$ by using the inverse transform method.

Step 5 The sampler becomes $p_i \sim p(p|\lambda_{i-1}, \theta_{i-1}, \underset{\sim}{x})$, $\theta_i \sim p(\theta|\lambda_{i-1}, p_i, \underset{\sim}{x})$, $\lambda_i \sim p(\lambda|\theta_i, p_i, \underset{\sim}{x})$, $i = 2, 3, \cdots, N$.

Step 6 The Bayes estimates of $\lambda, \theta$ and $p$ under square error loss function are

$$\hat{\lambda}_G = \hat{E}(\lambda|\underset{\sim}{x}) = \frac{1}{N}\sum_{i=1}^{N}\lambda_i,\ \hat{\theta}_G = \hat{E}(\theta|\underset{\sim}{x}) = \frac{1}{N}\sum_{i=1}^{N}\theta_i,\ \text{and}\ \hat{p}_G = \hat{E}(p|\underset{\sim}{x}) = \frac{1}{N}\sum_{i=1}^{N}p_i$$

where $(\lambda_i, \theta_i, p_i)$ is the $i$-th MCMC sample, $i = 1, 2, \cdots, N$.

## 5. Bias-Reduction

In what follows, we shall advise the simple bias-reduction methods which are the bootstrap resamplingand the Jackknife technique from Efron [3].

### 5.1 Bootstrap Resampling

Suppose now that $B$ bootstrap samples $(\underset{\sim}{x}^{*1}, \underset{\sim}{x}^{*2}, ..., \underset{\sim}{x}^{*B})$ are independent from the original sample $\underset{\sim}{x}$ and also we compute the estimators $(\hat{\alpha}^{*1}, \hat{\alpha}^{*2}, ..., \hat{\alpha}^{*B})$ from each bootstrap replication. Thus we can approximate the expected of the estimators by the bootstrap resampling as

$$\hat{\alpha}^{*(\cdot)} \approx \frac{1}{B}\sum_{b=1}^{B}\hat{\alpha}^{*b}\ ,\ b = 1, 2, ..., B$$

Hence, the bootstrap bias estimation obtained from the $B$ replicates of $\hat{\theta}$ leads to

$$\hat{B}(\hat{\alpha}, \alpha) = \hat{\alpha}^{*(\cdot)} - \hat{\alpha}\ .$$

Thus, the bias-corrected estimator by using the bootstrap resampling is

$$\hat{\alpha}_{br} = 2\hat{\alpha} - \hat{\alpha}^{*(\cdot)}\ . \tag{20}$$

### 5.2 Jackknife Technique

In Jackknifing, we remove sample point $x_j$, $j = 1, 2, ..., n$ respectively from the data set and recompute the estimator $\hat{\alpha}_{(j)}$ from the reduced sample of size $n-1$. Let us now be given

$$\hat{\alpha}_{(\cdot)} = \frac{1}{n}\sum_{j=1}^{n}\hat{\alpha}_{(j)}\ .$$

Thus, the bias-corrected estimator by using the jackknife technique is

$$\hat{\alpha}_J = n\hat{\alpha} - (n-1)\hat{\alpha}_{(\cdot)} \tag{21}$$

where $\hat{\alpha}$ is the estimator obtained from the data set.

## 6. Simulation Study

In order to estimate and compare the performanceof the three parameters of the various estimation methods described above. First of all, we consider the performance of the three estimators from the maximum likelihood estimation, we take the samples of size $n=10, 100$ and $500$ from $CR(1,1,0.1)$ and $CR(1,5,0.3)$ which we show in Table 1 by observing from the estimators in each replication.

**Table 1** The maximum likelihood estimators for different sample sizes where $\lambda=1,\ \theta=1,\ p=0.1$ and $\lambda=1, \theta=5,\ p=0.3$ in 5 replications. The first entry in each cell corresponds to $\lambda$, the second to $\theta$ and the third to $p$ .

| $n$ | $it-rep$ | $\lambda=1, \theta=1,\ p=0.1$ | | | $\lambda=1, \theta=5,\ p=0.3$ | | |
|---|---|---|---|---|---|---|---|
| **10** | 1 | 2.9694 | 0.4924 | 0.3759 | 1.1100 | 7.9052 | 0.7898 |
| | 2 | 0.6314 | 1.3000 | 0.1259 | 1.2767 | 1.9995 | 0.4810 |
| | 3 | 1.2160 | 0.8176 | 0.6923 | 1.1098 | 4.9528 | 0.8006 |
| | 4 | 1.1878 | 1.0775 | 0.5544 | 1.4332 | 1.7263 | 0.0043 |
| | 5 | 1.3043 | 0.4477 | 0.0010 | 0.9129 | 3.0521 | 0.0033 |
| **100** | 1 | 0.9280 | 1.0810 | 0.0094 | 1.0412 | 5.4144 | 0.0030 |
| | 2 | 0.9565 | 1.5053 | 0.4992 | 1.2058 | 3.5507 | 0.0114 |
| | 3 | 0.9507 | 0.9975 | 0.0043 | 0.9976 | 6.1686 | 0.3554 |
| | 4 | 0.8547 | 1.0914 | 0.0037 | 1.4073 | 3.7852 | 0.0017 |
| | 5 | 1.3523 | 1.3738 | 0.8749 | 1.0431 | 8.5845 | 0.7718 |
| **500** | 1 | 1.0674 | 1.0565 | 0.1909 | 0.9469 | 4.4585 | 0.0153 |
| | 2 | 0.8263 | 1.1981 | 0.1994 | 1.0833 | 4.6583 | 0.0055 |
| | 3 | 1.0028 | 1.0518 | 0.3617 | 1.0083 | 4.9457 | 0.0103 |
| | 4 | 0.8905 | 0.9057 | 0.0036 | 1.0607 | 4.8544 | 0.0940 |
| | 5 | 0.9489 | 0.9328 | 0.0057 | 1.0616 | 5.0001 | 0.0211 |

The second, we estimate the three parameters by the maximum likelihood method for the samples of size $n=1,000$ and $10,000$ from $CR(1,1,0.3)$ and $CR(1,1,0.7)$. The average value of the estimates, the bias and the MSE, based on 1,000 replications are reported in Table 2.

The third, we consider the Bayesian estimation forthe threeparameters. We take the samples of size $n=10,20,50$ and $70$ from $CR(1,1,0.3)$, and $CR(1,1,0.7)$ by assuming an informative priors on the parameters which are $Gamma(1,1)$ as the prior distribution of $\lambda=1$, $Gamma(1,1)$ as the prior distribution of $\theta=1$, $Beta(3,7)$ and $Beta(7,3)$ as the prior distribution of $p=0.3$ and $0.7$ respectively. MCMC samples of size 1,000 were taken for computation. Table 3 gives the average value of the estimates, the bias and the MSE, based on 1,000 replications.

**Table 2** Average, Bias and MSE of the maximum likelihood estimators for different sample sizes of $\lambda = 1, \theta = 1$ and $p = 0.3, 0.7$. The first entry in each cell corresponds to $\lambda$, the second to $\theta$ and the third to $p$.

| $\lambda$ | $\theta$ | $p$ | $n$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | **1,000** | | | **10,000** | | |
| 1 | 1 | 0.3 | 0.9949<br>−0.005<br>0.0026 | 1.0142<br>0.0142<br>0.0070 | 0.3835<br>0.0835<br>0.0212 | 1.0010<br>0.0009<br>0.0003 | 1.0037<br>0.0037<br>0.0005 | 0.2914<br>−0.008<br>0.0015 |
| 1 | 1 | 0.7 | 0.9898<br>−0.010<br>0.0064 | 1.0325<br>0.0325<br>0.0367 | 0.7975<br>0.0975<br>0.0277 | 0.9990<br>−0.001<br>0.0002 | 1.0041<br>0.0041<br>0.0014 | 0.6853<br>−0.014<br>0.0013 |

**Table 3** Average, Bias and MSE of the Bayes estimators for different sample sizes of $\lambda = 1, \theta = 1$ and $p = 0.3, 0.7$. The first entry in each cell corresponds to $\lambda$, the second to $\theta$ and the third to $p$.

| $n$ | $\lambda = 1, \theta = 1, p = 0.3$ | | | $\lambda = 1, \theta = 1, p = 0.7$ | | |
|---|---|---|---|---|---|---|
| **10** | 1.1755 | 1.2617 | 0.3749 | 1.1797 | 1.2741 | 0.7658 |
| | 0.1755 | 0.2617 | 0.0749 | 0.1797 | 0.2741 | 0.0658 |
| | 0.0323 | 0.0706 | 0.0056 | 0.0332 | 0.0760 | 0.0043 |
| **20** | 1.1447 | 1.2288 | 0.3689 | 1.1485 | 1.2357 | 0.7531 |
| | 0.1447 | 0.2288 | 0.0689 | 0.1485 | 0.2357 | 0.0531 |
| | 0.0215 | 0.0534 | 0.0047 | 0.0224 | 0.0561 | 0.0028 |
| **50** | 1.0944 | 1.1506 | 0.3481 | 1.0961 | 1.1523 | 0.7315 |
| | 0.0944 | 0.1506 | 0.0481 | 0.0961 | 0.1523 | 0.0315 |
| | 0.0092 | 0.0230 | 0.0023 | 0.0094 | 0.0235 | 0.0010 |
| **70** | 1.0539 | 1.1038 | 0.3280 | 1.0500 | 1.1086 | 0.7287 |
| | 0.0539 | 0.1038 | 0.0280 | 0.0500 | 0.1086 | 0.0287 |
| | 0.0032 | 0.0110 | 0.0008 | 0.0028 | 0.0120 | 0.0008 |

Lastly, we estimate and compare the performance of the three parameters with fixed $p$ by taking samples of size $n = 10, 20, 50$ and 70 from $CR(1,1,0.3)$ and by assuming informative priors on the parameters which are $Gamma(1,1)$ as the prior distribution of $\lambda = 1$ and $Gamma(1,1)$ as the prior distribution of $\theta = 1$ respectively. MCMC samples of size 1,000 were taken for computation. The number of bootstrap replications we provide is based on 1,000 replications. The average value of the estimates, the bias and the MSE, based on 1,000 replications, are reported in Table 4.

From Table 1, simulation results for the three estimators by the maximum likelihood estimation for moderate sample sizes, show that the estimators of $p$ are far from the true value. Then, we evaluated the three parameters by the maximum likelihood estimation for the very large sample sizes which are reported in Table 2, it is observed that the MSE decrease as the sample size increases and the estimators are performing very well. From Table 3, as expected it is observed that the sample size increases as the MSE decreases. The Bayes estimators obtained from the Gibbs sampling procedure are performing very well.

**Table 4** Average, Bias and MSE of different estimators for different sample sizes of $\lambda = 1, \theta = 1, p = 0.3$. The first entry in each cell corresponds to $\lambda$ and the second to $\theta$

| *n* | **10** | | **20** | | **50** | | **70** | |
|---|---|---|---|---|---|---|---|---|
| **MLE** | 1.447 | 0.916 | 1.204 | 0.949 | 1.067 | 0.993 | 1.054 | 0.984 |
| | 0.447 | –0.08 | 0.204 | –0.05 | 0.067 | –0.00 | 0.054 | –0.01 |
| | 0.985 | 0.213 | 0.268 | 0.108 | 0.061 | 0.050 | 0.041 | 0.032 |
| **MME** | 1.886 | 0.802 | 1.430 | 0.891 | 1.168 | 0.964 | 1.123 | 0.971 |
| | 0.886 | –0.19 | 0.430 | –0.10 | 0.168 | –0.03 | 0.123 | –0.02 |
| | 0.270 | 0.260 | 0.692 | 0.152 | 0.184 | 0.075 | 0.137 | 0.058 |
| **BMLE** | 0.700 | 0.991 | 0.982 | 0.988 | 0.994 | 1.010 | 1.004 | 0.996 |
| | –0.29 | –0.00 | –0.01 | –0.01 | –0.00 | 0.010 | 0.004 | –0.00 |
| | 0.685 | 0.235 | 0.160 | 0.114 | 0.050 | 0.052 | 0.035 | 0.032 |
| **BMME** | 0.944 | 0.908 | 1.094 | 0.960 | 1.034 | 1.001 | 1.021 | 0.999 |
| | –0.05 | –0.09 | 0.094 | –0.03 | 0.034 | 0.001 | 0.021 | –0.00 |
| | 1.241 | 0.318 | 0.518 | 0.176 | 0.178 | 0.085 | 0.138 | 0.064 |
| **JMLE** | 0.748 | 1.009 | 0.970 | 0.993 | 0.991 | 1.011 | 1.000 | 0.997 |
| | –0.25 | 0.009 | –0.02 | –0.00 | –0.00 | 0.011 | 0.000 | –0.00 |
| | 0.914 | 0.246 | 0.164 | 0.115 | 0.050 | 0.052 | 0.035 | 0.032 |
| **JMME** | 0.922 | 0.951 | 1.013 | 0.979 | 1.000 | 1.007 | 0.997 | 1.004 |
| | –0.07 | –0.04 | 0.013 | –0.02 | 0.000 | 0.007 | –0.00 | 0.004 |
| | 1.669 | 0.387 | 0.677 | 0.193 | 0.212 | 0.091 | 0.158 | 0.067 |
| **Bayes (Lindley)** | 0.905 | 1.186 | 1.064 | 1.092 | 1.032 | 1.052 | 1.031 | 1.027 |
| | –0.09 | 0.186 | 0.064 | 0.092 | 0.032 | 0.052 | 0.031 | 0.027 |
| | 0.138 | 0.194 | 0.096 | 0.104 | 0.046 | 0.051 | 0.033 | 0.031 |
| **Bayes (Gibbs)** | 1.168 | 1.252 | 1.139 | 1.221 | 1.092 | 1.149 | 1.052 | 1.059 |
| | 0.168 | 0.252 | 0.139 | 0.221 | 0.092 | 0.149 | 0.052 | 0.059 |
| | 0.029 | 0.066 | 0.020 | 0.050 | 0.008 | 0.022 | 0.003 | 0.003 |

From Table 4, it is observed that as expected, the MSE decreases as the sample size increases. We observed the bias in the absolute values of the maximum likelihood estimators, the method of moment estimators, the Bayes estimators obtained from Lindley's approximation and Gibbs sampling procedure. We found that the sample size increases as the absolute values of the bias decreases which are the asymptotically unbiased estimators. The absolute values of the bias from the bias-reduction of the maximum likelihood estimators are less biased than the maximum likelihood estimators. Similarly, the absolute values of the bias from the bias-reduction of the method of moment estimators are less biased than the method of moment estimators. The Bayes estimators obtained from Gibbs sampling contributed the lowest MSE, the estimators that contributed the next lower variance and MSE is the Bayes estimators obtained from Lindley's approximation. Thus, we can conclude that the Bayes estimators obtained from Lindley's approximation and Gibbs sampling procedure are observed to perform much better than the estimators from the classical estimations. Moreover, the maximum likelihood estimators perform better than the method of moment estimators. Overall, the numerical study reveals that the performance of the Bayes estimators obtained from Gibbs sampling procedure are very well in this situation.

## 7. Conclusion

In this paper we consider the new algorithm of the maximum likelihood estimation and the Bayesian estimation for the three-parameter of the $CR(\lambda,\theta,p)$ distribution. Moreover, we proposed the bias-reduction methods for reduction of the bias of the estimators that obtained from the classical estimation methods. We assumed independent gamma priors and beta prior for the three-parameter in this study and provide the Bayes estimators under assumptions of squared errors loss functions. Simulation results suggest that the Bayes estimates with informative priors behave much better than the maximum likelihood estimates and the method of moment estimates.The Gibbs sampling procedure show how techniques based on MCMC can easily deal with the issue of awkward posterior and perform well compared to another available methods of estimation. Nevertheless, perhaps the procedure of evaluation the Bayes estimators obtained from Gibbs sampling based on the inverse transform method is cumbersome, it is difficult to derive estimators from the complex function. In the future research, the question of interval estimation, the test of hypothesis and the estimation based on the censoring data still remains to be considered.

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## References

[1] Bowonrattanaset, P. and Budsaba, K. (2011). Some properties of the three-parameter Crack distribution, *Journal of Thailand Statistician*, **9**(2), 195-203.

[2] Bowonrattanaset, P. (2011). Point estimation for the Crack lifetime distribution, Ph.D. dissertation, Department of Mathematics and Statistics Faculty of Science and Technology, Thammasat University.

[3] Efron, B. (1982). The Jackknife, the Bootstrap and Other Resampling Plans, CBMS-NSF Regional Conference Series in Applied Mathematics, Monograph 38, SIAM, Philadelphia.

[4] Geman, S. and Geman, D. (1984). Stochastic relaxation, Gibbs distributions and Bayesian restoration of images, *IEEE Transaction Pattern Analysis and Machine Intelligence*, **6**, 721-741.

[5] Kohansal, A. and Rezakhah, S. (2012). Parameter estimation of type-II hybrid censored weighted exponential distribution, *Eprint arXiv*: 1203.0094, Cornel University.

[6] Kundu, D. and Gupta, R. (2008). Generalized exponential distribution: Bayesian estimations, *Computational Statistics and Data Analysis*, **52(4)**, 1873-1883.

[7] Lindley, D. V. (1961). The use of prior probability distribution in statistical inference and decision, *Proceedings of the 4th Berkeley Symposium*, **1**, 453-488.

[8] Miller, R. G., Jr (1981). Survival Analysis, New York: Wiley.

[9] Pandey, B. D. and Bandyopadhyay, P. (2012). Bayesian estimation of Inverse Gaussian distribution, *Eprint arXiv*: 1210.4524, Cornel University.

[10] Pradhan, B. and Kundu, D. (2011). Bayes estimation and prediction of the two-parameter Gamma distribution, *Journal of Statistical Computation and Simulation*, **81**(9), 1187-1198.

[11] Robert, K. and Odong, L. O. (2007). Bayesian estimation the parameters of the Birnbaum-Saunders distribution, *East African Journal of Statistic*, **1**(3), 248-260.

[12] Volodin, I. N. and Dzhungurova, O. A. (2012). On limit distribution emerging in the generalized Birnbaum-Saunders model, *Journal of Mathematical Sciences*, **99**(3), 1348- 1366.

# Appendix

Let $L(\lambda,\theta,p)$ denote the log of likelihood function and $\rho(\lambda,\theta,p)$ denote the log of the prior density. Thus, we get

$$L(\lambda,\theta,p) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\theta + n\lambda - \frac{1}{2\theta}\sum_{i=1}^{n} x_i - \frac{\lambda^2\theta}{2}\sum_{i=1}^{n}\frac{1}{x_i} + \sum_{i=1}^{n}\ln\frac{1}{x_i^{\frac{1}{2}}}\left[1 - p + \frac{\lambda\theta p}{x_i}\right] \quad (22)$$

and

$$\rho(\lambda,\theta,p) = \ln\left[\frac{a^{\gamma} b^{\eta}\Gamma(\alpha+\beta)}{\Gamma(\gamma)\Gamma(\eta)\Gamma(\alpha)\Gamma(\beta)}\right] + (\gamma-1)\ln\lambda - a\lambda$$
$$+(\eta-1)\ln\theta - b\theta + (\alpha-1)\ln p + (\beta-1)\ln(1-p). \quad (23)$$

The third derivative of the log of likelihood function is

$$\frac{\partial^3 L(\lambda,\theta,p)}{\partial i\partial j\partial k}$$

where $i, j, k = \lambda,\theta,p$ and the first derivative of the joint prior density is

$$\rho_i = \frac{\partial\rho(\lambda,\theta,p)}{\partial i}$$

where $i = \lambda,\theta,p$ . We substitute the maximum likelihood estimators to the three unknown parameters $\lambda,\theta$ and $p,$ then we obtain an estimate of $L_{ijk}$ and $\rho_i$ which denoted by $\hat{L}_{ijk}$ and $\hat{\rho}_i$, $i, j, k = \lambda, \theta, p$.

Let $I(\lambda,\theta,p)$ be the fisher information matrix of the unknown parameters $\lambda,\theta$ and $p$. The element of $3\times 3$ matrix $I(\lambda,\theta,p)$ are

$$I_{ij}(\lambda,\theta,p) = -\frac{\partial^2 L(\lambda,\theta,p)}{\partial i \partial j} \quad , i, j = \lambda, \theta, p.$$

By Miller [8], we obtain the asymptotic distribution of the maximum likelihood estimators $(\hat{\lambda},\hat{\theta},\hat{p})$ given by

$$\begin{pmatrix} \hat{\lambda} \\ \hat{\theta} \\ \hat{p} \end{pmatrix} \sim N\left( \begin{pmatrix} \lambda \\ \theta \\ p \end{pmatrix}, \begin{pmatrix} \sigma_{\lambda\lambda} & \sigma_{\lambda\theta} & \sigma_{\lambda p} \\ \sigma_{\theta\lambda} & \sigma_{\theta\theta} & \sigma_{\theta p} \\ \sigma_{p\lambda} & \sigma_{p\theta} & \sigma_{pp} \end{pmatrix} \right) \tag{24}$$

with the variance-covariance matrix as the inverse of the information matrix,

$$\Sigma = I^{-1} = \begin{pmatrix} \sigma_{\lambda\lambda} & \sigma_{\lambda\theta} & \sigma_{\lambda p} \\ \sigma_{\theta\lambda} & \sigma_{\theta\theta} & \sigma_{\theta p} \\ \sigma_{p\lambda} & \sigma_{p\theta} & \sigma_{pp} \end{pmatrix}. \tag{25}$$

We substitute the maximum likelihood estimators of the three parameters to the three unknown parameters, so we obtain an estimator of $\Sigma$ which is denoted by $\hat{\Sigma}$ and defined as follows:

$$\hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_{\lambda\lambda} & \hat{\sigma}_{\lambda\theta} & \hat{\sigma}_{\lambda p} \\ \hat{\sigma}_{\theta\lambda} & \hat{\sigma}_{\theta\theta} & \hat{\sigma}_{\theta p} \\ \hat{\sigma}_{p\lambda} & \hat{\sigma}_{p\theta} & \hat{\sigma}_{pp} \end{pmatrix}. \tag{26}$$