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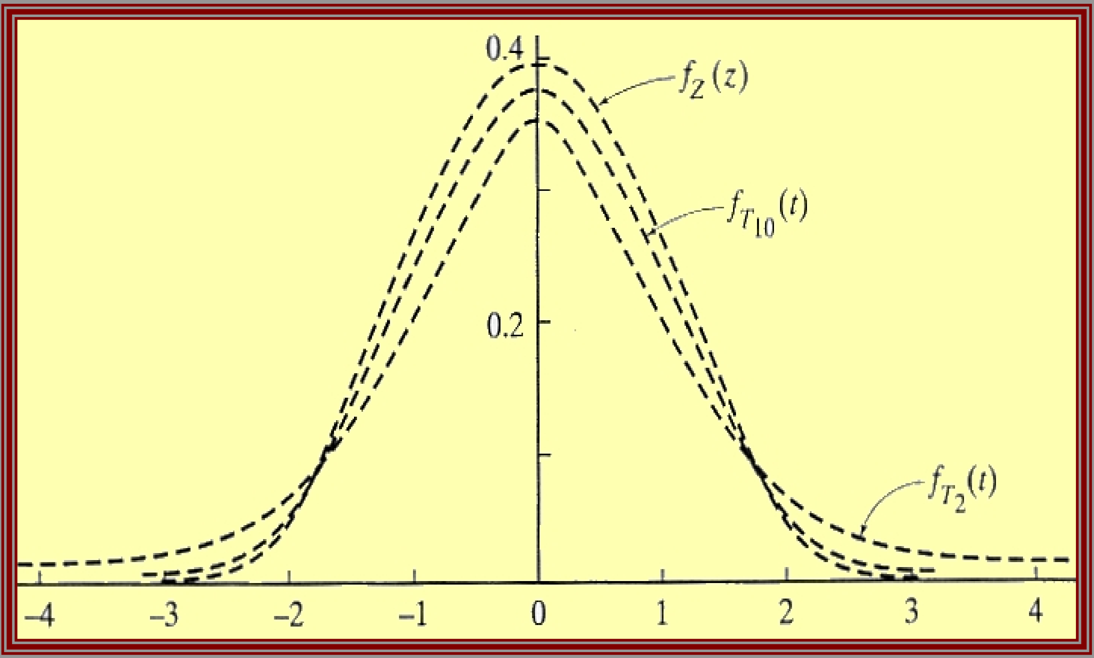

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Weighted Likelihood Estimator of Scale Parameter for the Two-parameter Weibull Distribution with a Contamination

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ABSTRACT In this article, we propose the weighted likelihood estimator for the scale parameter of the two-parameter Weibull distribution when the data set has a contamination. This method assigns zero weights to observations with small likelihood. We also examine the robust properties of the MLE and WLE for the parameters of the Weibull distribution. To examine the performance of the WLE method as compared to the MLE methods, and found that the WLE method out-performs the MLE method based on the relative bias and quadratic risk values.

Keywords Weibull distribution; Weighted likelihood estimator; Maximum likelihood estimator; Contamination; Outlier.

1. Introduction

The Weibull distribution is widely used in many fields such as engineering [1], biomedical sciences [2], ecology [3], etc. The Weibull distribution is named after the Swedish scientist Waloddi Weibull who associated this distribution with the strength of materials in 1939. This distribution is useful in describing wear-out or fatigue failures [4]. As further practical applications of the two-parameter Weibull distribution we mention wind energy assessment, rainfall amount, prediction of water levels, and analysis of lifetime of materials. The Weibull distribution is often used as the first step for modeling some real phenomenon.

An assumption of known shape parameter is appropriate for many real lifetime analysis problems. For example, in engineering applications we often assume that the shape parameter is

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one and get an exponential distribution [7]. Therefore, we would like to study two-parameter Weibull distribution assuming that the shape parameter is known.

As we mentioned above, Weibull distribution plays a central role in lifetime models. If a dataset is contaminated with outliers, the maximum likelihood estimator (MLE) can be very unreliable [2]. The problem of estimating the parameters of the Weibull distribution when a proportion of the observations are outliers is quite important to reliability applications. The weighted likelihood estimator (WLE) was proposed for robust estimation of the exponential distribution parameters by [1]. The WLE was introduced by [4] and it has been applied to a problem of robust estimation of parameters. The weighted likelihood method was introduced as a generalization of the local likelihood method and it can be global, as demonstrated in [5].

The weighted likelihood method from [1] yields α–trimmed mean type estimators of the parameter of interest. We continue this investigation applying this technique to the Weibull distribution in order to obtain robust estimation of the scale parameter. We assume that the shape parameter is known and a dataset shows a contamination. This method assigns zero weights to observations with small likelihood. We also examine the robust properties of the MLE and WLE for the parameters of the Weibull distribution with contamination. The simulation studies are extended to compare the MLE and WLE methods based on the relative bias and quadratic risk values.

The rest of this paper is organized as follows. In Section 2 we define the WLE for the Weibull distribution and the robustness of this estimator show in Section 3. In Section 4, we compare the WLE and MLE in terms of relative bias and quadratic risk. An example is given in Section 5. Some conclusions remarks are finally made in Section 6.

2. Proposed Weighted Likelihood Estimator

The distribution function for the two-parameter Weibull distribution is

$$F(x;\delta,\beta) = 1 - e^{-(x/\delta)^{\beta}}; \quad x \geq 0, \quad \delta > 0, \quad \beta > 0,$$

and the probability density function is

$$f(x;\delta,\beta) = \frac{\beta}{\delta^{\beta}} x^{\beta-1} e^{-(x/\delta)^{\beta}}; \quad x \geq 0, \quad \delta > 0, \quad \beta > 0,$$

where δ is the scale parameter and β is the shape parameter of the distribution.

Let $x^{(n)} = \{x_1, x_2, \ldots, x_n\}$ be a sample values from a distribution with a density function $f(x;\delta,\beta)$. The *weighted likelihood estimators* (WLE) of $\{\delta,\beta\}$ are obtained by maximizing the weighted likelihood function

$$L\big(\delta,\beta \mid x^{(n)}\big) = \sum_{i=1}^{n} w_i(x^{(n)}) \ln\big(f(x_i;\delta,\beta)\big),$$

where $w_i(x^{(n)}), 1 \leqslant i \leqslant n$ are the weights which depend on the sample. If all the weights are equal to one, then the resulting estimator is the *maximum likelihood estimator* (MLE).

Our goal is to estimate the scale parameter δ of the Weibull distribution, we assume the shape parameter β is known. Following the idea presented in [1] we let the weight w_i that corresponds to the i^{th} observation to be 1, if its estimated likelihood is sufficiently large, and 0 elsewhere. To be more precise, let

$$w_i = \begin{cases} 1 & \text{if} \quad f(x_i; \widehat{\delta}, \beta) > C \\ 0 & \quad \text{otherwise,} \end{cases}$$

where $\widehat{\delta}$ be the MLE of the parameter δ and we remind that β is assumed to be known. This means that we delete all improbable observations from the sample, we reject only extreme order statistics.

We now have the issue of the choice of C. Following the ideas of [1], we suggest this not be considered as a constant. Rather assume that C is chosen from the condition of a small probability of rejection of an observation when we sample from the non contamination Weibull distribution with cumulative distribution function, $F(x; \delta, \beta)$. Hence, we define C by the given pre-assigned small probability α as

$$P\Big[f\Big(\max_{1 \leqslant i \leqslant n} X_i; \widehat{\delta}, \beta\Big) < C\Big] = \alpha.$$

So we get, $C \approx \alpha\beta/(n\widehat{\delta})$.

Let the weighted likelihood estimator $\widetilde{\delta}$ of the parameter δ be defined as the solution of the equation

$$\frac{\sum_{k=1}^{m} \partial f(x_{i_k}; \delta, \beta)}{\partial \delta} = 0,$$

where $x_{i_1}, x_{i_2}, ..., x_{i_m}$ are the remaining observations in the sample after applying our procedure ($w_{i_k} = 1$). The WLE of δ is $\widetilde{\delta} = \big(\frac{1}{m} \sum_{k=1}^{m} x_{i_k}^{\beta}\big)^{1/\beta}$.

3. Robust Properties of the WLE for the Weibull Distribution with a Contamination

Assume that the sample $(x_1, x_2, ..., x_n)$ is taken from a population that follows a distribution with the distribution function $G_\varepsilon(x)$ to be defined now. We define the *ε–contamination model* as

$$\begin{aligned} G_\varepsilon(x) &= (1-\varepsilon)F(x;\delta,\beta) + \varepsilon F_1(x;\delta_1,\beta_1) = (1-\varepsilon)\big(1 - e^{-(\frac{x}{\delta})^\beta}\big) + \varepsilon\big(1 - e^{-(\frac{x}{\delta_1})^{\beta_1}}\big) \\ &= 1 - e^{-(\frac{x}{\delta})^\beta} - \varepsilon\Big(e^{-\big(\frac{x}{\delta(1+\Delta_1)}\big)^{\beta(1+\Delta_2)}} - e^{-(\frac{x}{\delta})^\beta}\Big). \end{aligned}$$

where $F(x;\delta,\beta)$ is the Weibull distribution with parameters (δ,β), and contamination $F_1(x;\delta_1,\beta_1)$ is the Weibull distribution with parameters (δ_1,β_1), where $\delta_1=\delta(1+\Delta_1),\beta_1=\beta(1+\Delta_2)$, $\Delta_1,\Delta_2>0$, and ε denotes the contamination proportion, $0\leqslant\varepsilon\leqslant 1$. Under the ε–contamination model we assume β is known. Let $\widehat{\delta}=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^{\beta}\right)^{1/\beta}$ be the MLE of the scale parameter δ. We assume

$$\delta_\varepsilon=E[G_\varepsilon(x)]=\delta\Big((1-\varepsilon)\Gamma\big(1+\frac{1}{\beta}\big)+\varepsilon(1+\Delta_1)\Gamma\Big(1+\frac{1}{\beta(1+\Delta_2)}\Big)\Big).$$

By the strong law of large numbers,

$$\lim_{n\to\infty}\widehat{\delta}=\delta_\varepsilon=\delta\Big((1-\varepsilon)\Gamma\big(1+\frac{1}{\beta}\big)+\varepsilon(1+\Delta_1)\Gamma\Big(1+\frac{1}{\beta(1+\Delta_2)}\Big)\Big).$$

The estimate $\widetilde{\delta}$ converges in probability to some value $\overline{\delta_\varepsilon}$ which can be calculated as a limit of the expected values truncated at the point $A=-\delta_\varepsilon\left(\ln\frac{\alpha}{n}\right)^{1/\beta}$ of the distribution $\overline{G_\varepsilon(x)}=G(x)/G(A)$ where $0\leqslant x\leqslant A$, so we get

$$\begin{aligned}\overline{\delta_\varepsilon}&=\lim_{n\to\infty}E\left[\overline{G_\varepsilon(x)}\right]\\&=\lim_{n\to\infty}\frac{1}{G(A)}\Big((1-\varepsilon)\int_0^A\frac{\beta}{\delta^\beta}x^\beta e^{-\left(\frac{x}{\delta}\right)^\beta}dx+\varepsilon\int_0^A\frac{\beta_1}{\delta_1^{\beta_1}}x^{\beta_1}e^{-\left(\frac{x}{\delta_1}\right)^{\beta_1}}dx\Big).\end{aligned}$$

Then

$$\overline{\delta_\varepsilon}=\frac{\delta}{G(A)}\Big((1-\varepsilon)IG_1+\varepsilon(1+\Delta_1)IG_2\Big)$$

where

$$IG_1=\Gamma\Big(\frac{1}{\beta}+1\Big)P\Big(\frac{1}{\beta}+1,A_u\Big),\quad IG_2=\Gamma\Big(\frac{1}{\beta_1}+1\Big)P\Big(\frac{1}{\beta_1}+1,A_v\Big),$$

$$G(A)=1-e^{-\left(\frac{A}{\delta}\right)^\beta}-\varepsilon\Big(e^{-\left(\frac{A}{\delta_1}\right)^{\beta_1}}-e^{-\left(\frac{A}{\delta}\right)^\beta}\Big),\quad A_u=\frac{\delta_\varepsilon^\beta\ln\frac{n}{\alpha}}{\delta^\beta},\quad A_v=\frac{\delta_\varepsilon^{\beta_1}\ln\frac{n}{\alpha}}{\delta_1^{\beta_1}},$$

$$P\Big(\frac{1}{\beta}+1,A_u\Big)=\frac{\gamma\big(\frac{1}{\beta}+1,A_u\big)}{\Gamma\big(\frac{1}{\beta}+1\big)}\quad \textit{and}\quad P\Big(\frac{1}{\beta_1}+1,A_v\Big)=\frac{\gamma\big(\frac{1}{\beta_1}+1,A_v\big)}{\Gamma\big(\frac{1}{\beta_1}+1\big)}$$

are a normalized function of a lower incomplete gamma function $\gamma\big(1/\beta+1,A_u\big)$ and $\gamma\big(1/\beta_1+1,A_v\big)$, respectively.

Therefore, an exact calculation of the gain in the bias and reduction of the risk of the proposed estimator in comparison with the MLE cannot be possible. Indeed, these integrals cannot be evaluated in the closed form. Moreover the solution is cumbersome. Thus, we cannot easily compare the relative bias of $\widetilde{\delta}$ with the relative bias of $\widehat{\delta}$. A relevant conclusion may not be possible concerning the gain in bias using these precise formulas even if we expand $\overline{\delta_\varepsilon}$ in powers of ε. Thus, we shall confine ourselves to asymptotic analysis. Recall that we reject observation with $X_k > -\widehat{\delta}\big(\ln\frac{\alpha}{n}\big)^{1/\beta}$ and note that $-\widehat{\delta}\big(\ln\frac{\alpha}{n}\big)^{1/\beta} \sim -\delta_\varepsilon\big(\ln\frac{\alpha}{n}\big)^{1/\beta} = \big(\delta_\varepsilon^\beta \ln\frac{n}{\alpha}\big)^{1/\beta}$. Hence the probability of rejecting an observation in the contamination model is asymptotically equal to

$$P[X_1 > A] = 1 - P[X_1 \leqslant A] = 1 - G(A) \approx \frac{\alpha}{n}.$$

Therefore the asymptotic distribution of $\widetilde{\delta}$ equals to the distribution of the α–generalized trimmed sample mean where we defined α–generalized trimmed sample mean as

$$Y = \left(\frac{1}{n(1-\alpha)}\sum_{k=1}^{n(1-\alpha)} Y_k^\beta\right)^{\frac{1}{\beta}}$$

of the random sample of size $n(1-\alpha)$ from the distribution concentrated on the interval $(0, A)$. The probability density of this distribution is positive only the interval $(0, A)$ and has the form

$$\begin{aligned} f_A(x;\delta,\beta) &= \left(\overline{G_\varepsilon(x)}\right)' = \frac{G(x)'}{G(A)} \quad ;0 < x \leqslant A \\ &= \frac{(1-\varepsilon)f(x;\delta,\beta) + \varepsilon f(x;\delta_1,\beta_1)}{G_\alpha};\ \text{where } G_\alpha \approx 1 - \frac{\alpha}{n}. \end{aligned}$$

Result 1 *Under the Weibull distribution contamination has the relative bias of the maximum likelihood estimator* $(\widehat{\delta})$ *as*

$$(1-\varepsilon)\Gamma\Big(1 + \frac{1}{\beta}\Big) + \varepsilon(1+\Delta_1)\Gamma\Big(1 + \frac{1}{\beta_1}\Big) - 1$$

Result 2 *Under the Weibull distribution contamination has the relative bias of the weighted likelihood estimator* $(\widetilde{\delta})$ *as*

$$\frac{1}{G_\alpha}\Big((1-\varepsilon)IG_1 + \varepsilon(1+\Delta_1)IG_2\Big) - 1$$

where

$$IG_1 = \Gamma\Big(\frac{1}{\beta}+1\Big)P\Big(\frac{1}{\beta}+1, A_u\Big), \quad IG_2 = \Gamma\Big(\frac{1}{\beta_1}+1\Big)P\Big(\frac{1}{\beta_1}+1, A_v\Big),$$

$$G_\alpha \approx 1 - \frac{\alpha}{n}, \quad A_u = \frac{\delta_\varepsilon^\beta \ln\frac{n}{\alpha}}{\delta^\beta}, \quad A_v = \frac{\delta_\varepsilon^{\beta_1}\ln\frac{n}{\alpha}}{\delta_1^{\beta_1}},$$

$$P\Big(\frac{1}{\beta}+1, A_u\Big) = \frac{\gamma\big(\frac{1}{\beta}+1, A_u\big)}{\Gamma\big(\frac{1}{\beta}+1\big)}, \quad P\Big(\frac{1}{\beta_1}+1, A_v\Big) = \frac{\gamma\big(\frac{1}{\beta_1}+1, A_v\big)}{\Gamma\big(\frac{1}{\beta_1}+1\big)}.$$

Result 3 *Under the Weibull distribution contamination has the quadratic risk of the maximum likelihood estimator* $(\widehat{\delta})$ *as*

$$\frac{\delta^2}{n}\big(E_2 - E_1^2 + (1-E_1)^2\big)$$

where

$$E_1 = (1-\varepsilon)\Gamma\Big(1+\frac{1}{\beta}\Big) + \varepsilon(1+\Delta_1)\Gamma\Big(1+\frac{1}{\beta(1+\Delta_2)}\Big)$$
$$E_2 = (1-\varepsilon)\Gamma\Big(1+\frac{2}{\beta}\Big) + \varepsilon(1+\Delta_1)^2\Gamma\Big(1+\frac{2}{\beta(1+\Delta_2)}\Big).$$

Result 4 *Under the contamination is the Weibull distribution has the quadratic risk of the weighted likelihood estimator* $(\widetilde{\delta})$ *as*

$$\frac{\delta^2((1-\varepsilon)IG_3 + \varepsilon(1+\Delta_1)^2 IG_4)}{n(1-\alpha)G_\alpha} + \frac{\delta^2}{n(1-\alpha)} - \frac{2\delta^2((1-\varepsilon)IG_1 + \varepsilon(1+\Delta_1)IG_2)}{n(1-\alpha)G_\alpha}$$

where

$$IG_1 = \Gamma\Big(\frac{1}{\beta}+1\Big)P\Big(\frac{1}{\beta}+1, A_u\Big), \quad IG_2 = \Gamma\Big(\frac{1}{\beta_1}+1\Big)P\Big(\frac{1}{\beta_1}+1, A_v\Big),$$
$$G_\alpha \approx 1-\frac{\alpha}{n}, \quad A_u = \frac{\delta_\varepsilon^\beta \ln\frac{n}{\alpha}}{\delta^\beta}, \quad A_v = \frac{\delta_\varepsilon^{\beta_1}\ln\frac{n}{\alpha}}{\delta_1^{\beta_1}},$$
$$IG_3 = \Gamma\Big(\frac{2}{\beta}+1\Big)P\Big(\frac{2}{\beta}+1, A_u\Big), \quad IG_4 = \Gamma\Big(\frac{2}{\beta_1}+1\Big)P\Big(\frac{2}{\beta_1}+1, A_v\Big),$$
$$P\Big(\frac{1}{\beta}+1, A_u\Big) = \frac{\gamma\big(\frac{1}{\beta}+1, A_u\big)}{\Gamma\big(\frac{1}{\beta}+1\big)}, \quad P\Big(\frac{1}{\beta_1}+1, A_v\Big) = \frac{\gamma\big(\frac{1}{\beta_1}+1, A_v\big)}{\Gamma\big(\frac{1}{\beta_1}+1\big)},$$
$$P\Big(\frac{2}{\beta}+1, A_u\Big) = \frac{\gamma\big(\frac{2}{\beta}+1, A_u\big)}{\Gamma\big(\frac{2}{\beta}+1\big)}, \quad P\Big(\frac{2}{\beta_1}+1, A_v\Big) = \frac{\gamma\big(\frac{2}{\beta_1}+1, A_v\big)}{\Gamma\big(\frac{2}{\beta_1}+1\big)}.$$

4. Simulation Study

A simulation study was carried out to study the performance of the MLE and WLE methods for the Weibull distribution contamination based on the simulated and asymptotic relative bias and quadratic risk of the estimation for the scale parameter δ.

We generated data sets of size $n = 25, 50, 100$ from the ε–contamination model. The central distribution to be the Weibull with scale parameter $\delta = 1$ and shape parameter $\beta = 0.5, 1, 2$. The contamination has the Weibull distribution with scale parameter $\delta_1 = \delta(1+\Delta_1)$ and shape parameter $\beta_1 = \beta(1+\Delta_2)$ where $\Delta_1, \Delta_2 = 1, 3, 5$ and the contamination proportion $\varepsilon = 0.05$ and the values of $\alpha = 0.01, 0.03, 0.05, 0.07, 0.09$. All the following simulations results are based on 10,000 replicates by using programs written in `R` statistical software [10].

The results are presented in Tables 1-3 which show the simulated(sim) and the asymptotic(asy) relative bias and quadratic risk of the MLE and WLE for the central distribution is the Weibull$(1, \beta)$ when distribution of the contamination is Weibull$\big(1(1+\Delta_1), \beta(1+\Delta_2)\big)$ and the contamination proportion $\varepsilon = 0.05$. Note that the simulated and the asymptotic relative bias and quadratic risk of the MLE do not depend on α for all sample sizes. From these results, we observe the following.

The simulated and the asymptotic relative bias of the MLE and WLE are smallest at $\Delta_1 = 1$. The same conclusion can be seen for the quadratic risk values of the MLE and WLE. In the case when $\beta = 0.5$, relative bias of the WLE method provides smaller than those of the MLE method for small values of α, but the quadratic risk of the WLE method is higher than those of the MLE method. In the case of Weibull$\big(1(1+\Delta_1), 1(1+\Delta_2)\big)$ contamination, the WLE method provides smaller quadratic risk δ than those of the MLE method and quadratic risks of the WLE is close to those of the MLE method when α is small. When Δ_1 gets large, the WLE provides smaller relative bias and quadratic risk values than those of the MLE when α is increasing. In the case when $\beta = 2$, the WLE outperforms the MLE based on the simulated and the asymptotic relative bias and quadratic risk when α is small.

Table 1 The simulated(sim) and the asymptotic(asy) relative bias and quadratic risk of MLE and WLE for $W(1,0.5) + 0.05W(1(1+\Delta_1), 0.5(1+\Delta_2))$.

(Δ_1, Δ_2)	α	relative bias						quadratic risk					
		n=25		n=50		n=100		n=25		n=50		n=100	
		sim	asy	sim	asy	sim	asy	sim	asy	sim	asy	sim	asy
(1 , 1)	MLE	0.066	1.000	0.046	1.000	0.037	1.000	0.180	0.808	0.087	0.404	0.044	0.202
	0.01	0.064	1.000	0.045	1.000	0.036	1.000	0.181	0.842	0.088	0.421	0.044	0.210
	0.03	0.059	1.000	0.041	1.000	0.034	1.000	0.181	0.844	0.088	0.423	0.044	0.211
	0.05	0.053	0.999	0.038	1.000	0.032	1.000	0.180	0.845	0.088	0.424	0.044	0.212
	0.07	0.048	0.999	0.035	1.000	0.030	1.000	0.180	0.847	0.088	0.426	0.044	0.213
	0.09	0.043	0.998	0.031	0.999	0.028	1.000	0.180	0.848	0.087	0.428	0.044	0.215
(3 , 1)	MLE	0.071	1.100	0.055	1.100	0.047	1.100	0.187	0.848	0.091	0.424	0.045	0.212
	0.01	0.069	1.100	0.053	1.100	0.046	1.100	0.187	0.850	0.091	0.425	0.045	0.213
	0.03	0.064	1.100	0.050	1.100	0.044	1.100	0.187	0.869	0.091	0.435	0.045	0.218
	0.05	0.059	1.100	0.047	1.100	0.042	1.100	0.187	0.889	0.091	0.446	0.045	0.223
	0.07	0.054	1.099	0.043	1.100	0.040	1.100	0.187	0.910	0.090	0.457	0.045	0.229
	0.09	0.048	1.099	0.040	1.100	0.038	1.100	0.187	0.931	0.090	0.468	0.045	0.234
(5 , 1)	MLE	0.085	1.200	0.067	1.200	0.056	1.200	0.194	0.920	0.094	0.460	0.046	0.230
	0.01	0.084	1.200	0.066	1.200	0.055	1.200	0.194	0.864	0.094	0.432	0.046	0.216
	0.03	0.079	1.200	0.063	1.200	0.052	1.200	0.194	0.915	0.094	0.458	0.046	0.229
	0.05	0.075	1.200	0.059	1.200	0.050	1.200	0.193	0.966	0.094	0.484	0.046	0.242
	0.07	0.070	1.200	0.056	1.200	0.048	1.200	0.193	1.020	0.094	0.511	0.046	0.256
	0.09	0.065	1.199	0.053	1.200	0.046	1.200	0.192	1.076	0.094	0.540	0.046	0.270
(1 , 3)	MLE	0.064	0.989	0.044	0.989	0.037	0.989	0.182	0.801	0.089	0.400	0.043	0.200
	0.01	0.062	0.989	0.042	0.989	0.036	0.989	0.182	0.840	0.090	0.420	0.043	0.210
	0.03	0.058	0.988	0.039	0.989	0.034	0.989	0.183	0.839	0.090	0.420	0.043	0.210
	0.05	0.054	0.988	0.035	0.988	0.032	0.989	0.182	0.837	0.090	0.420	0.043	0.211
	0.07	0.048	0.987	0.032	0.988	0.030	0.989	0.183	0.835	0.090	0.420	0.043	0.211
	0.09	0.043	0.987	0.029	0.988	0.028	0.988	0.182	0.833	0.090	0.420	0.043	0.211
(3 , 3)	MLE	0.073	1.077	0.059	1.077	0.047	1.077	0.186	0.818	0.090	0.409	0.045	0.204
	0.01	0.072	1.077	0.058	1.077	0.046	1.077	0.186	0.844	0.090	0.422	0.045	0.211
	0.03	0.066	1.077	0.055	1.077	0.043	1.077	0.185	0.851	0.090	0.426	0.045	0.213
	0.05	0.061	1.077	0.052	1.077	0.041	1.077	0.185	0.857	0.090	0.430	0.045	0.215
	0.07	0.056	1.076	0.049	1.077	0.039	1.077	0.184	0.864	0.090	0.434	0.045	0.217
	0.09	0.051	1.076	0.045	1.077	0.038	1.077	0.184	0.870	0.089	0.438	0.044	0.220
(5 , 3)	MLE	0.084	1.166	0.064	1.166	0.057	1.166	0.187	0.851	0.091	0.425	0.047	0.213
	0.01	0.083	1.166	0.062	1.166	0.056	1.166	0.187	0.850	0.091	0.425	0.047	0.213
	0.03	0.079	1.166	0.059	1.166	0.054	1.166	0.188	0.871	0.091	0.436	0.047	0.218
	0.05	0.073	1.166	0.056	1.166	0.052	1.166	0.187	0.893	0.091	0.447	0.047	0.224
	0.07	0.068	1.165	0.052	1.166	0.050	1.166	0.186	0.915	0.091	0.459	0.047	0.230
	0.09	0.063	1.165	0.050	1.166	0.048	1.166	0.186	0.938	0.090	0.471	0.047	0.236

Table 1: Continued

(Δ_1, Δ_2)	α	relative bias						quadratic risk					
		n=25		n=50		n=100		n=25		n=50		n=100	
		sim	asy	sim	asy	sim	asy	sim	asy	sim	asy	sim	asy
(1 , 5)	MLE	0.067	0.989	0.046	0.989	0.037	0.989	0.182	0.800	0.088	0.400	0.044	0.200
	0.01	0.065	0.989	0.045	0.989	0.036	0.989	0.182	0.840	0.088	0.420	0.044	0.210
	0.03	0.060	0.989	0.042	0.989	0.034	0.989	0.181	0.839	0.088	0.420	0.044	0.210
	0.05	0.055	0.989	0.038	0.989	0.032	0.989	0.182	0.837	0.088	0.420	0.044	0.210
	0.07	0.049	0.988	0.035	0.989	0.029	0.989	0.180	0.834	0.088	0.420	0.044	0.210
	0.09	0.044	0.987	0.031	0.989	0.027	0.989	0.180	0.832	0.088	0.420	0.043	0.211
(3 , 5)	MLE	0.074	1.079	0.055	1.079	0.049	1.079	0.185	0.815	0.087	0.407	0.044	0.204
	0.01	0.073	1.079	0.053	1.079	0.048	1.079	0.186	0.843	0.087	0.422	0.044	0.211
	0.03	0.069	1.078	0.050	1.079	0.046	1.079	0.186	0.849	0.087	0.425	0.044	0.213
	0.05	0.064	1.078	0.047	1.078	0.044	1.079	0.186	0.854	0.087	0.428	0.044	0.214
	0.07	0.060	1.078	0.044	1.078	0.042	1.079	0.187	0.859	0.087	0.431	0.044	0.216
	0.09	0.055	1.077	0.041	1.078	0.040	1.078	0.187	0.864	0.087	0.435	0.044	0.218
(5 , 5)	MLE	0.083	1.168	0.065	1.168	0.054	1.168	0.184	0.844	0.096	0.422	0.046	0.211
	0.01	0.082	1.168	0.064	1.168	0.053	1.168	0.184	0.849	0.096	0.425	0.046	0.212
	0.03	0.078	1.168	0.061	1.168	0.051	1.168	0.185	0.867	0.096	0.434	0.046	0.217
	0.05	0.073	1.168	0.058	1.168	0.048	1.168	0.185	0.885	0.095	0.444	0.046	0.222
	0.07	0.069	1.167	0.055	1.168	0.046	1.168	0.185	0.904	0.095	0.454	0.046	0.227
	0.09	0.064	1.167	0.052	1.168	0.044	1.168	0.185	0.924	0.095	0.464	0.045	0.232

Table 2 The simulated(sim) and the asymptotic(asy) relative bias and quadratic risk of MLE and WLE for $W(1,1) + 0.05W(1(1+\Delta_1), 1(1+\Delta_2))$.

(Δ_1, Δ_2)	α	relative bias						quadratic risk					
		n=25		n=50		n=100		n=25		n=50		n=100	
		sim	asy	sim	asy	sim	asy	sim	asy	sim	asy	sim	asy
(1 , 1)	MLE	0.016	0.039	0.018	0.039	0.017	0.039	0.041	0.041	0.020	0.020	0.010	0.010
	0.01	0.015	0.036	0.017	0.037	0.017	0.038	0.041	0.040	0.020	0.020	0.010	0.010
	0.03	0.013	0.032	0.016	0.035	0.016	0.037	0.041	0.040	0.020	0.020	0.010	0.010
	0.05	0.010	0.027	0.014	0.033	0.015	0.035	0.042	0.040	0.020	0.021	0.010	0.010
	0.07	0.007	0.024	0.012	0.031	0.014	0.034	0.042	0.041	0.021	0.021	0.010	0.010
	0.09	0.005	0.020	0.010	0.028	0.013	0.033	0.042	0.042	0.021	0.022	0.010	0.010
(3 , 1)	MLE	0.034	0.127	0.036	0.127	0.034	0.127	0.045	0.058	0.023	0.029	0.012	0.013
	0.01	0.033	0.122	0.035	0.125	0.034	0.126	0.046	0.043	0.023	0.022	0.012	0.011
	0.03	0.029	0.111	0.033	0.119	0.032	0.124	0.046	0.049	0.023	0.025	0.012	0.012
	0.05	0.026	0.102	0.031	0.114	0.031	0.121	0.046	0.054	0.023	0.029	0.012	0.014
	0.07	0.022	0.093	0.029	0.109	0.030	0.118	0.046	0.059	0.023	0.032	0.012	0.016
	0.09	0.019	0.085	0.026	0.104	0.028	0.115	0.046	0.063	0.023	0.035	0.012	0.018
(5 , 1)	MLE	0.046	0.216	0.053	0.216	0.053	0.216	0.051	0.091	0.028	0.045	0.015	0.023
	0.01	0.041	0.170	0.047	0.185	0.046	0.196	0.050	0.047	0.028	0.024	0.015	0.012
	0.03	0.032	0.135	0.040	0.159	0.041	0.177	0.050	0.056	0.027	0.030	0.014	0.016
	0.05	0.025	0.114	0.036	0.142	0.038	0.164	0.049	0.062	0.026	0.035	0.014	0.019
	0.07	0.020	0.099	0.032	0.129	0.035	0.154	0.048	0.068	0.026	0.039	0.014	0.022
	0.09	0.016	0.086	0.029	0.119	0.033	0.146	0.048	0.072	0.025	0.043	0.014	0.024
(1 , 3)	MLE	0.020	0.041	0.017	0.041	0.019	0.041	0.041	0.040	0.020	0.020	0.014	0.022
	0.01	0.019	0.038	0.017	0.039	0.019	0.040	0.041	0.040	0.020	0.020	0.012	0.011
	0.03	0.017	0.034	0.015	0.037	0.018	0.039	0.041	0.039	0.020	0.020	0.012	0.013
	0.05	0.015	0.030	0.013	0.035	0.017	0.038	0.042	0.039	0.021	0.020	0.011	0.014
	0.07	0.012	0.026	0.012	0.033	0.016	0.036	0.042	0.039	0.021	0.020	0.011	0.015
	0.09	0.010	0.022	0.010	0.031	0.015	0.035	0.042	0.040	0.021	0.021	0.011	0.016
(3 , 3)	MLE	0.032	0.131	0.036	0.131	0.037	0.131	0.043	0.054	0.022	0.027	0.012	0.013
	0.01	0.032	0.130	0.035	0.131	0.037	0.131	0.043	0.043	0.022	0.021	0.012	0.011
	0.03	0.030	0.127	0.034	0.129	0.036	0.130	0.044	0.049	0.022	0.025	0.012	0.012
	0.05	0.027	0.125	0.033	0.128	0.035	0.130	0.044	0.055	0.022	0.028	0.012	0.014
	0.07	0.025	0.122	0.031	0.127	0.034	0.129	0.044	0.062	0.022	0.031	0.012	0.016
	0.09	0.022	0.119	0.030	0.125	0.033	0.128	0.044	0.068	0.023	0.035	0.012	0.018

Table 2: Continued

(Δ_1, Δ_2)	α	relative bias						quadratic risk					
		n=25		n=50		n=100		n=25		n=50		n=100	
		sim	asy	sim	asy	sim	asy	sim	asy	sim	asy	sim	asy
(5 , 3)	MLE	0.056	0.222	0.053	0.222	0.054	0.222	0.053	0.082	0.027	0.041	0.015	0.021
	0.01	0.055	0.220	0.052	0.222	0.053	0.222	0.053	0.049	0.027	0.024	0.015	0.012
	0.03	0.050	0.207	0.049	0.218	0.052	0.221	0.054	0.064	0.027	0.033	0.015	0.017
	0.05	0.044	0.186	0.045	0.211	0.050	0.219	0.055	0.077	0.027	0.042	0.015	0.021
	0.07	0.039	0.166	0.041	0.202	0.047	0.217	0.055	0.087	0.027	0.049	0.015	0.026
	0.09	0.033	0.146	0.038	0.192	0.045	0.213	0.055	0.095	0.027	0.056	0.015	0.031
(1 , 5)	MLE	0.017	0.043	0.019	0.043	0.019	0.043	0.041	0.040	0.020	0.020	0.012	0.014
	0.01	0.016	0.040	0.019	0.041	0.019	0.042	0.041	0.040	0.020	0.020	0.012	0.011
	0.03	0.014	0.036	0.017	0.039	0.018	0.041	0.041	0.039	0.020	0.020	0.012	0.013
	0.05	0.011	0.032	0.015	0.037	0.016	0.040	0.041	0.039	0.020	0.020	0.012	0.015
	0.07	0.009	0.028	0.014	0.035	0.015	0.039	0.042	0.039	0.020	0.020	0.012	0.017
	0.09	0.007	0.024	0.012	0.033	0.015	0.038	0.042	0.040	0.021	0.021	0.012	0.018
(3 , 5)	MLE	0.039	0.136	0.035	0.136	0.037	0.136	0.044	0.054	0.023	0.027	0.020	0.040
	0.01	0.038	0.134	0.034	0.135	0.036	0.135	0.045	0.043	0.023	0.021	0.013	0.012
	0.03	0.036	0.132	0.033	0.134	0.035	0.135	0.045	0.049	0.023	0.025	0.012	0.014
	0.05	0.033	0.129	0.031	0.132	0.034	0.134	0.045	0.055	0.023	0.028	0.012	0.016
	0.07	0.031	0.126	0.030	0.131	0.033	0.133	0.045	0.061	0.023	0.031	0.012	0.017
	0.09	0.028	0.124	0.028	0.130	0.033	0.133	0.045	0.068	0.023	0.035	0.012	0.019
(5 , 5)	MLE	0.059	0.228	0.054	0.228	0.056	0.228	0.052	0.082	0.027	0.041	0.015	0.021
	0.01	0.058	0.228	0.054	0.228	0.055	0.228	0.052	0.049	0.027	0.024	0.015	0.012
	0.03	0.054	0.226	0.052	0.227	0.054	0.228	0.053	0.066	0.028	0.033	0.015	0.017
	0.05	0.049	0.219	0.049	0.226	0.053	0.227	0.054	0.084	0.028	0.043	0.015	0.022
	0.07	0.044	0.204	0.046	0.225	0.051	0.227	0.054	0.099	0.028	0.053	0.015	0.026
	0.09	0.040	0.186	0.043	0.221	0.050	0.227	0.055	0.110	0.028	0.062	0.016	0.032

Table 3 The simulated(sim) and the asymptotic(asy) relative bias and quadratic risk of MLE and WLE for $W(1,2) + 0.05W(1(1+\Delta_1), 2(1+\Delta_2))$.

(Δ_1, Δ_2)	α	relative bias						quadratic risk					
		n=25		n=50		n=100		n=25		n=50		n=100	
		sim	asy	sim	asy	sim	asy	sim	asy	sim	asy	sim	asy
(1 , 1)	MLE	0.011	-0.067	0.015	-0.067	0.016	-0.067	0.010	0.010	0.005	0.005	0.003	0.003
	0.01	0.011	-0.074	0.015	-0.071	0.015	-0.070	0.010	0.009	0.006	0.005	0.003	0.002
	0.03	0.009	-0.084	0.014	-0.077	0.015	-0.073	0.011	0.010	0.006	0.005	0.003	0.003
	0.05	0.007	-0.091	0.013	-0.082	0.014	-0.076	0.011	0.010	0.006	0.005	0.003	0.003
	0.07	0.005	-0.097	0.012	-0.086	0.013	-0.078	0.011	0.011	0.006	0.006	0.003	0.003
	0.09	0.004	-0.103	0.010	-0.089	0.012	-0.081	0.011	0.011	0.006	0.006	0.003	0.003
(3 , 1)	MLE	0.059	0.023	0.061	0.023	0.065	0.023	0.028	0.025	0.017	0.012	0.011	0.006
	0.01	0.015	-0.134	0.014	-0.129	0.016	-0.123	0.014	0.010	0.007	0.005	0.003	0.002
	0.03	0.009	-0.143	0.009	-0.138	0.012	-0.132	0.012	0.010	0.006	0.005	0.003	0.003
	0.05	0.005	-0.148	0.007	-0.142	0.010	-0.136	0.012	0.011	0.006	0.006	0.003	0.003
	0.07	0.003	-0.151	0.005	-0.145	0.009	-0.139	0.012	0.012	0.006	0.006	0.003	0.003
	0.09	0.001	-0.154	0.004	-0.147	0.008	-0.141	0.011	0.013	0.006	0.007	0.003	0.003
(5 , 1)	MLE	0.117	0.114	0.134	0.114	0.139	0.114	0.085	0.053	0.060	0.026	0.042	0.013
	0.01	0.006	-0.152	0.011	-0.150	0.009	-0.148	0.013	0.010	0.006	0.005	0.003	0.002
	0.03	0.002	-0.155	0.008	-0.153	0.007	-0.151	0.012	0.010	0.006	0.005	0.003	0.003
	0.05	0.000	-0.157	0.006	-0.154	0.006	-0.152	0.011	0.011	0.006	0.006	0.003	0.003
	0.07	-0.002	-0.158	0.005	-0.156	0.005	-0.153	0.011	0.012	0.005	0.006	0.003	0.003
	0.09	-0.003	-0.160	0.004	-0.156	0.005	-0.154	0.011	0.013	0.005	0.007	0.003	0.003
(1 , 3)	MLE	0.013	-0.064	0.015	-0.064	0.017	-0.064	0.010	0.010	0.005	0.005	0.003	0.003
	0.01	0.013	-0.066	0.015	-0.065	0.016	-0.064	0.010	0.009	0.005	0.005	0.003	0.002
	0.03	0.011	-0.069	0.014	-0.067	0.016	-0.065	0.010	0.010	0.005	0.005	0.003	0.003
	0.05	0.010	-0.073	0.013	-0.068	0.015	-0.066	0.011	0.011	0.005	0.005	0.003	0.003
	0.07	0.009	-0.077	0.012	-0.070	0.015	-0.067	0.011	0.011	0.005	0.006	0.003	0.003
	0.09	0.007	-0.082	0.011	-0.072	0.015	-0.068	0.011	0.012	0.005	0.006	0.003	0.003

Table 3: Continued

(Δ_1, Δ_2)	α	relative bias						quadratic risk					
		n=25		n=50		n=100		n=25		n=50		n=100	
		sim	asy	sim	asy	sim	asy	sim	asy	sim	asy	sim	asy
(3 , 3)	MLE	0.061	0.030	0.063	0.030	0.067	0.030	0.027	0.025	0.016	0.012	0.011	0.006
	0.01	0.015	-0.151	0.011	-0.147	0.013	-0.142	0.015	0.010	0.007	0.005	0.004	0.002
	0.03	0.006	-0.157	0.005	-0.153	0.009	-0.150	0.012	0.010	0.006	0.005	0.003	0.003
	0.05	0.003	-0.160	0.003	-0.156	0.007	-0.152	0.011	0.011	0.006	0.006	0.003	0.003
	0.07	0.000	-0.162	0.002	-0.157	0.006	-0.154	0.011	0.012	0.006	0.006	0.003	0.003
	0.09	-0.002	-0.164	0.001	-0.159	0.005	-0.155	0.011	0.013	0.006	0.007	0.003	0.003
(5 , 3)	MLE	0.125	0.124	0.132	0.124	0.140	0.124	0.085	0.053	0.055	0.027	0.041	0.013
	0.01	0.006	-0.158	0.005	-0.158	0.005	-0.157	0.015	0.009	0.006	0.005	0.003	0.002
	0.03	0.002	-0.159	0.003	-0.158	0.004	-0.158	0.012	0.010	0.005	0.005	0.003	0.003
	0.05	0.000	-0.160	0.002	-0.159	0.003	-0.158	0.011	0.011	0.005	0.006	0.003	0.003
	0.07	-0.002	-0.161	0.001	-0.160	0.003	-0.159	0.011	0.012	0.005	0.006	0.003	0.003
	0.09	-0.004	-0.163	0.001	-0.160	0.003	-0.159	0.011	0.013	0.005	0.006	0.003	0.003
(1 , 5)	MLE	0.013	-0.062	0.016	-0.062	0.018	-0.062	0.010	0.010	0.005	0.005	0.003	0.003
	0.01	0.013	-0.064	0.015	-0.063	0.018	-0.063	0.010	0.009	0.005	0.005	0.003	0.002
	0.03	0.012	-0.067	0.015	-0.065	0.017	-0.064	0.010	0.010	0.005	0.005	0.003	0.003
	0.05	0.010	-0.070	0.014	-0.066	0.017	-0.064	0.011	0.011	0.005	0.005	0.003	0.003
	0.07	0.009	-0.072	0.013	-0.068	0.016	-0.065	0.011	0.012	0.005	0.006	0.003	0.003
	0.09	0.008	-0.075	0.012	-0.069	0.016	-0.066	0.011	0.012	0.006	0.006	0.003	0.003
(3 , 5)	MLE	0.065	0.034	0.065	0.034	0.068	0.034	0.028	0.025	0.017	0.013	0.011	0.006
	0.01	0.015	-0.157	0.011	-0.155	0.011	-0.152	0.016	0.009	0.007	0.005	0.003	0.002
	0.03	0.006	-0.160	0.005	-0.158	0.006	-0.156	0.013	0.010	0.006	0.005	0.003	0.003
	0.05	0.002	-0.162	0.003	-0.159	0.005	-0.157	0.012	0.011	0.006	0.006	0.003	0.003
	0.07	-0.001	-0.164	0.002	-0.160	0.004	-0.158	0.011	0.012	0.005	0.006	0.003	0.003
	0.09	-0.003	-0.165	0.001	-0.161	0.003	-0.159	0.011	0.013	0.005	0.006	0.003	0.003
(5 , 5)	MLE	0.124	0.129	0.133	0.129	0.145	0.129	0.084	0.054	0.056	0.027	0.042	0.014
	0.01	0.003	-0.158	0.002	-0.158	0.004	-0.158	0.014	0.009	0.005	0.005	0.003	0.002
	0.03	0.000	-0.159	0.000	-0.159	0.003	-0.158	0.012	0.010	0.005	0.005	0.003	0.003
	0.05	-0.003	-0.160	-0.001	-0.159	0.003	-0.159	0.012	0.011	0.005	0.006	0.003	0.003
	0.07	-0.004	-0.162	-0.001	-0.160	0.002	-0.159	0.011	0.012	0.005	0.006	0.003	0.003
	0.09	-0.006	-0.163	-0.002	-0.160	0.002	-0.159	0.011	0.013	0.005	0.006	0.003	0.003

5. An Example

In this section we apply the MLE and WLE methods to the data set on lifetime in years at the system, this data is taken from [3]. The lifetimes are
30.20, 36.55, 25.11, 39.35, 27.57, 25.91, 31.50, 29.24, 18.39, 16.65, 21.85, 24.88, 31.61, 18.74, 19.63, 28.98, 11.10, 21.66, 26.04, 25.07, 23.48, 28.21, 25.21, 25.12, 27.76, 23.47, 23.51, 24.39, 21.93, 37.63, 20.32, 28.17, 24.66, 30.13, 21.42, 17.21, 19.98, 33.09, 16.04, 17.96, 19.57, 22.91, 25.69, 23.47, 16.91, 27.20, 27.23.

We use the MLE method to fit the this data set. We fit the data set for the Weibull distribution by using the function fitdist in the *R* package. The MLE of scale and shape parameter are 27.007 and 4.579 with the standard error 0.911 and 0.493, respectively. Then we create 7% of the contamination in the data set with the Weibull distribution with shape parameter $27.007(1 + 1)$ and shape parameter $4.579(1 + 1)$. We change the first three of the observations 30.20, 36.55, 25.11, as 50.76, 61.53, 58.21, respectively. Then we apply the MLE and WLE methods to the new data set, assuming that the shape parameter is 4.579. The MLE of the scale parameter is 34.042 with the standard error 1.997, but the WLE of the scale parameter is 26.159

with the standard error 0.869. The result of WLE method are closed to the MLE obtained from the data set without any outlier. Therefore, the WLE method outperforms the MLE when the contamination is present in this data set.

6. Conclusion

The Weibull distribution plays a central role in lifetime models in medical and biological sciences as well as in engineering. When data are contaminated with outliers, the maximum likelihood estimator can be very unreliable. In this study the weighted likelihood estimator is applied to the Weibull distribution for the more robust estimation of the scale parameter, assuming that the shape parameter is known, when the data set shows the Weibull contamination that is the presence of outliers. This method assigns zero weights to observations with small likelihood. We also examine the robust properties of the MLE and WLE for the parameters of the Weibull distribution when the data set shows a contamination. To examine the performance of the WLE method in comparison with the MLE methods, we found that the WLE method outperforms the MLE method based on the relative bias and quadratic risk values. And in the most of the cases, the relative bias and quadratic risk of the WLE method decrease as sample increase. This is expected because most estimator in statistical theory perform better when sample size increases. The gain in terms of the relative bias and the quadratic risk of the WLE methods decreases as α increase.

As future work we may mention that the WLE method can be extended to some further modifications of the Weibull distribution. A robust estimator obtained should be compared to the other estimators of the Weibull parameter such as the estimations from the moment method, the least squares method, the method of percentiles and the Bayesian estimation method. In addition, in this study we were considered only an estimator of the scale parameter of the Weibull distribution and assume that the shape parameter is known. The WLE method can be extended to estimate the two-parameter Weibull distribution when we assume both of the scale and shape parameter are unknown. Next, the WLE method can be extended to estimate the three-parameter Weibull distribution.

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Appendix

A.1 Proof of Result 1

Proof. By definition the relative bias is

$$\frac{bias(\delta)}{\delta}=\frac{E[\widehat{\delta}]-\delta}{\delta}=\frac{\delta\big((1-\varepsilon)\Gamma\big(1+\frac{1}{\beta}\big)+\varepsilon(1+\Delta_1)\Gamma\big(1+\frac{1}{\beta_1}\big)\big)-\delta}{\delta}$$
$$=(1-\varepsilon)\Gamma\Big(1+\frac{1}{\beta}\Big)+\varepsilon(1+\Delta_1)\Gamma\Big(1+\frac{1}{\beta_1}\Big)-1$$

A2. Proof of Result 2

Proof. For the asymptotic bias of estimator $\widetilde{\delta}$, we have

$$\begin{aligned}E[\widetilde{\delta}]&\sim E[Y]\\&=\int_0^A xf_A(x;,\delta,\beta)dx=\frac{1}{G_\alpha}\int_0^A x\Big((1-\varepsilon)\frac{\beta}{\delta^\beta}x^{\beta-1}e^{-(\frac{x}{\delta})^\beta}+\varepsilon\frac{\beta_1}{\delta_1^{\beta_1}}x^{\beta_1-1}e^{-\left(\frac{x}{\delta_1}\right)^{\beta_1}}\Big)dx\\&=\frac{\delta}{G_\alpha}\big((1-\varepsilon)IG_1+\varepsilon(1+\Delta_1)IG_2\big),\end{aligned}$$

where $IG_1=\Gamma\big(1/\beta+1\big)P\big(1/\beta+1,A_u\big)$ and $IG_2=\Gamma\big(1/\beta_1+1\big)P\big(1/\beta_1+1,A_v\big)$. So we have relative bias as

$$\frac{bias(\delta)}{\delta}=\frac{E[\widetilde{\delta}]-\delta}{\delta}=\frac{1}{G_\alpha}\Big((1-\varepsilon)IG_1+\varepsilon(1+\Delta_1)IG_2\Big)-1$$

A3. Proof of Result 3

Proof. By the definition of the quadratic risk, we have

$$\begin{aligned}R(\widehat{\delta})&=\frac{E[(X-\hat{\delta})^2]}{n}=\frac{\delta^2}{n}\left(E\Big[\Big(\frac{X}{\delta}-\frac{E[X]}{\delta}\Big)^2\Big]+\Big(\frac{\delta-\delta_\varepsilon}{\delta}\Big)^2\right)\\&=\frac{\delta^2}{n}\left(Var\Big(\frac{X}{\delta}\Big)+\Big(\frac{\delta-\delta_\varepsilon}{\delta}\Big)^2\right)=\frac{\delta^2}{n}\left(\frac{1}{\delta^2}\big(E[X^2]-E[X])^2\big)+\Big(\frac{\delta-\delta_\varepsilon}{\delta}\Big)^2\right).\end{aligned}$$

From the obstructing model

$$G_\varepsilon(x)=(1-\varepsilon)\Big(1-e^{-(\frac{x}{\delta})^\beta}\Big)+\varepsilon\Big(1-e^{-\left(\frac{x}{\delta_1}\right)^{\beta_1}}\Big).$$

We get

$$\begin{aligned}E[X]&=\int_0^\infty x\Big((1-\varepsilon)\frac{\beta}{\delta^\beta}x^{\beta-1}e^{-(\frac{x}{\delta})^\beta}+\varepsilon\frac{\beta_1}{\delta_1^{\beta_1}}x^{\beta_1-1}e^{-\left(\frac{x}{\delta_1}\right)^{\beta_1}}\Big)dx\\&=(1-\varepsilon)\int_0^\infty x\frac{\beta}{\delta^\beta}x^{\beta-1}e^{-(\frac{x}{\delta})^\beta}dx+\varepsilon\int_0^\infty x\frac{\beta_1}{\delta_1^{\beta_1}}x^{\beta_1-1}e^{-\left(\frac{x}{\delta_1}\right)^{\beta_1}}dx\end{aligned}$$

$$
\begin{aligned}
&= (1-\varepsilon)\delta\Gamma\Big(1+\frac{1}{\beta}\Big) + \varepsilon\delta_1\Gamma\Big(1+\frac{1}{\beta_1}\Big)\\
&= \delta\left((1-\varepsilon)\Gamma\Big(1+\frac{1}{\beta}\Big) + \varepsilon(1+\Delta_1)\Gamma\Big(1+\frac{1}{\beta(1+\Delta_2)}\Big)\right) = \delta E_1,\\
E[X^2] &= \int_0^\infty x^2\Big((1-\varepsilon)\frac{\beta}{\delta^\beta}x^{\beta-1}e^{-(\frac{x}{\delta})^\beta} + \varepsilon\frac{\beta_1}{\delta_1^{\beta_1}}x^{\beta_1-1}e^{-\left(\frac{x}{\delta_1}\right)^{\beta_1}}\Big)dx\\
&= (1-\varepsilon)\delta^2\Gamma\Big(1+\frac{2}{\beta}\Big) + \varepsilon\delta_1^2\Gamma\Big(1+\frac{2}{\beta_1}\Big) = \delta^2E_2,
\end{aligned}
$$

where

$$
\begin{aligned}
E_1 &= (1-\varepsilon)\Gamma\Big(1+\frac{1}{\beta}\Big) + \varepsilon(1+\Delta_1)\Gamma\Big(1+\frac{1}{\beta(1+\Delta_2)}\Big),\\
E_2 &= (1-\varepsilon)\Gamma\Big(1+\frac{2}{\beta}\Big) + \varepsilon(1+\Delta_1)^2\Gamma\Big(1+\frac{2}{\beta(1+\Delta_2)}\Big).
\end{aligned}
$$

Therefore the quadratic risk is

$$
R(\widehat{\delta}) = \frac{\delta^2}{n}\Big(\frac{1}{\delta^2}\big(E[X^2]-(E[x])^2\big) + \Big(\frac{\delta-\delta_\varepsilon}{\delta}\Big)^2\Big) = \frac{\delta^2}{n}\big(E_2 - E_1^2 + (1-E_1)^2\big).
$$

A.4 Proof of Result 4

Proof. Since, the asymptotic distribution of $\widetilde{\delta}$ equals the distribution of the $\alpha-$ generalized trimmed sample mean where we defined

$$
Y = \Big(\frac{1}{n(1-\alpha)}\sum_{k=1}^{n(1-\alpha)} Y_k^\beta\Big)^{\frac{1}{\beta}}.
$$

The asymptotic $\widetilde{\delta}$ has the quadratic risk is

$$
\begin{aligned}
R(\widetilde{\delta}) &= \frac{E_\alpha[(Y-\delta)^2]}{n(1-\alpha)} = \frac{1}{n(1-\alpha)}E_\alpha\big[\big((Y-\mu)-(\delta-\mu)\big)^2\big]\\
&= \frac{1}{n(1-\alpha)}\Big(E_\alpha[(Y-\mu)^2] + E_\alpha[(\delta-\mu)^2] - 2(\delta-\mu)E_\alpha[(Y-\mu)]\Big)\\
&= \frac{1}{n(1-\alpha)}\Big(E_\alpha[(Y-\mu)^2] + E_\alpha[(\delta-\mu)^2]\Big)\\
&= \frac{1}{n(1-\alpha)}\Big(Var_\alpha(Y) + E_\alpha[(\delta-\mu)^2]\Big).
\end{aligned}
$$

The asymptotic distribution of $\widetilde{\delta}$ equals the distribution of the α-generalized trimmed sample mean of the random sample of size $n(1-\alpha)$ from the distribution concentrated on the interval $(0, A)$. The probability density of this distribution is positive only the interval and has the form

$$
f_A(x;\delta,\beta) = \frac{(1-\varepsilon)f(x;\delta,\beta) + \varepsilon f(x;\delta_1,\beta_1)}{G_\alpha}.
$$

Then

$$\mu = E_\alpha[Y] = \int_0^A x f_A(x;,\delta,\beta)dx$$
$$= \frac{1}{G_\alpha}\int_0^A x\Big((1-\varepsilon)\frac{\beta}{\delta^\beta}x^{\beta-1}e^{-(\frac{x}{\delta})^\beta} + \varepsilon\frac{\beta_1}{\delta_1^{\beta_1}}x^{\beta_1-1}e^{-\left(\frac{x}{\delta_1}\right)^{\beta_1}}\Big)dx$$
$$= \frac{\delta}{G_\alpha}\big((1-\varepsilon)IG_1 + \varepsilon(1+\Delta_1)IG_2\big),$$
$$E_\alpha[Y^2] = \int_0^A x^2 f_A(x;,\delta,\beta)dx$$
$$= \frac{1}{G_\alpha}\int_0^A x^2\Big((1-\varepsilon)\frac{\beta}{\delta^\beta}x^{\beta-1}e^{-(\frac{x}{\delta})^\beta} + \varepsilon\frac{\beta_1}{\delta_1^{\beta_1}}x^{\beta_1-1}e^{-\left(\frac{x}{\delta_1}\right)^{\beta_1}}\Big)dx$$
$$= \frac{1}{G_\alpha}\left((1-\varepsilon)\delta^2 IG_3 + \varepsilon\delta_1^2 IG_4)\right) = \frac{1}{G_\alpha}\left((1-\varepsilon)\delta^2 IG_3 + \varepsilon\delta^2(1+\Delta_1)^2 IG_4)\right)$$
$$= \frac{\delta^2}{G_\alpha}\left((1-\varepsilon)IG_3 + \varepsilon(1+\Delta_1)^2 IG_4)\right),$$

where $IG_1 = \Gamma\big(1/\beta+1\big)P\big(1/\beta+1, A_u\big)$, $IG_2 = \Gamma\big(1/\beta_1+1\big)P\big(1/\beta_1+1, A_v\big)$, $IG_3 = \Gamma\big(2/\beta+1\big)P\big(2/\beta+1, A_u\big)$, $P\big(2/\beta+1, A_u\big) = \gamma\big(2/\beta+1, A_u\big)/\Gamma\big(2/\beta+1\big)$ is a normalized function of a lower incomplete gamma function $\gamma\big(2/\beta+1, A_u\big)$, $IG_4 = \Gamma\big(2/\beta_1 + 1\big)P\ (2/\beta_1+1, A_v)$, $P\ (2/\beta_1+1, A_v) = \gamma\big(2/\beta_1+1, A_v\big)/\Gamma\big(2/\beta_1+1\big)$ is a normalized function of a lower incomplete gamma function $\gamma\big(2/\beta_1+1, A_v\big)$. Therefore the quadratic risk is

$$R(\widetilde{\delta}) = \frac{1}{n(1-\alpha)}\left(E_\alpha[Y^2] - (E_\alpha[Y])^2 + E_\alpha[(\delta-\mu)^2]\right)$$
$$= \frac{1}{n(1-\alpha)}\left(E_\alpha[Y^2] - (E_\alpha[Y])^2 + (\delta-\mu)^2\right)$$
$$= \frac{1}{n(1-\alpha)}\left(E_\alpha[Y^2] - \mu^2 + \delta^2 - 2\delta\mu + \mu^2\right)$$
$$= \frac{E_\alpha[Y^2]}{n(1-\alpha)} + \frac{\delta^2}{n(1-\alpha)} - \frac{2\delta\mu}{n(1-\alpha)}$$
$$= \frac{\delta^2\big((1-\varepsilon)IG_3 + \varepsilon(1+\Delta_1)^2 IG_4\big)}{n(1-\alpha)G_\alpha} + \frac{\delta^2}{n(1-\alpha)} - \frac{2\delta^2\big((1-\varepsilon)IG_1 + \varepsilon(1+\Delta_1)IG_2\big)}{n(1-\alpha)G_\alpha}.$$

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