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Some Theoretical Properties and Parameter Estimation for the Two-Sided Length Biased Inverse Gaussian Distribution

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ABSTRACT The new lifetime distribution based on non-classical parametrization model called the two-sided length biased inverse Gaussian distribution is introduced. The physical phenomena of this situation can be explained in the case when a crack develops from two sides. Some statistical properties of the distribution such as reciprocal properties and the first four moments are investigated. The conventional point estimation, method of moment, is developed for estimating the parameters of the distribution together with asymptotic property of the proposed estimators. In order to evaluate the performance of the suggested estimators, Monte Carlo simulation studies are conducted. Additionally, real data sets in a practical setting are used to illustrate the presented estimation method. Concluding remarks and discussions are also provided.

Keywords Asymptotic property; Lifetime distribution; Method of moment estimate; Parametrization; Reciprocal property.

1. Introduction

Lifetime distributions are frequently studied in reliability aspects. It is easy to consider a lifetime or failure time of physical objects such as coins, electric light bulbs, some pieces of machines, etc. They provide useful information on certain practical problems. Since some machines or systems are very important and extremely expensive, this information motivates practitioners to prevent financial or industrial damages occurring after the failure time terminates. One of the interesting views of lifetime distributions in reliability analysis is in the situation when a failure of the object under consideration occurs from a fatigue crack development. The common distributions used in practical applications of this area are Birnbaum-Saunders, inverse Gaussian, and length biased inverse Gaussian [7, 13].

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These distributions had been studied in various cases. Birnbaum and Saunders [2] introduced the two-parameter Birnbaum-Saunders (BS) distribution as a lifetime distribution for fatigue failure caused by periodic loading. Ahmed *et al.* [1] proposed the new parametrization of BS distribution. Importantly, the physical situation under this study are fitted by this reparametrization since the suggested parameters correspond to the thickness of the object under study and the nominal treatment loading on the object, respectively. The original parameters of the distribution do not give these characteristics. Several studies regarding on the inverse Gaussian (IG) distribution are often referred to Chhikara and Folks [4], Seshadri [15, 16], Johnson *et al.* [10] and Tweedi [17, 18]. Recently, Lisawadi [13] presented two new distributions based on the re-parametrization model proposed in [1] and they were called the two-sided Birbaum-Saunders and inverse Gaussian lifetime distributions. These distributions are considered in the situation of a crack develops from two sides. A review of applications of length biased distributions was given in Gupta and Kirmani [9]. Akman and Gupta [2] had studied the length biased inverse Gaussian (LBIG) distribution. They provided comparative simulation studies of different estimators for the mean of data from IG and LBIG distributions. Gupta and Akman [8] offered statistical properties involving the arithmetic and harmonic means of the LBIG distribution. Essentially, as seen in the reviewed literature, all of them were considered in the term of usual parameters except the studies of [1] and [13]. In this article, we introduce a new lifetime distribution based on non-classical parameters presented by Ahmed *et al.* [1]. The new distribution is called the two-sided length biased inverse Gaussian lifetime distribution denoted as TS-LBIG distribution. Interestingly, our contribution is in the investiga- tion process. The probability model of the TS-LBIG distribution is formed by applying the approach of Lisawadi [13]. The new distribution is considered in the case when a crack develops from two sides. For example, on a metallic rectangular object which is fixed on two sides, a pressure is applied to both upper and lower sides of the object that leads to a crack development from two sides. Some statistical properties of the distribution such as reciprocal properties and the first four moments are investigated. The traditional point estimation, method of moment, is developed together with the asymptotic analysis of the proposed estimators. Monte Carlo simulations are utilized to study the efficiency of the suggested estimators, and illustrative examples for explaining the given estimation method are also provided.

The article is organized as follows. The pdf of IG and LBIG distributions based on non-classical parameters are provided in Section 2. Probability model of the TS-LBIG is introduced in Section 3. Theoretical results are given in Section 4. Numerical results are shown in Section 5. Finally, conclusions and discussion are reported in Section 6.

2. Materials and Methods

2.1 Inverse Gaussian Distribution

The usual pdf of inverse Gaussian distribution of a continuous random variable X is

$$f(x;\mu,\beta)=\sqrt{\frac{\beta}{2\pi}}x^{-3/2}\exp\left\{-\frac{\beta(x-\mu)^2}{2\mu^2 x}\right\};\quad x>0, \tag{1}$$

where $\mu > 0$ and $\beta > 0$. The parameter μ stands for the mean and β represents the scale parameter. The proposed parameters provided by Ahmed *et. al.* [1] are $\lambda > 0$ and $\theta > 0$ standing for the thickness of the object under consideration and the nominal treatment loading on the object, correspondingly. The relations between the classical and proposed parameters are:

$$\lambda = \frac{\beta}{\mu}, \quad \theta = \frac{\mu^2}{\beta}, \quad \mu = \lambda\theta, \quad \text{and} \quad \beta = \lambda^2\theta. \tag{2}$$

Hence, the pdf of IG distribution based on the new parametrization is define as

$$f\left(x;\lambda,\theta\right) = \frac{\lambda}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{3/2} \exp\left\{-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right)^2\right\}; \quad x > 0, \tag{3}$$

where $\lambda > 0$ and $\theta > 0$.

2.2 Length Biased Inverse Gaussian Distribution

We start with the definition of a length biased pdf presented by Khattree [11].

Definition 2.2.1 Let X be a non-negative random variable having an absolutely continuous pdf, $f(x)$. Assuming $\mu = E(X) < \infty$, the length biased random variable Y has a pdf defined as

$$g(\cdot) = \frac{xf(x)}{\mu}; \quad x > 0. \tag{4}$$

It is known that $\mu = \lambda\theta$, and by (3) and (4), finally, pdf of the length biased inverse Gaussian (LBIG) distribution is define by

$$f\left(x;\lambda,\theta\right) = \frac{1}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{1/2} \exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{x}{\theta}} - \sqrt{\frac{\theta}{x}}\right)^2\right\}; \quad x > 0, \tag{5}$$

where $\lambda > 0$ and $\theta > 0$.

3. Probability Model

The physical phenomena of this situation can be explained in the case when a crack develops from two sides of the object under consideration. Let $F(t)$, $t > 0$, be the distribution function of the moment of the object breakdown τ for one-sided loading. We consider $F(t) = F_{LBIG}\left(t;\lambda,\theta\right)$. Let $Y = k/\tau$ be the random variable interpreted as a speed of the crack evolution. At the bottom side of the metallic block, a crack is developing with the distribution function of the time to reach the length k. Simultaneously, at the top side of the block, a crack is developing with the same distribution function as the bottom side. Then, we have two random variables τ_1 and τ_2, and they are assumed to be independent and identically distributed. The speed of the crack development for this two-sided case is $Y_1 + Y_2 = k\tau_1^{-1} + k\tau_2^{-1}$, and the random variable $\tau = k/(Y_1 + Y_2) = (\tau_1^{-1} + \tau_2^{-1})^{-1}$ corresponds to a moment of the object under consideration break down. The distribution function and pdf of the two-sided length biased inverse Gaussian distribution are presented in the following theorems.

Theorem 3.1 A random variable τ has a two-sided length biased inverse Gaussian distribution denoted as TS-LBIG(λ,θ), if it has a distribution function in the form:

$$F_\tau(u)=1-\int_0^{u^{-1}} f\left(\frac{1}{t}\right)\frac{dt}{t^2}\int_0^{u^{-1}-t} f\left(\frac{1}{s}\right)\frac{ds}{s^2},$$

and a density function in the following form:

$$f_\tau(u)=u^{-2}\int_0^{u^{-1}} f\left(\frac{1}{t}\right)f\left(\frac{1}{u^{-1}-t}\right)\frac{dt}{t^2(u^{-1}-t)^2}.$$

Proof. Let τ be a random variable defined above, $T=1/\tau_1$ and $S=1/\tau_2$. Then

$$F_T(t)=P(T\le t)=P(\tau_1>1/t)=1-F_{\tau_1}(1/t) \quad \text{and} \quad f_T(t)=F_T{}'(t)=f(1/t)/t^2.$$

Similarly,

$$F_S(s)=1-F_{\tau_2}(1/s) \quad \text{and} \quad f_s(s)=f(1/s)/s^2.$$

Thus,

$$F_\tau(u)=P\left(T+S>u^{-1}\right)=\iint_{t+s>u^{-1}} f_T(t)f_S(s)dtds=1-\iint_{t+s\le u^{-1}} f\left(\frac{1}{t}\right)f\left(\frac{1}{s}\right)\frac{dtds}{t^2s^2}$$
$$=1-\int_0^{u^{-1}} f\left(\frac{1}{t}\right)\frac{dt}{t^2}\int_0^{u^{-1}-t} f\left(\frac{1}{s}\right)\frac{ds}{s^2}.$$

The density function is obtained by differentiating the $F_\tau(u)$. Therefore,

$$f_\tau(u)=u^{-2}\int_0^{u^{-1}} f\left(\frac{1}{t}\right)f\left(\frac{1}{u^{-1}-t}\right)\frac{dt}{t^2(u^{-1}-t)^2}.$$

Finally, we achieve the required expressions which complete the proof. □

However, the pdf of τ has no explicit form. This may be difficult to find main functions such as a characteristic function and a moment generating function. Figures 1-2 show variety of

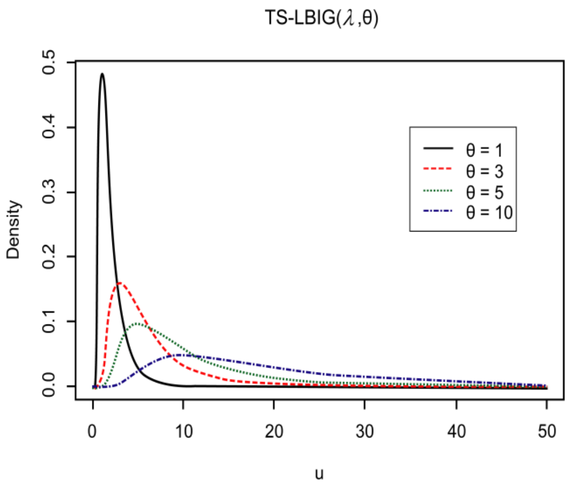

Figure 1 The two-sided length biased inverse Gaussian density functions for $\lambda=2$.

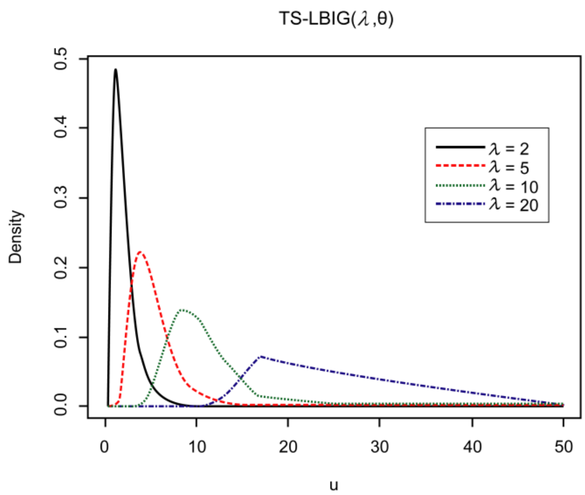

Figure 2 The two-sided length biased inverse Gaussian density functions for $\theta=1$.

the probability density functions for TS-LBIG. It is indicated that the TS-LBIG is the positively skewed distribution which is a useful choice in this framework.

4. Theoretical Results

4.1 Reciprocal Properties

Proposition 4.1.1 If random variable $\tau > 0$ has the density function $f_\tau(x)$, then the reciprocal random variable $1/\tau$ has the pdf $f_{1/\tau}(x) = x^{-2} f_\tau(1/x)$.

Proof. For the reciprocal random variable $1/\tau$, applying a definition of a cumulative distribution function, then its distribution function is defined by

$$F_{1/\tau}(x) = P\left(\frac{1}{\tau} \le x\right) = P\left(\tau \ge \frac{1}{x}\right) = 1 - F_\tau\left(\frac{1}{x}\right),$$

and applying the chain rule, the density function is $f_{1/\tau}(x) = F'_{1/\tau}(x) = x^{-2} f_\tau(1/x)$. □

Proposition 4.1.2 If random variable $\tau > 0$ has LBIG(λ, θ) distribution, then the reciprocal random variable $1/\tau$ is IG$[\lambda, 1/(\lambda^2\theta)]$ distributed.

Proof. By Proposition 4.1.1,

$$f_{1/\tau}(x) = x^{-2} f_{LBIG}\left(\frac{1}{x}; \lambda, \theta\right) = \frac{x^{-3/2}\theta^{-1/2}}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\lambda(\theta x)^{1/2} - (\theta x)^{-1/2}\right)^2\right\}$$

$$= \frac{\lambda(1/\lambda^2\theta)^{1/2} x^{-3/2}}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{1/(\lambda^2\theta)}{x}} - \sqrt{\frac{x}{1/(\lambda^2\theta)}}\right)^2\right\} = IG[x; \lambda, 1/(\lambda^2\theta)].$$ □

4.2 The First Four Moments

Finding the first four moments of the TS-LBIG distribution, we deal with the following way. The reciprocal property is necessarily needed. The characteristic function of the IG distribution is required. Because of the difficulty of direct derivation, Maclaurin expansion is applied. The first four terms are considered to obtain the first four cumulants. Then, they are modified to gain the first four moments. The required theorems are presented as follows.

Theorem 4.2.1 If a continuous random variable X is inverse Gaussian distributed with parameters λ and θ denoted as IG(λ, θ), then its characteristic function is

$$\varphi_{IG}(x; \lambda, \theta) = \exp\{\lambda[1 - (1 - 2i\theta t)^{-1/2}]\}.$$

Proof. By definition, the characteristic function of the IG(λ, θ) is defined as

$$\varphi_{IG}(x; \lambda, \theta) = \int_{-\infty}^{\infty} \exp(itx) f_{IG}(x; \lambda, \theta) dx$$

$$= \lambda\sqrt{\frac{\theta}{2\pi}} \int_0^\infty \exp(itx) x^{-3/2} \exp\left\{-\frac{\lambda^2\theta}{2x} + \lambda - \frac{x}{2\theta}\right\} dx$$

$$= \lambda e^{\lambda} \sqrt{\frac{\theta}{2\pi}} \int_0^{\infty} x^{-3/2} \exp\left\{ -\left(\frac{1-2i\theta t}{2\theta} \right) x - \frac{\lambda^2 \theta}{2x} \right\} dx.$$

By applying the formula 3.472.5 from Gradsthteyn and Ryzhik [6], (p. 369), we have

$$p = \frac{1-2i\theta t}{2\theta} \quad \text{and} \quad q = \frac{\lambda^2 \theta}{2},$$

then we get

$$\varphi_{IG}(x;\lambda,\theta) = \lambda e^{\lambda} \sqrt{\frac{\theta}{2\pi}} \left[\sqrt{\frac{2\pi}{\lambda^2 \theta}} \exp\left\{ -2\sqrt{\left(\frac{1-2i\theta t}{2\theta} \right) \cdot \left(\frac{\lambda^2 \theta}{2} \right)} \right\} \right]$$

$$= e^{\lambda} \exp\left\{ -\lambda \sqrt{1-2i\theta t} \right\} = \exp\left\{ \lambda [1-(1-2i\theta t)^{-1/2}] \right\}. \qquad \square$$

Theorem 4.2.2 If a random variable X is two-sided length biased inverse Gaussian distributed with parameters λ and θ denoted as TS-LBIG(λ,θ), then the first four moments are

$$\mu(X) = k_1(X) = \frac{2}{\lambda\theta},\ \sigma^2(X) = k_2(X) = \frac{2}{\lambda^3\theta^2},\ \mu_3(X) = k_3(X) = \frac{6}{\lambda^5\theta^3},$$

and

$$\mu_4(X) = k_4(X) + 3\sigma^4(X) = \frac{30+12\lambda}{\lambda^7\theta^4}.$$

Proof. If $\varphi(t)$ is a characteristic function, then its cumulants $k_j; j = 1,2,\cdots,m$, are defined from the Maclaurin expansion of the logarithm of the characteristic function:

$$\ln \varphi(t) = \sum_{j=1}^{m} \frac{k_j}{j!}(it)^j + o(|t|^m).$$

Hence, the logarithm of the characteristic function of IG(λ,θ) distribution is

$$\ln \varphi(t) = \lambda - \lambda(1-2i\theta t)^{1/2}. \tag{6}$$

For the Maclaurin expansion, we use

$$(1-x)^{1/2} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - O(x^5), \tag{7}$$

where $|x| < 1$. In this case, $x = 2i\theta t$ and we assume that $0 < t < 1/(2\theta)$. Applying equations (6) and (7), we obtain

$$\ln \varphi(t) = \lambda - \lambda\left(1 - \frac{2(i\theta t)}{2} - \frac{4(i\theta t)^2}{8} - \frac{8(i\theta t)^3}{16} - \frac{5\cdot 16(i\theta t)^4}{128} - O(t^5) \right)$$

$$= \lambda(i\theta t) + \frac{\lambda}{2}(i\theta t)^2 + \frac{\lambda}{2}(i\theta t)^3 + \frac{5\lambda}{8}(i\theta t)^4 + O(t^5)$$

$$= \frac{(it)}{1!}\lambda\theta + \frac{(it)^2}{2!}\lambda\theta^2 + \frac{(it)^3}{3!}3\lambda\theta^3 + \frac{(it)^4}{4!}15\lambda\theta^4 + O(t^5)$$

This expression gives us the cumulants of the IG distribution:

$$k_1 = \mu = \lambda\theta,\ k_2 = \sigma^2 = \lambda\theta^2,\ k_3 = \mu_3 = 3\lambda\theta^3,\ \text{and}\ k_4 = \mu_4 - 3\sigma^4 = 15\lambda\theta^4.$$

By proposition 4.1.2, if $\tau \sim \text{LBIG}(\lambda, \theta)$, then the reciprocal $1/\tau \sim \text{IG}[\lambda, 1/(\lambda^2\theta)]$. Substituting θ by $1/(\lambda^2\theta)$, $1/\tau$ has the following cumulants:

$$k_1(1/\tau) = \frac{1}{\lambda\theta},\ k_2(1/\tau) = \frac{1}{\lambda^3\theta^2},\ k_3(1/\tau) = \frac{3}{\lambda^5\theta^3},\ \text{and}\ k_4(1/\tau) = \frac{15}{\lambda^7\theta^4}.$$

Because of the i.i.d. property, the moments for the random variable $X = \tau_1^{-1} + \tau_2^{-1}$ will be obtained by combining the cumulants for $1/\tau$. Finally, the first four moments for the two-sided length biased inverse Gaussian distribution are:

$$\mu(X) = k_1(X) = \frac{2}{\lambda\theta},\ \sigma^2(X) = k_2(X) = \frac{2}{\lambda^3\theta^2},\ \mu_3(X) = k_3(X) = \frac{6}{\lambda^5\theta^3},$$

and

$$\mu_4(X) = k_4(X) + 3\sigma^4(X) = \frac{30 + 12\lambda}{\lambda^7\theta^4}. \qquad \square$$

4.3 Method of Moment Estimation

Estimation of the parameters λ and θ by the method of moment for the TS-LBIG distribution can be derived in the following way. In the previous section, the first four moments are investigated, and thus we obtain formulas for the expectation and variance;

$$E(X) = \mu(X) = \frac{2}{\lambda\theta} \quad \text{and} \quad Var(X) = \sigma^2(X) = \frac{2}{\lambda^3\theta^2}.$$

Hence, the method of moment estimation is

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{2}{\lambda\theta} \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{2}{\lambda^3\theta^2}. \qquad (8)\ \&\ (9)$$

Solving equations (8) and (9) for λ and θ, we obtain the desired estimates. Let $T = S^2/\bar{X}^2$. Then,

$$T = \frac{2}{\lambda^3\theta^2} \cdot \frac{\lambda^2\theta^2}{4} = \frac{1}{2\lambda}.$$

Therefore, the method of moment estimators are

$$\hat{\lambda} = \frac{1}{2T} \quad \text{and} \quad \hat{\theta} = \frac{2}{\hat{\lambda}\bar{X}}.$$

The asymptotic variances and covariance are presented in the theorem below.

<u>Theorem 4.3.1</u> As $n \to \infty$, $Var(\hat{\lambda})$, $Var(\hat{\theta})$, and $Cov(\hat{\lambda}, \hat{\theta})$ are given by

$$Var(\hat{\lambda}) = \frac{7\lambda + 6\lambda^2 - \lambda^5\theta^2}{2n} + O\left(\frac{1}{n^2}\right),$$

$$Var(\hat{\theta}) = \frac{6\theta^2 + 6\lambda\theta^2 - \lambda^4\theta^4}{2\lambda n} + O\left(\frac{1}{n^2}\right), \text{ and}$$

$$Cov(\hat{\lambda}, \hat{\theta}) = \frac{9\theta - 6\lambda\theta + \lambda^4\theta^3}{2n} + O\left(\frac{1}{n^2}\right).$$

Proof. The asymptotic distribution of an estimate which smoothly depends on sample moments is commonly obtained by their decomposition into the Taylor series expansion. For our case, $\lambda = \hat{\lambda}(\bar{X}, S^2)$, $\theta = \hat{\theta}(\bar{X}, S^2)$, $\mu(X)$ is the true value of $\bar{X}$, and $\sigma^2(X)$ is that of S^2. Let a_1 and a_2 be the values of partial derivatives of $\hat{\lambda}$ by $\bar{X}$ and S^2 at the point $(\mu(X), \sigma^2(X))$, correspondingly. The b_1 and b_2 are denoted as the values of analogous derivatives of $\hat{\theta}$. Therefore, the Taylor expansion can be written as

$$\hat{\lambda} = \lambda + a_1\left(\bar{X} - \mu(X)\right) + a_2\left(S^2 - \sigma^2(X)\right) + O_p(1/n),$$

$$\hat{\theta} = \theta + b_1\left(\bar{X} - \mu(X)\right) + b_2\left(S^2 - \sigma^2(X)\right) + O_p(1/n).$$

The $X_n = O_p(1/n)$, where $\{X_n, n \geq 1\}$ is a sequence of random variables, means that there exists a constant $C > 0$ such that

$$\lim_{x\to\infty} \sup_{n\geq 1} P\left|X_n - (C/n)\right| > t = 0.$$

These expressions are understood in terms of convergence in probability. Thus, the vector $(\sqrt{n}(\hat{\lambda} - \lambda), \sqrt{n}(\hat{\theta} - \theta))$ has limit in distribution, the two-dimensional normal distribution with zero means, variances

$$Var(\hat{\lambda}) = a_1^2 Var(\bar{X}) + a_2^2 Var(S^2) + 2a_1 a_2 Cov(\bar{X}, S^2), \tag{10}$$

$$Var\left(\hat{\theta}\right) = b_1^2 Var(\bar{X}) + b_2^2 Var(S^2) + 2b_1 b_2 Cov(\bar{X}, S^2), \text{ and} \tag{11}$$

$$Cov(\hat{\lambda}, \hat{\theta}) = a_1 b_1 Var(\bar{X}) + a_2 b_2 Var(S^2) + 2\left(a_1 b_2 + a_2 b_1\right) Cov(\bar{X}, S^2). \tag{12}$$

As presented results in section 4.2, we apply the formulas outlined in Cramér [5], (p. 352-358) and we obtain:

$$Var(\bar{X}) = \frac{2}{\lambda^3\theta^2 n},\quad Var(S^2) = \frac{30 + 12\lambda - 2\lambda^4\theta^2}{\lambda^7\theta^4 n} + O\left(\frac{1}{n^2}\right),$$

and (13)

$$Cov(\bar{X}, S^2) = \frac{6}{\lambda^5\theta^3 n} + O\left(\frac{1}{n^2}\right).$$

We firstly consider the derivatives of the estimate $\hat{\lambda}$ and we have

$$\frac{\partial\hat{\lambda}}{\partial\bar{X}} = \frac{d\hat{\lambda}}{dT}\cdot\frac{\partial T}{\partial\bar{X}} = -\frac{1}{2T^2}\cdot\frac{\partial T}{\partial\bar{X}}.$$

Exchanging $\bar{X}$ and S^2 on μ and σ^2, we get

$$a_1 = \frac{\partial\hat{\lambda}}{\partial\bar{X}} = \left(\frac{4\lambda^2}{2}\right)\cdot\left(\frac{\theta}{2}\right) = \lambda^2\theta.$$

Similarly, based on the appropriate substitution, we obtain

$$a_2 = \frac{\partial\hat{\lambda}}{\partial S^2} = \left(-\frac{4\lambda^2}{2}\right)\cdot\left(\frac{\lambda^2\theta^2}{4}\right) = -\frac{\lambda^2\theta^2}{2}.$$

We secondly provide the analogous derivations of the derivatives of the $\hat{\theta}$ and we have

$$\frac{\partial\hat{\theta}}{\partial\overline{X}}=\frac{d\hat{\theta}}{d\hat{\lambda}}\cdot\frac{\partial\hat{\lambda}}{\partial\overline{X}}=\left(d\left(\frac{2}{\hat{\lambda}}\right)\Big/d\hat{\lambda}\right)\cdot\frac{\partial\hat{\lambda}}{\partial\overline{X}}\cdot\overline{X}^{-1}+\left(\frac{2}{\hat{\lambda}}\right)(-1)\overline{X}^{-2}=-\left(\frac{2}{\hat{\lambda}^2\overline{X}}\cdot\frac{\partial\hat{\lambda}}{\partial\overline{X}}+\frac{2}{\hat{\lambda}\overline{X}^2}\right).$$

Note that $\frac{\partial\hat{\lambda}}{\partial\overline{X}}(\mu,\sigma^2)=a_1$, and hence

$$b_1=\frac{\partial\hat{\theta}}{\partial\overline{X}}(\mu,\sigma^2)=-\frac{3}{2}\lambda\theta^2.$$

Equivalently,

$$b_2=\frac{\partial\hat{\theta}}{\partial S^2}(\mu,\sigma^2)=\frac{\lambda^3\theta^3}{2}.$$

Then, we substitute a_1,a_2,b_1,b_2 and (13) into equations (10)-(12) to obtain $Var(\hat{\lambda})$, $Var(\hat{\theta})$, and $Cov(\hat{\lambda},\hat{\theta})$. Hence,

$$\begin{aligned}Var(\hat{\lambda})&=(\lambda^2\theta)^2\left(\frac{2}{\lambda^3\theta^2n}\right)+\left(-\frac{\lambda^2\theta^2}{2}\right)^2\left(\frac{30+12\lambda-2\lambda^4\theta^2}{\lambda^7\theta^4n}\right)\\&\quad+2(\lambda^2\theta)\left(-\frac{\lambda^2\theta^2}{2}\right)\left(\frac{6}{\lambda^5\theta^3n}\right)+O\left(\frac{1}{n^2}\right)\\&=\frac{2\lambda}{n}+\frac{15\lambda+6\lambda^2-\lambda^5\theta^2}{2n}-\frac{6\lambda}{n}+O\left(\frac{1}{n^2}\right)=\frac{7\lambda+6\lambda^2-\lambda^5\theta^2}{2n}+O\left(\frac{1}{n^2}\right).\end{aligned}$$

Thus,

$$\begin{aligned}Var(\hat{\theta})&=\left(-\frac{3\lambda\theta^2}{2}\right)^2\left(\frac{2}{\lambda^3\theta^2n}\right)+\left(\frac{\lambda^3\theta^3}{2}\right)^2\left(\frac{30+12\lambda-2\lambda^4\theta^2}{\lambda^7\theta^4n}\right)\\&\quad+2\left(-\frac{3\lambda\theta^2}{2}\right)\left(\frac{\lambda^3\theta^3}{2}\right)\left(\frac{6}{\lambda^5\theta^3n}\right)+O\left(\frac{1}{n^2}\right)\\&=\frac{9\theta^2}{2\lambda n}+\frac{15\theta^2+6\lambda\theta^2-\lambda^4\theta^4}{2\lambda n}-\frac{9\theta^2}{\lambda n}+O\left(\frac{1}{n^2}\right)=\frac{6\theta^2+6\lambda\theta^2-\lambda^4\theta^4}{2\lambda n}+O\left(\frac{1}{n^2}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}Cov(\hat{\lambda},\hat{\theta})&=(\lambda^2\theta)\left(-\frac{3\lambda\theta^2}{2}\right)\left(\frac{2}{\lambda^3\theta^2n}\right)+\left(-\frac{\lambda^2\theta^2}{2}\right)\left(\frac{\lambda^3\theta^3}{2}\right)\left(\frac{30+12\lambda-2\lambda^4\theta^2}{\lambda^7\theta^4n}\right)\\&\quad+2\left\{(\lambda^2\theta)\left(\frac{\lambda^3\theta^3}{2}\right)+\left(-\frac{\lambda^2\theta^2}{2}\right)\left(-\frac{3\lambda\theta^2}{2}\right)\right\}\left(\frac{6}{\lambda^5\theta^3n}\right)+O\left(\frac{1}{n^2}\right)\\&=-\frac{3\theta}{n}-\frac{15\theta+6\lambda\theta-\lambda^4\theta^3}{2n}+\frac{15\theta}{n}+O\left(\frac{1}{n^2}\right)=\frac{9\theta-6\lambda\theta+\lambda^4\theta^3}{2n}+O\left(\frac{1}{n^2}\right).\end{aligned}$$

Finally, the proof is complete. □

5. Numerical Results

5.1 Simulation Study

We studied the properties of the presented estimators by using the numerical method. The results were reported to investigate the behavior of the estimators via calculating the estimated bias and mean square error. Monte Carlo simulations were performed for different sample sizes. The R program version 3.1.2 was used to generate and analyze the data. The number of iterations was fixed at 5,000 for each combination of λ, θ and sample sizes *n*. We considered all combinations of the following values of λ,θ, and *n* as follows: λ = 2, 5, 10 and 50, θ = 1, 5, 10 and 50, *n* = 10, 50 and 100. Tables 1-3 show the estimated bias and MSE of $\hat{\lambda}$ and $\hat{\theta}$ for *n* = 10, 50 and 100, respectively.

Table 1 The estimated bias and mean square error of $\hat{\lambda}$ and $\hat{\theta}$ for *n* = 10

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda-\hat{\lambda}$	$\theta-\hat{\theta}$	$MSE(\hat{\lambda})$	$MSE(\hat{\theta})$
2	1	2.8783	0.9308	−0.8783	0.0692	3.9637	0.2492
	5	2.8448	4.7403	−0.8448	0.2597	4.1692	6.6210
	10	2.8604	9.3805	−0.8604	0.6195	4.0240	25.5914
	50	2.9545	45.9168	−0.9545	4.0832	5.1910	634.6735
5	1	6.8814	0.9548	−1.8814	0.0452	22.3718	0.2329
	5	6.7424	4.9214	−1.7424	0.0786	24.4203	6.6715
	10	6.6686	9.8587	−1.6686	0.1413	20.4223	24.9115
	50	6.7746	48.3445	−1.7746	1.6555	20.3529	608.9861
10	1	13.1557	0.9892	−3.1557	0.0108	72.8913	0.2430
	5	13.2763	4.8915	−3.2763	0.1085	82.1559	5.6023
	10	13.2732	9.8483	−3.2732	0.1517	89.7769	22.9481
	50	13.0669	49.5379	−3.0669	0.4621	70.9626	585.4694
50	1	64.0534	1.0045	−14.0534	−0.0045	1962.9841	0.2195
	5	64.3390	4.9736	−14.3390	0.0264	1748.4027	5.4419
	10	64.9925	9.9752	−14.9925	0.0248	1980.6301	22.6549
	50	64.9923	49.6432	−14.9923	0.3568	1937.7596	556.1875

Table 2 The estimated bias and mean square error of $\hat{\lambda}$ and $\hat{\theta}$ for *n* = 50

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda-\hat{\lambda}$	$\theta-\hat{\theta}$	$MSE(\hat{\lambda})$	$MSE(\hat{\theta})$
2	1	2.1807	0.9806	−0.1807	0.0194	0.3447	0.0630
	5	2.1668	4.9312	−0.1668	0.0688	0.3365	1.5770
	10	2.1602	9.9191	−0.1602	0.0809	0.3432	6.8895
	50	2.1794	48.9890	−0.1794	1.0110	0.3513	158.2939
5	1	5.3235	0.9891	−0.3235	0.0109	1.6022	0.0490
	5	5.2835	4.9861	−0.2835	0.0139	1.5832	1.2768
	10	5.3022	9.9285	−0.3022	0.0715	1.6239	5.0893
	50	5.3330	49.4576	−0.3330	0.5424	1.6896	126.1433
10	1	10.5357	0.9946	−0.5357	0.0054	5.7267	0.0454
	5	10.5397	4.9685	−0.5397	0.0315	5.6970	1.1347
	10	10.5112	9.9506	−0.5112	0.0494	5.4637	4.4754
	50	10.4860	49.9723	−0.4860	0.0277	5.6338	114.6892
50	1	52.2225	0.9995	−2.2225	0.0005	128.4855	0.0420
	5	52.2654	4.9916	−2.2654	0.0084	130.2534	1.0312
	10	51.9926	10.0367	−1.9926	-0.0367	127.8785	4.2046
	50	52.3120	49.8948	−2.3120	0.1052	130.7992	104.0902

Table 3 The estimated bias and mean square error of $\hat{\lambda}$ and $\hat{\theta}$ for $n = 100$

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$\lambda-\hat{\lambda}$	$\theta-\hat{\theta}$	$MSE(\hat{\lambda})$	$MSE(\hat{\theta})$
2	1	2.0878	0.9919	−0.0878	0.0081	0.1596	0.0341
	5	2.0869	4.9573	−0.0869	0.0427	0.1534	0.8364
	10	2.0744	9.9796	−0.0744	0.0204	0.1554	3.4912
	50	2.0874	49.5103	−0.0874	0.4897	0.1554	80.8199
5	1	5.1549	0.9951	−0.1549	0.0049	0.7316	0.0253
	5	5.1407	4.9881	−0.1407	0.0119	0.7177	0.6355
	10	5.1734	9.9265	−0.1734	0.0735	0.7493	2.5712
	50	5.1715	49.6719	−0.1715	0.3281	0.7477	62.9094
10	1	10.2766	0.9975	−0.2766	0.0025	2.6691	0.0235
	5	10.2824	4.9792	−0.2824	0.0208	2.5877	0.5644
	10	10.2369	10.0057	−0.2369	−0.0057	2.5639	2.2955
	50	10.2854	49.7860	−0.2854	0.2140	2.6673	57.8074
50	1	51.2834	0.9958	−1.2834	0.0042	59.2753	0.0206
	5	51.0023	5.0004	−1.0023	−0.0004	53.4545	0.4927
	10	51.1610	9.9823	−1.1610	0.0177	58.8880	2.1036
	50	51.0919	50.0134	−1.0919	−0.0134	59.8056	54.1736

As seen in Tables 1-3, importantly, the simulated bias for $\hat{\lambda}$ has a negative bias for all situations, so it is an overestimate. In contrast, that for $\hat{\theta}$ are systematically positive, then it is an underestimate. When sample sizes are small, the amount of bias is quite large particularly when the true value of at least one parameter is sufficient big. Furthermore, the bigger parameters, the larger are the bias. For instance, at $n = 100$, $\lambda = 2$ and $\theta = 5$, the bias of $\hat{\lambda}$ is -0.0869, while for $\lambda = 50$ and $\theta = 50$, the bias of $\hat{\lambda}$ is -1.0919. However, the magnitude of the bias may be assumed to be relatively small. Tables 1-3 reveal that the more increasing values of the parameters, the more growing are the mean square error. As the results from the numerical study, it is observed that $\hat{\lambda}$ and $\hat{\theta}$ are consistent. Thus, they are asymptotically unbiased estimators. The simulated bias corresponds to the theoretical background as it is a decreasing function of sample sizes n. That is, when sample sizes increase, the amount of the bias decreases and tends to zero as $n \to \infty$. Similarly, the MSE is a decreasing function of n. The larger sample size, the smaller is the MSE, and it approaches to zero as $n \to \infty$.

5.2 Illustrative Examples

The practical applications of the suggested estimators are illustrated in this section. Two real data sets are considered as the followings.

Example 1 The following data set was provided by Lieblein and Zelen [12] on the fatigue life of the 23 deep groove ball bearings:

17.88 28.92 33.00 41.52 42.12 45.60 48.48 51.84 51.96 54.12 55.56 67.80
68.64 68.64 68.88 84.12 93.12 98.64 105.12 105.84 127.92 128.04 173.40.

The above data had been analyzed by Gupta and Akman [8] and the result indicated that the data set comes from the length biased inverse Gaussian distribution. For two-sided case, we divide all data in pairs, we have only 11 pairs of observations, and the last one is dropped. We obtain 11 observations: $(y_1, y_2), (y_3, y_4), \cdots, (y_{21}, y_{22})$. Let

$$u_1 = \frac{1}{y_1} + \frac{1}{y_2},\ u_2 = \frac{1}{y_3} + \frac{1}{y_4},\ \cdots,\ u_{11} = \frac{1}{y_{21}} + \frac{1}{y_{22}},$$

be the observations drawn from the TS-LBIG distribution. In this example, the point estimates are reported in Table 4. Moreover, using the relationship given in Section 2.1, the point estimates for parameters μ and β are also computed.

Example 2 This example was taken from Nichols and Padgett [14] consisting of 100 observations on breaking stress of carbon fibers (in GPa). These data had been analyzed by assuming the Weibull distribution. In our case, the data was ascendingly ordered. Dealing with the analogous manner of the example 1, finally, we obtain 50 observations drawn from the TS-LBIG distribution. The point estimates for the proposed and original parameters are shown in Table 5.

Most importantly, the estimators $\hat{\lambda}$ and $\hat{\theta}$ represent the thickness of the object under study and the nominal treatment pressure on the object, correspondingly. On the other hand, the estimators $\hat{\mu}$ and $\hat{\beta}$ lack this physical explanation.

Table 4 Point estimates for Example 1 **Table 5** Point estimates for Example 2

Proposed Parameters		Usual Parameters		Proposed Parameters		Usual Parameters	
λ	θ	μ	β	λ	θ	μ	β
1.5648	34.1267	53.4029	83.5672	1.3736	1.5579	2.1399	2.9395

6. Conclusions and Discussion

The new lifetime distribution based on re-parametrization model called the two-sided length biased inverse Gaussian distribution is proposed. The reciprocal properties and the first four moments of the distribution are investigated. The conventional point estimation, method of moment, is developed to estimate the parameters of the distribution and the asymptotic variances and covariance of the suggested estimators are also provided.

In this article, we discussed on some statistical properties of the distribution. The results ensure us that the method of moment estimators works and provides consistent statistics. Importantly, we estimate the parameters which reflect the physical nature of an object analyzed by a statistical view of the distribution. Accordingly, one could continue investigations for other point estimation schemes such as maximum likelihood estimators, although we expect to encounter mathematical difficulties. The density function of the TS-LBIG distribution involves an integral sign, and finding a maximum of their products even numerically is not easy work. Furthermore, we may consider new other estimators, i.e., a regression-quantile (least square). This method is dealt with the regression analysis of sample quantiles. However, the method of moment estimation can be examined as a preliminary topic of studying for the TS-LBIG distribution since it has a more satisfying property, e.g. simple computation. Interval estimation and hypothesis testing issues remain to be interesting for further investigation. As presented results of the asymptotic analysis, tests and confident interval estimation procedures will be explored regarding on a power of the test and coverage probabilities. Nevertheless, they are above the scope of this article and the investigation will be separately communicated.

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