Journal of Probability and Statistical Science
16(2), 183-195, Aug. 2018

Characterization of Two-Sided Topp-Leone Family and Its Applications

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ABSTRACT In this article, we introduce a new family of lifetime distributions, called the two-sided Topp-Leone generated family of distributions. A special case of the new family is the two-sided Topp-Leone Weibull distribution. This distribution uses the two-sided Topp-Leone distribution as a generator. The two-sided Topp-Leone Weibull distribution can obey several shapes: decreasing, unimodal and bimodal. This makes this distribution more flexible than the Weibull distribution. Its quantile, hazard rate and moment functions are derived. The parameter estimation procedure by the method using maximum likelihood estimation is discussed. The proposed distribution can be applied for the remission times of bladder cancer patients, time to failure of turbocharger and strength data set. We compare the proposed distribution to the Topp-Leone generated Weibull and two-sided generalized Weibull distribution. In conclusion, the two-sided Topp-Leone Weibull distribution and two-sided generalized Weibull distribution have better fit to the dataset comparing to the Topp-Leone generated Weibull.

Keywords Lifetime data Analysis; Parameter-adding method; Topp-Leone family; Two-sided family.

1. Introduction

The Weibull distribution is the lifetime distribution which has been widely used in reliability analysis. It can be used to model a variety of failure characteristics such as increasing, decreasing, and bathtub shapes. If X has Weibull distribution with parameters $\lambda \geq 0$ and $k \geq 0$, then the cumulative distribution function (cdf) and probability density function (pdf) are

$$F(x;\lambda,k) = 1 - \exp\{-(x/\lambda)^k\}, \quad x \geq 0$$

□Received April 2018, revised June 2018, in final form July 2018.

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 JPSS: ISSN 1726-3328

and

$$f(x;\lambda,k)=\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\exp\left\{-\left(\frac{x}{\lambda}\right)^{k}\right\},\ \ x\geq 0$$

where λ is the scale parameter and k is the shape parameter.

In the last two decades, many researchers have proposed some new classes of distributions, which are modifications of the distribution functions by adding extra parameters [9], for example, Sangsanit and Bodhisuwan [13] as well as Rezaei *et al.* [12] presented the Topp-Leone generator of distributions. The authors also studied the properties of Topp-Leone generalized exponential and Topp-Leone gamma distributions.

In this paper, a new two-sided generalized class of distributions is proposed, called the two-sided Topp-Leone generated family of distribution. It provides some sub models, namely the two-sided Topp-Leone Weibull (TSTLW) distribution. It can be used to describe some lifetime data which is commonly used in reliability analysis. The new families of distributions using the Topp-Leone (TL) distribution are introduced by Topp and Leone [14], and the two-sided (TS) distribution is introduced by Van Dorp and Kotz [15]. The TS distribution has its cdf and pdf as

$$F(x;\alpha,\beta)=\begin{cases}\beta\left(\dfrac{x}{\beta}\right)^{\alpha}, & 0<x\leq\beta\\[2ex] 1-(1-\beta)\left(\dfrac{1-x}{1-\beta}\right)^{\alpha}, & \beta\leq x<1\end{cases}$$

and

$$f(x;\alpha,\beta)=\begin{cases}\alpha\left(\dfrac{x}{\beta}\right)^{\alpha-1}, & 0<x\leq\beta\\[2ex] \alpha\left(\dfrac{1-x}{1-\beta}\right)^{\alpha-1}, & \beta\leq x<1\end{cases}.$$

Next, the TL distribution can be written in the forms of the cdf and pdf, respectively, as

$$f(x;a,b)=\left(\frac{2a}{b}\right)\left(1-\frac{x}{b}\right)\left(\frac{x}{b}\right)^{a-1}\left(2-\frac{x}{b}\right)^{a-1},$$

$0<x<b<\infty,\ 0<a$, and

$$F(x;a,b)=\left(\frac{x}{b}\right)^{a}\left(2-\frac{x}{b}\right)^{a},\ \ 0<x<b.$$

This paper is organized as follows. The two-sided Topp-Leone generated family of distribution is introduced in Section 2. The TSTLW distribution and its properties are studied in

Section 3. The parameter estimation of the TSTLW distribution by the method of maximum likelihood is shown in Section 4. Finally, some applications of the TSTLW distribution to three real datasets are shown in Section 5.

2. The Two-Sided Topp-Leone Generated Family of Distributions

The procedure of creating a new family of distribution consists of two principal components, which are generator and parent distributions, see [1]. Eugene, Lee, and Famoye [6] described a general class of distributions generated by the beta random variable, which is called beta generated (BG) distribution. The beta-normal distribution is an example of using the beta distribution as a generator and normal distribution as the parent distribution. Many researcher follow the same idea by taking a different parent distributions, see [7].

The new family of Kumaraswamy generalized (Kw-G) distribution is introduced in [5]. A special cases of this family are Kw-normal, Kw-Weibull, Kw-gamma, Kw-Gumbel and Kw-inverse Gaussian distributions. According to this procedure, the new TS distribution is based on the TL distribution with parameter $b=1$ and it is developed similar BG of distribution. Let TS distribution be a generator and the TL distribution be a parent distribution. Therefore, the cdf and pdf of two-sided Topp-Leone (TSTL) distribution can be defined, respectively, as

$$F(x;\alpha,\beta,\theta)=\begin{cases}\beta^{1-\alpha}[x^{\theta}(2-x)^{\theta}]^{\alpha}, & 0<x\le 1-\sqrt{1-\beta^{1/\theta}}\\ 1-(1-\beta)^{1-\alpha}[1-x^{\theta}(2-x)^{\theta}]^{\alpha}, & 1-\sqrt{1-\beta^{1/\theta}}\le x<1\end{cases} \tag{1}$$

and

$$f(x;\alpha,\beta,\theta)=\begin{cases}\alpha\beta^{1-\alpha}(2\theta)x^{\theta-1}(1-x)(2-x)^{\theta-1}[x^{\theta}(2-x)^{\theta}]^{\alpha-1}, & 0<x\le 1-\sqrt{1-\beta^{1/\theta}}\\ \alpha(1-\beta)^{1-\alpha}(2\theta)x^{\theta-1}(1-x)(2-x)^{\theta-1}[1-x^{\theta}(2-x)^{\theta}]^{\alpha-1}, & 1-\sqrt{1-\beta^{1/\theta}}\le x<1\end{cases}.$$

In the same manner, the TSTL family of distribution is obtained by substituting x with $G(x)$ in (1). Thus, its cdf can be obtained as follows:

$$F(x;\alpha,\beta,\theta)=\begin{cases}\beta^{1-\alpha}\{[G(x)]^{\theta}[2-G(x)]^{\theta}\}^{\alpha}, & 0<x\le G^{-1}\left(1-\sqrt{1-\beta^{1/\theta}}\right)\\ 1-(1-\beta)^{1-\alpha}\{1-[G(x)]^{\theta}[2-G(x)]^{\theta}\}^{\alpha}, & G^{-1}\left(1-\sqrt{1-\beta^{1/\theta}}\right)\le x<1\end{cases}. \tag{2}$$

By differentiating Eq. (2), the pdf of the TSTL family of distribution can be obtained as

$$f(x;\alpha,\beta,\theta)=\alpha\beta^{1-\alpha}(2\theta)g(x)[1-G(x)][G(x)]^{\theta-1}[2-G(x)]^{\theta-1}\{[G(x)]^{\theta}[2-G(x)]^{\theta}\}^{\alpha-1}$$

if $0<x\le G^{-1}\left(1-\sqrt{1-\beta^{1/\theta}}\right)$, and

$$f(x;\alpha,\beta,\theta)=\alpha(1-\beta)^{1-\alpha}(2\theta)g(x)[1-G(x)][G(x)]^{\theta-1}[2-G(x)]^{\theta-1}\{1-[G(x)]^{\theta}[2-G(x)]^{\theta}\}^{\alpha-1}$$

if $G^{-1}\left(1-\sqrt{1-\beta^{1/\theta}}\right)\le x<1$. (3)

Consequently, the quantile function of the TSTL distribution is

$$F^{-1}(x;\alpha,\beta,\theta)=\begin{cases}1-\sqrt{1-\left(\dfrac{x}{\beta^{1-\alpha}}\right)^{1/(\alpha\theta)}}, & 0<x\le\beta\\ 1-\sqrt{1-\left(1-\left(\dfrac{1-x}{(1-\beta)^{1-\alpha}}\right)^{1/\alpha}\right)^{1/\theta}}, & \beta\le x<1\end{cases}.$$

Hence the quantile function of the TSTL family is

$$Q(x)=\begin{cases}G^{-1}\left(1-\sqrt{1-\left(\dfrac{x}{\beta^{1-\alpha}}\right)^{1/(\alpha\theta)}}\right), & 0<x\le\beta\\ G^{-1}\left(1-\sqrt{1-\left(1-\left(\dfrac{1-x}{(1-\beta)^{1-\alpha}}\right)^{1/\alpha}\right)^{1/\theta}}\right), & \beta\le x<1\end{cases}.$$

Finally, the hazard rate function of the TSTL family is given by

$$h(x;\alpha,\beta,\theta)=\frac{\alpha\beta^{1-\alpha}(2\theta)g(x)[1-G(x)][G(x)]^{\theta-1}[2-G(x)]^{\theta-1}\{[G(x)]^{\theta}[2-G(x)]^{\theta}\}^{\alpha-1}}{1-\beta^{1-\alpha}\{[G(x)]^{\theta}[2-G(x)]^{\theta}\}^{\alpha}}$$

if $0<x\le G^{-1}\left(1-\sqrt{1-\beta^{1/\theta}}\right)$, and

$$h(x;\alpha,\beta,\theta)=\frac{\alpha(1-\beta)^{1-\alpha}(2\theta)g(x)[1-G(x)][G(x)]^{\theta-1}[2-G(x)]^{\theta-1}\{1-[G(x)]^{\theta}[2-G(x)]^{\theta}\}^{\alpha-1}}{1-\left[1-(1-\beta)^{1-\alpha}\{1-[G(x)]^{\theta}[2-G(x)]^{\theta}\}^{\alpha}\right]}$$

if $G^{-1}\left(1-\sqrt{1-\beta^{1/\theta}}\right)\le x<\infty$.

3. The Two-Sided Topp-Leone Weibull Distribution

In this section, we introduce a special case of the TSTL family of distribution, which is called the TSTLW distribution. Besides, some properties are studied. The Weibull distribution is used as a parent distribution with cdf

$$G(x)=1-e^{-(x/\lambda)^k},\quad x\ge 0$$

and is associated with pdf

$$g(x)=\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-(x/\lambda)^k},\quad x\ge 0.$$

By substituting $G(x)$ into (2), the cdf of the TSTLW distribution with parameter Θ, where $\Theta=(\alpha,\beta,\theta,\lambda,k)^T$, is

$$F_{\text{TSTLW}}(x)=\begin{cases}\beta^{1-\alpha}\left[\left(1-e^{-(x/\lambda)^k}\right)^{\theta}\left(1+e^{-(x/\lambda)^k}\right)^{\theta}\right]^{\alpha}, & 0<x\le\eta\\ 1-(1-\beta)^{1-\alpha}\left[1-\left(1-e^{-(x/\lambda)^k}\right)^{\theta}\left(1+e^{-(x/\lambda)^k}\right)^{\theta}\right]^{\alpha}, & \eta<x\le\infty\end{cases}$$

where $\eta=\lambda\left(-\log\sqrt{1-\beta^{1/\theta}}\right)^{1/k}$. The pdf of the TSTLW distribution is obtained by substituting $G(x)$ and $g(x)$ into (3):

$$f_{\text{TSTLW}}(x)=\alpha\beta^{1-\alpha}(2\theta)\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-2(x/\lambda)^k}\left[\left(1-e^{-(x/\lambda)^k}\right)\left(1+e^{-(x/\lambda)^k}\right)\right]^{\theta\alpha-1}$$

if $0<x\le\eta$, and

$$f_{\text{TSTLW}}(x)=\alpha(1-\beta)^{1-\alpha}(2\theta)\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\left(e^{-(x/\lambda)^k}\right)^{2+\theta(\alpha-1)}\left(1-e^{-(x/\lambda)^k}\right)^{\theta-1}\left(1+e^{-(x/\lambda)^k}\right)^{\theta\alpha-1}$$

if $\eta<x\le\infty$.

The hazard rate function of the TSTLW distribution is

$$h_{\text{TSTLW}}(x)=\frac{\alpha\beta^{1-\alpha}(2\theta)\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-2(x/\lambda)^k}\left[\left(1-e^{-(x/\lambda)^k}\right)\left(1+e^{-(x/\lambda)^k}\right)\right]^{\theta\alpha-1}}{1-\beta^{1-\alpha}\left[\left(1-e^{-(x/\lambda)^k}\right)^{\theta}\left(1+e^{-(x/\lambda)^k}\right)^{\theta}\right]^{\alpha}}$$

if $0<x\le\eta$, and

$$h_{\text{TSTLW}}(x)=\frac{\alpha\left(1-\beta\right)^{1-\alpha}(2\theta)\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\left(e^{-(x/\lambda)^k}\right)^{2+\theta(\alpha-1)}\left(1-e^{-(x/\lambda)^k}\right)^{\theta-1}\left(1+e^{-(x/\lambda)^k}\right)^{\theta\alpha-1}}{1-\left\{1-(1-\beta)^{1-\alpha}\left[1-\left(1-e^{-(x/\lambda)^k}\right)^{\theta}\left(1+e^{-(x/\lambda)^k}\right)^{\theta}\right]^{\alpha}\right\}}$$

if $\eta<x\le\infty$. Thus, the quantile function of the TSTLW distribution is given by

$$F^{-1}{}_{\text{TSTLW}}(x)=\begin{cases}\lambda\left(-\log\left(-\sqrt{1-\left(\dfrac{x}{\beta^{1-\alpha}}\right)^{1/(\alpha\theta)}}\right)\right)^{1/k}, & 0<x\le\beta\\ \lambda\left(-\log\left(-\sqrt{1-\left(1-\left(\dfrac{1-x}{(1-\beta)^{1-\alpha}}\right)^{1/\alpha}\right)^{1/\theta}}\right)\right)^{1/k}, & \beta<x\le\infty\end{cases}.$$

The rth moment about origin of X can be derived by the binomial expansions:

$$E(X^r \mid \Theta)$$

$$= \beta^{1-\alpha}\sum_{j,v=0}^{\infty}\binom{\theta\alpha}{j}(-1)^{j+v}2^{\theta\alpha-j}\binom{\theta\alpha+j}{v}\left(-\frac{1}{\lambda}\right)kv\int_0^{\eta}x^r e^{-\left(\frac{x}{\lambda}\right)^{kv}}\left[-\left(\frac{x}{\lambda}\right)\right]^{kv-1}dx$$

$$+1-\frac{(1-\beta)^{1-\alpha}}{(-\lambda)}\sum_{u,m,q=0}^{\infty}\binom{\alpha}{m}\binom{\theta m}{u}\binom{\theta m+u}{q}(-1)^{m+u+q}2^{\theta m-u}kq\int_{\eta}^{\infty}x^r e^{-\left(\frac{x}{\lambda}\right)^{kq}}\left[-\left(\frac{x}{\lambda}\right)\right]^{kq-1}dx.$$

Using the incomplete gamma function, the rth moment about origin of X is

$$E(X^r \mid \Theta)$$

$$=\left[\beta^{1-\alpha}\sum_{j,v=0}^{\infty}\binom{\theta\alpha}{j}(-1)^{j+v}2^{\theta\alpha-j}\binom{\theta\alpha+j}{v}\left(-\frac{1}{\lambda}\right)\lambda^{r+1}(-1)^{kv}\gamma\left(\frac{kv+r}{kv},\frac{x^{kv}}{\lambda^{kv}}\right)\right]$$

$$+\left[1-(1-\beta)^{1-\alpha}\sum_{u,m,q=0}^{\infty}\binom{\alpha}{m}\binom{\theta m}{u}\binom{\theta m+u}{q}(-1)^{m+u+q}2^{\theta m-u}\left(-\frac{1}{\lambda}\right)\lambda^{r+1}(-1)^{kq}\Gamma\left(\frac{kq+r}{kq},\frac{x^{kq}}{\lambda^{kq}}\right)\right]$$

where k is a parameter, v is a non-negative integer, and the incomplete gamma functions are defined as

$$\gamma(a,\eta)=\int_0^{\eta}(x^{a-1}-e^{x})dx \quad \text{and} \quad \Gamma(a,\eta)=\int_{\eta}^{\infty}(x^{a-1}-e^{x})dx,$$

$a>0,\ \eta\ge 0.$ Some pdf and hazard rate function plots of the TSTLW distribution are illustrated in Figures 1 and 2.

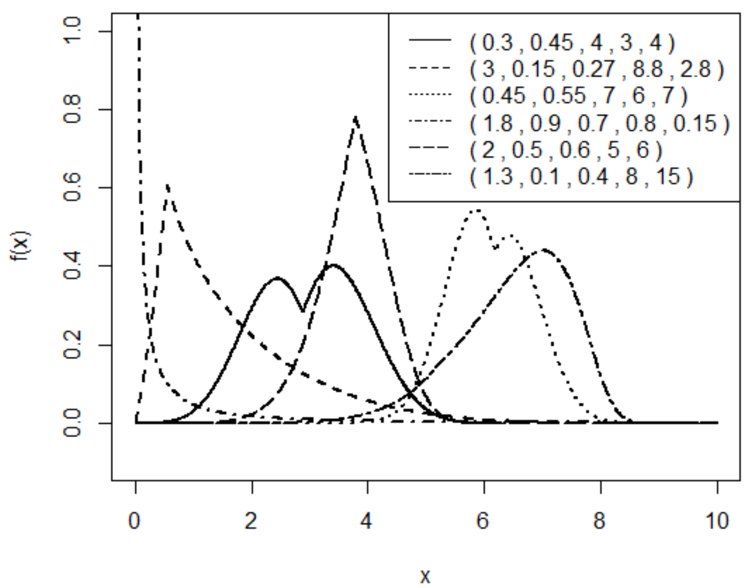

Figure 1 Density plots of the TSTLW distribution for specified parameters $\alpha,\beta,\theta,\lambda$ and k

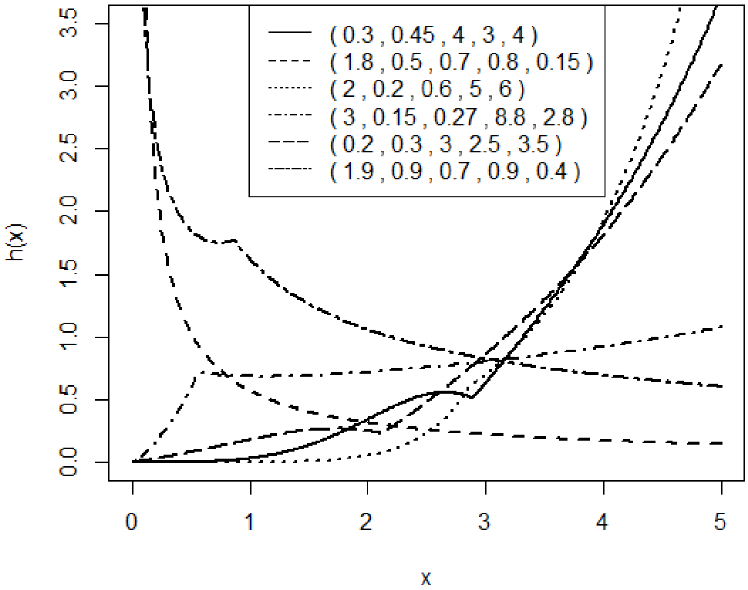

Figure 2 Hazard rate function of the TSTLW distribution for specified parameters $\alpha, \beta, \theta, \lambda$ and k

4. Parameter Estimation

Let $\vec{X} = (X_1, X_2, \cdots X_n)$ be a sequence of independent and identically distributed TSTLW random variables (sample of size n). The log-likelihood function of $\vec{X}$, ℓ, is

$$\ell = r\log\alpha + r(1-\alpha)\log\beta + r\log 2\theta + r\log k - r\log\lambda + (k-1)\sum_{i=1}^{r}\log\left(\frac{x_i}{\lambda}\right) + 2\sum_{i=1}^{r}\left(\frac{x_i}{\lambda}\right)^k$$

$$+(\theta\alpha-1)\sum_{i=1}^{r}\log\left(1-e^{-(x_i/\lambda)^k}\right) + (\theta\alpha-1)\sum_{i=1}^{r}\log\left(1+e^{-(x_i/\lambda)^k}\right) + (n-r)\log\alpha$$

$$+(n-r)(1-\alpha)\log(1-\beta) + (n-r)\log 2\theta + (n-r)\log k - (n-r)\log\lambda + (k-1)\sum_{i=r+1}^{n}\log\left(\frac{x_i}{\lambda}\right)$$

$$-[2+\theta(\alpha-1)]\sum_{i=r+1}^{n}\left(\frac{x_i}{\lambda}\right)^k + (\theta-1)\sum_{i=r+1}^{n}\left(1-e^{-(x_i/\lambda)^k}\right) + (\theta\alpha-1)\sum_{i=r+1}^{n}\log\left(1+e^{-(x_i/\lambda)^k}\right).$$

The first partial derivatives of ℓ with respect to parameters $\alpha, \beta, \theta, \lambda$ and k, called the score functions, are

$$\frac{\partial\ell}{\partial\alpha} = \frac{r}{\alpha} - r\log\beta + \theta\sum_{i=1}^{r}\log\left(1-e^{-(x_i/\lambda)^k}\right) + \theta\sum_{i=1}^{r}\log\left(1+e^{-(x_i/\lambda)^k}\right) + \frac{(n-r)}{\alpha} - (n-r)\log(1-\beta)$$

$$-\theta\sum_{i=r+1}^{n}\left(\frac{x_i}{\lambda}\right)^k + \theta\sum_{i=r+1}^{n}\log\left(1+e^{-(x_i/\lambda)^k}\right),$$

$$\frac{\partial\ell}{\partial\beta} = \frac{r(1-\alpha)}{\beta} - \frac{(n-\gamma)(1-\alpha)}{1-\beta},$$

$$\frac{\partial \ell}{\partial \theta}=\frac{r}{\theta}+\alpha\sum_{i=1}^{r}\log\left(1-e^{-(x_i/\lambda)^k}\right)+\alpha\sum_{i=1}^{r}\log\left(1+e^{-(x_i/\lambda)^k}\right)+\frac{n-r}{\theta}-(\alpha-1)\sum_{i=r+1}^{n}\left(\frac{x_i}{\lambda}\right)^k$$

$$+\sum_{i=r+1}^{n}\left[1-e^{-(x_i/\lambda)^k}+\alpha\sum_{i=r+1}^{n}\log\left(1+e^{-(x_i/\lambda)^k}\right)\right],$$

$$\frac{\partial \ell}{\partial \lambda}=-\frac{r}{\lambda}-(k-1)\sum_{i=1}^{r}\frac{\lambda}{x_i}\cdot\frac{x_i}{\lambda^2}+2k\sum_{i=1}^{r}x_i^k\lambda^{-k-1}-\frac{(\theta\alpha-1)}{1-e^{-(x_i/\lambda)^k}}\cdot e^{-(x_i/\lambda)^k}\left(kx_i^k\lambda^{-k-1}\right)$$

$$+\frac{(\theta\alpha-1)}{1+e^{-(x_i/\lambda)^k}}\cdot e^{-(x_i/\lambda)^k}\left(kx_i^k\lambda^{-k-1}\right)-\frac{n-r}{\lambda}-(k-1)\sum_{i=r+1}^{n}\frac{\lambda}{x_i}\cdot\frac{x_i}{\lambda^2}-[2+\theta(\alpha-1)]\sum_{i=r+1}^{n}x_i^k\lambda^{-k-1}$$

$$-\frac{(\theta-1)}{1-e^{-(x_i/\lambda)^k}}\cdot e^{-(x_i/\lambda)^k}\left(kx_i^k\lambda^{-k-1}\right)+\frac{(\theta\alpha-1)}{1+e^{-(x_i/\lambda)^k}}\cdot e^{-(x_i/\lambda)^k}\left(kx_i^k\lambda^{-k-1}\right),$$

and

$$\frac{\partial \ell}{\partial k}=\frac{r}{k}+\sum_{i=1}^{r}\log\left(\frac{x_i}{\lambda}\right)-2\sum_{i=1}^{r}\left(\frac{x_i}{\lambda}\right)^k\log\left(\frac{x_i}{\lambda}\right)+\frac{(\theta\alpha-1)}{1-e^{-(x_i/\lambda)^k}}\sum_{i=1}^{r}e^{-(x_i/\lambda)^k}\left(\frac{x_i}{\lambda}\right)^k\log\left(\frac{x_i}{\lambda}\right)$$

$$-(\theta\alpha-1)\sum_{i=1}^{r}\frac{e^{-(x_i/\lambda)^k}}{1+e^{-(x_i/\lambda)^k}}\left(\frac{x_i}{\lambda}\right)^k\log\left(\frac{x_i}{\lambda}\right)+\frac{n-r}{k}+\sum_{i=r+1}^{n}\log\left(\frac{x_i}{\lambda}\right)$$

$$-[2+\theta(\alpha-1)]\sum_{i=r+1}^{n}\left(\frac{x_i}{\lambda}\right)^k\log\left(\frac{x_i}{\lambda}\right)+(\theta-1)\sum_{i=r+1}^{n}e^{-(x_i/\lambda)^k}\left(\frac{x_i}{\lambda}\right)^k\log\left(\frac{x_i}{\lambda}\right)$$

$$-(\theta\alpha-1)\sum_{i=r+1}^{n}\frac{e^{-(x_i/\lambda)^k}}{1+e^{-(x_i/\lambda)^k}}\left(\frac{x_i}{\lambda}\right)^k\log\left(\frac{x_i}{\lambda}\right).$$

The maximum likelihood estimates (MLE) of parameters $\alpha,\beta,\theta,\lambda,$ and k can be obtained by setting the score functions to zero. Then the above system of score equations can solved by using *optim* function in R language, see [11].

5. Applications

In this section, the Topp-Leone generated Weibull (TLGW) [3] and two-sided generalized Weibull (TSGW) [8] distributions are applied for three real datasets. The MLE is used to estimate parameters, and the performance of fitting data is evaluated based on the goodness of fit test using Kolmogorov-Smirnov (KS) test statistic and the model selection using minus log-likelihood (-LL) function to calculated Akaike's information criterion (AIC) and Bayesian information criterion (BIC).

5.1 The Strength Data

The first data set provides the strength data of single carbon fibers can be found in [4]. The experiment is tested under tension at gauge lengths of 1, 10, 20 and 50 mm. We consider only the data set consisting of the single fibers of 20 mm. Table 1 shows the MLE of the model parameters, KS test, -LL, AIC, and BIC.

Table 1 Parameter estimates, KS test results and -LL, AIC, and BIC values for the strength measured in GPA data set

Distribution	MLE	KS test (*p*-value)	–LL	AIC	BIC
TSTLW	$\hat{\alpha}=1.833$, $\hat{\beta}=0.004$, $\hat{\theta}=5.048$, $\hat{\lambda}=3.631$, $\hat{k}=3.105$	0.1138 (0.6487)	29.22	68.44	77.13
TLGW	$\hat{\alpha}=112.441$, $\hat{\beta}=0.492$, $\hat{\theta}=149.055$, $\hat{\eta}=18.029$	0.1149 (0.6364)	27.23	62.46	69.41
TSGW	$\hat{\alpha}=0.252$, $\hat{\gamma}=5.056$, $\hat{\theta}=2.524$, $\hat{\beta}=1.000$	0.0875 (0.7199)	61.98	131.96	140.54

The plots of fitted TSTLW, TLGW and TSGW with the strength dataset are shown in Figure 3.

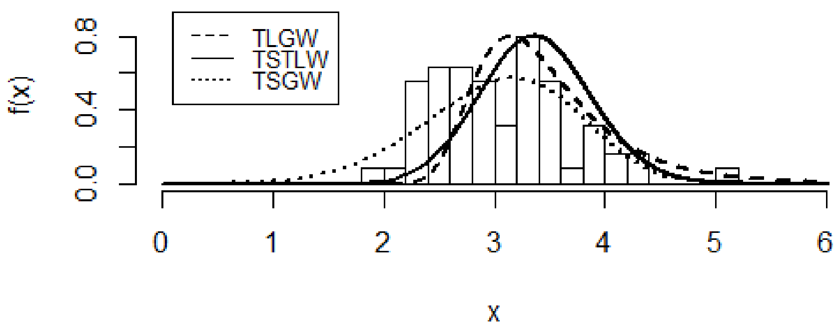

Figure 3 Estimated densities of the TSTLW, TLGW and TSW models for the strength data set

Based on *p*-value of the KS test, the TSGW distribution outperforms the TSTLW and TLGW distributions.

5.2 Remission Times of Bladder Cancer Patients

The second dataset comes from Lemonte and Cordeiro [2] and Lee and Wang [10]. They studied the remission times (in months) of the cancer patients. Unimodal and right skewed distributions are considered. Table 2 exhibits the MLE of the model parameters, KS test, -LL, AIC, and BIC.

Table 2 Parameter estimates, KS test results and -LL, AIC, and BIC values for the remission time data set

Distribution	MLE	KS test (*p*-value)	–LL	AIC	BIC
TSTLW	$\hat{\alpha}=0.349$, $\hat{\beta}=0.003$, $\hat{\theta}=3.146$, $\hat{\lambda}=3.555$, $\hat{k}=0.769$	0.0393 (0.9889)	411.07	832.14	846.40
TLGW	$\hat{\alpha}=2.078$, $\hat{\beta}=0.438$, $\hat{\theta}=2.696$, $\hat{\eta}=0.360$	0.0468 (0.9420)	411.21	830.42	841.83
TSGW	$\hat{\alpha}=0.711$, $\hat{\gamma}=1.214$, $\hat{\theta}=11.527$, $\hat{\beta}=0.719$	0.2140 (0.0000)	426.79	861.58	872.99

The plots of fitted TSTLW, TLGW and TSGW with the remission time dataset are shown in Figure 4.

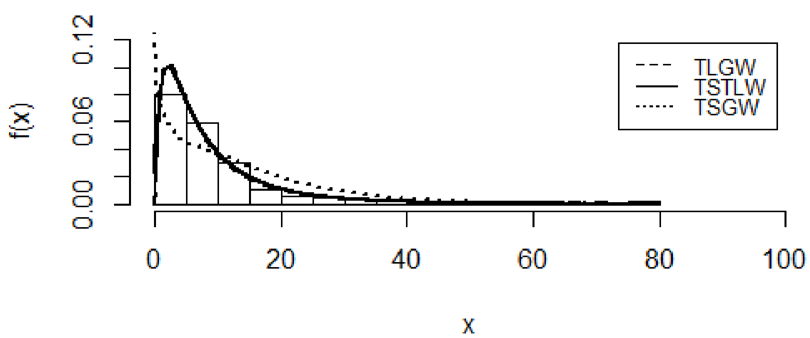

Figure 4 Estimated densities of the TSTLW, TLGW and TSW models for the remission times dataset

Based on *p*-value of KS test, the TSTLW distribution is outperform the TSGW and TLGW distributions.

5.3 Time-to-Failure of Turbocharger

The third dataset is about the time-to-failure (10^3 h) of turbocharger, which is one type of engine, and it is given by Xu *et al.* [16]. The MLE of the model parameters with some statistical value displayed in Table 3.

Table 3 Parameter Estimates, KS Test Results and -LL, AIC, and BIC Values for the Time-to-Failure Dataset

Distribution	MLE	KS test (*p*-value)	–LL	AIC	BIC
TSTLW	$\hat{\alpha}=0.165$, $\hat{\beta}=0.152$, $\hat{\theta}=5.469$, $\hat{\lambda}=4.875$, $\hat{k}=3.460$	0.096 (0.8543)	81.35	172.70	181.14
TLGW	$\hat{\alpha}=4.115$, $\hat{\beta}=0.630$, $\hat{\theta}=5.374$, $\hat{\eta}=0.703$	0.158 (0.2709)	91.50	191.00	197.75
TSGW	$\hat{\alpha}=0.009$, $\hat{\gamma}=3.949$, $\hat{\theta}=2.080$, $\hat{\beta}=0.995$	0.1083 (0.7359)	82.70	173.41	180.16

The plots of fitted TSTLW, TLGW and TSGW with the time-to-failure remission times of turbocharger dataset are shown in Figure 5.

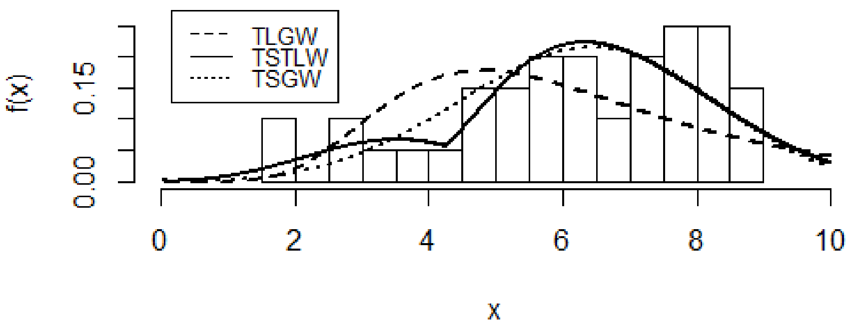

Figure 5 Estimated densities of the TSTLW, TLGW and TSW models for the time-to-failure remission times of turbocharger dataset

Based on *p*-value of KS test, the fitting by TSTLW distribution slightly better than others. However TSGW can be fairly used to fit this particular dataset.

6. Conclusion

In this article, the new class of lifetime distributions is introduced, namely the TSTL family. Moreover, the new lifetime distribution is also proposed and it is called the TSTLW distribution. Some properties of this distribution are discussed. The parameters estimation by the maximum likelihood principle is also presented. The three real datasets are used to expose the efficiency of the purposed distribution compared with the TLGW and TSGW distribution. In application study, we found that TSTLW and TSGW distribution provide better fit to the datasets comparing with TSGW distribution.

Acknowledgments

The authors are grateful to the reviewers for worthy comments. Moreover, the authors would like to thank the Department of Statistics, Faculty of Science, Kasetsart University and Graduate School at Kasetsart University.

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