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Asymptotic Properties and Parameter Estimation Based on Two-Sided Crack Distribution

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ABSTRACT In this paper we propose a new family of the two-sided crack distribution. The theoretical properties of the two-sided crack distribution is established. Also, we develop and investigate the method of moments of parameters estimation. A Monte Carlo simulation and real data study are conducted to appraise the performance of the proposed estimators for given sample sizes by using R program for evaluation.

Keywords Birnbaum-Saunders distribution; Inverse Gaussian distribution; Length-biased inverse Gaussian distribution; Maximum likelihood estimators.

1. Introduction

The crack distribution is a positively skewed model, which is widely applicable to model failure times of fatiguing materials. It is also known as the inverse Gaussian mixture distribution, was studied by Jorgensen *et al*. [11] and Bowonrattanaset and Budsaba [5]. Gupta and Akman [9] proposed the mixture of inverse Gaussian (IG) distribution and length biased inverse Gaussian (LBIG) distribution which given in Jorgensen *et al*. [11] in a reliability view point, and here called JSW distribution. Gupta and Akman [10] studied the mixture of IG distribution and LBIG distribution in the view of Bayes estimation. Balakrishnan *et al*. [2] discussed several aspects of the inverse Gaussian mixture distribution which is useful for modelling positive data. Specifically, they discussed transformations, the derivation of moments, fitting of models, and a shape analysis of the transformations. Bowonrattanaset and Budsaba [5] introduced the inverse Gaussian mixture distribution based on re-parametrization model presented in Ahmed *et al*. [1], and proposed the name crack for this distribution, it will be denoted by $CR(\lambda,\theta,p)$. They also established some deeper results especially function with rigorous proves. Gupta and Kundu [8] proposed to use the EM algorithm to estimate the unknown parameters of the inverse Gaussian

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mixture distribution for complete and censored samples. Duangsaphon [6] studied crack distribution in the view of regression-quantile estimation, Bayesian estimation and confidence interval estimation. Additionally, Saengthong and Bodhisuwan [14] proposed a two-parame-ter crack distribution which is obtained by adding a new weight parameter p to the inverse Gaussian mixture distribution.

The crack lifetime distribution depends on three parameters. It is formed by adding the weighted parameter and combining the two parameter IG distribution and two parameter *LBIG* distribution. This distribution contains special case three known distribution, namely, the Birnbaum and Saunders (BS) distribution, the IG distribution, and the *LBIG* distribution. The relevance of the probability density function (pdf.) of distributions mentioned above are as follows. Suppose X_1 and X_2 be independent random variables such that $X_1 \sim IG(\lambda,\theta)$ i.e., X_1 has IG distribution with the parameters $\lambda,\theta > 0$ and it has the pdf.:

$$f_{IG}(x_1;\lambda,\theta) = \begin{cases} \frac{\lambda}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x_1}\right)^{\frac{3}{2}} exp\Big[-\frac{1}{2}\Big(\sqrt{\frac{x_1}{\theta}}-\lambda\sqrt{\frac{\theta}{x_1}}\Big)^2\Big]; & x > 0 \\ 0; & otherwise. \end{cases} \tag{1}$$

In addition, suppose $X_2 \sim LBIG(\lambda,\theta)$. The pdf. of X_2 is given by:

$$f_{LBIG}(x_2;\lambda,\theta) = \begin{cases} \frac{1}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x_2}\right)^{\frac{1}{2}} exp\Big[-\frac{1}{2}\Big(\sqrt{\frac{x_2}{\theta}}-\lambda\sqrt{\frac{\theta}{x_2}}\Big)^2\Big]; & x > 0 \\ 0; & otherwise. \end{cases} \tag{2}$$

The variable X_2 is so-called the complementary reciprocal of X_1. For the crack distribution, we consider the new random variable X such that,

$$X = \begin{cases} X_1 \text{ with probability } p \\ X_2 \text{ with probability } 1-p \end{cases} \tag{3}$$

where $0 \le p \le 1$. Obviously, X is a mixture of X_1 and X_2 and the pdf. of X is given by the following formula

$$f_{CR}(x;\lambda,\theta,p) = p f_{IG}(x;\lambda,\theta) + (1-p) f_{LBIG}(x;\lambda,\theta) \tag{4}$$

From (4), the probability density function, can be expressed as

$$f_{CR}(x;\lambda,\theta,p) = \begin{cases} \frac{1}{\theta\sqrt{2\pi}}\Big[p\lambda\left(\frac{\theta}{x}\right)^{\frac{3}{2}} + (1-p)\left(\frac{\theta}{x}\right)^{\frac{1}{2}}\Big] exp\Big[-\frac{1}{2}\Big(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\Big)^2\Big]; & x > 0 \\ 0; & otherwise. \end{cases} \tag{5}$$

where $\lambda > 0, \theta > 0$ and $0 \le p \le 1$. Evidently, the crack distribution become IG distribution for $p = 1$, *LBIG* distribution for $p = 0$ and BS distribution by substituting $p = 0.5$.

However, all of published about the crack distribution can be used in model one sides crack growth data except the studies of Lisawadi [12]. Hence, the main aim of this paper is to introduce a new class of distributions based on crack distribution proposed in [5] by considering the case when a crack develops from two sides. The new distribution is call two-sided crack distribution. Additionally, this kind of study has not been studied before. We propose probability model of the two-sided crack distribution by applying the approach of Lisawadi [12]. The theoretical properties of the two-sided crack distribution are established. Also, we develop and investigate the method of moments of parameters point estimation for the new distribution. A Monte Carlo simulation and real data study are conducted to appraise the performance of the proposed estimators for given sample sizes by using R program for evaluation.

2. Probability Model of Two-Sided Crack Distribution

In this dessertation we consider the case when a crack develops from two sides of material with the same distribution function of the time reaching the critical value, e.g., fatigue limit loading applied to both upper and lower sides of the block.

Suppose $F(t), t > 0$, be the cumulative distribution function of break down time moment τ for one-sided loading. If at the top and the bottom side of the block a crack is developing with the same distribution function of the time of reaching the critical length a, then we have two, assumed to be independent, identically distributed random variables τ_1 and τ_2.

Therefore, the random variable $Y_1 = a/\tau_1$ can be interpreted as a speed of the crack evolution which has cumulative distribution function $F_{Y_1}(t) = 1 - F(at^{-1})$, that is the reliability function $R(t)$, and probability density function $f_{Y_1}(t) = at^{-2} f(at^{-1})$, which is a derivative of the $R(t)$.

In a similar way, we get $Y_2 = a/\tau_2$ which has cumulative distribution function $F_{Y_2}(t) = 1 - F(at^{-1})$ and probability density function $f_{Y_2}(t) = at^{-2} f(at^{-1})$. Then the speed of the crack evolution for this two-sided case equals $Y_1 + Y_2 = a\tau_1^{-1} + a\tau_2^{-1}$ and the random variable

$$\tau = \frac{a}{Y_1 + Y_2} = \left[\tau_1^{-1} + \tau_2^{-1}\right]^{-1}$$

corresponds to a moment of the block break down.

The cdf. of τ is

$$F_\tau(z) = \int\int_{t+s>z^{-1}} f\left(\frac{1}{t}\right) f\left(\frac{1}{s}\right) \frac{dtds}{t^2 s^2} \tag{6}$$

$$= 1 - \int_0^{z^{-1}} f\left(\frac{1}{t}\right) \frac{dt}{t^2} \int_0^{z^{-1}-t} f\left(\frac{1}{s}\right) \frac{ds}{s^2}. \tag{7}$$

Proof. Let $X = 1/\tau_1,\ Y = 1/\tau_2$. Recall on

$$P(X + Y \geq z^{-1}) = \iint_{x+y \geq z^{-1}} f_{X,Y}(x, y) dx dy$$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y).$$

Therefore,

$$F_X(x) = P(X \leq x) = P\Big(\frac{1}{\tau_1} \leq x\Big) = P(\tau_1 \geq x^{-1}) = 1 - F_{\tau_1}\Big(\frac{1}{x}\Big)$$
$$f_X(x) = F'_X(x) = f\Big(\frac{1}{x}\Big) \cdot \frac{1}{x^2}.$$

The same $f_Y(y) = f\ (1/y)\ /y^2$. If $z = \infty$, then $F_\tau(\infty) = 1$ and $1/z = 0$. So,

$$F_\tau(z) = 1 - \iint f_\tau(x,y)dxdy = 1 - \int_0^{z^{-1}}\int_0^{z^{-1}-y} f_\tau(x,y)dxdy$$
$$f_\tau(x,y) = f\Big(\frac{1}{x}\Big)f\Big(\frac{1}{y}\Big)\frac{1}{x^2}\frac{1}{y^2}.$$

Hence,

$$F_\tau(z) = 1 - \int_0^{z^{-1}}\int_0^{z^{-1}-t} f\Big(\frac{1}{t}\Big)f\Big(\frac{1}{s}\Big)\frac{1}{t^2s^2}dsdt.$$
$$= 1 - \int_0^{z^{-1}} f\Big(\frac{1}{t}\Big)\frac{dt}{t^2}\int_0^{z^{-1}-t} f\Big(\frac{1}{s}\Big)\frac{ds}{s^2}.$$

□

The pdf. of τ is

$$f_\tau(z) = z^{-2}\int_0^{z^{-1}} f\Big(\frac{1}{t}\Big)f\Big(\frac{1}{z^{-1}-t}\Big)\frac{1}{t^2\Big(z^{-1}-t\Big)^2}dt \tag{8}$$

Proof. Let

$$F_\tau(z) = 1 - \int_0^{z^{-1}}\int_0^{z^{-1}-t} f\Big(\frac{1}{t}\Big)f\Big(\frac{1}{s}\Big)\frac{1}{t^2s^2}dsdt.$$

Hence,

$$F_\tau(z)' = -\bigg(\int_0^{z^{-1}}\int_0^{z^{-1}-t} f\Big(\frac{1}{t}\Big)f\Big(\frac{1}{s}\Big)\frac{1}{t^2s^2}dsdt\bigg)'.$$

Let $g_z(z,t) = \int_0^{z^{-1}-t}\big[f\ (1/t)\ f\ (1/s)\ /(t^2s^2)\big]ds$. Therefore,

$$F_\tau(z)' = -\bigg(\int_0^{z^{-1}} g_z(z,t)dt\bigg)'.$$

Recall on Leibniz integral rule, hence,

$$F_\tau(z)' = -\int_0^{z^{-1}} g'_z(z,t)dt - g(z,z^{-1}) \cdot \Big(-\frac{1}{z^2}\Big) + 0.$$

We can see that

$$g(z,z^{-1}) = \int_0^{z^{-1}-z^{-1}} f\Big(\frac{1}{t}\Big) f\Big(\frac{1}{s}\Big) \frac{1}{t^2 s^2} ds = \int_0^{0} f\Big(\frac{1}{t}\Big) f\Big(\frac{1}{s}\Big) \frac{1}{t^2 s^2} ds = 0$$

Thus,

$$F_\tau(z)' = -\int_0^{z^{-1}} g_z'(z,t)dt, \qquad g_z'(z,t) = \Big(\int_0^{z^{-1}-t} f\Big(\frac{1}{t}\Big) f\Big(\frac{1}{s}\Big) \frac{1}{t^2 s^2} ds\Big)'$$

Let $h(z,s) = h(s) = f(1/t)\, f(1/s)\, /(t^2 s^2)$. Recall on Leibniz integral rule again, therefore,

$$\begin{aligned}
g_z'(z,t) &= \int_0^{z^{-1}-t} h'(z,s)ds + h(z,z^{-1}-t)\cdot\Big(-\frac{1}{z^2}\Big) - g(z,0)\cdot 0 \\
&= \int_0^{z^{-1}} 0\, ds + h(z,z^{-1}-t)\cdot\Big(-\frac{1}{z^2}\Big) - 0 \\
&= -\frac{1}{z^2} h(z,z^{-1}-t) \\
F_\tau(z)' &= \int_0^{z^{-1}} \frac{1}{z^2} h(z,z^{-1}-t)dt \\
&= z^{-2}\int_0^{z^{-1}} f\Big(\frac{1}{t}\Big) f\Big(\frac{1}{z^{-1}-t}\Big) \frac{1}{t^2\,(z^{-1}-t)^2} dt
\end{aligned}$$

For the case when we consider $F(t) = F_{CR}(t;\lambda,\theta,p)$, we can say that the random variable τ has two-sided crack distribution.

3. Cumulants and Moments

Theorem 3.1 *Let τ be a random variable of the two-sided crack distribution with parameters λ,θ and p. Then the first four cumulants of τ can be given as:*

$$\begin{aligned}
&1^{st} \quad \textit{cumulant} \quad K_1 = \frac{2}{\lambda^2\theta}(\lambda + p) \\
&2^{nd} \quad \textit{cumulant} \quad K_2 = \frac{2}{\lambda^4\theta^2}\left(\lambda + 3p - p^2\right) \\
&3^{th} \quad \textit{cumulant} \quad K_3 = \frac{2}{\lambda^6\theta^3}\left(3\lambda + 15p - 9p^2 + 2p^3\right) \\
&4^{th} \quad \textit{cumulant} \quad K_4 = \frac{2}{\lambda^8\theta^4}\left(15\lambda + 105p - 87p^2 + 36p^3 - 6p^4\right).
\end{aligned}$$

Proof. Let X be a random variable of the crack distribution with parameters λ,θ and p. The first four cumulants of a random variable X are defined as

$$\begin{aligned}
&1^{st} \quad \textit{cumulant} \quad K_1(X) = (\lambda + 1 - p)\theta \\
&2^{nd} \quad \textit{cumulant} \quad K_2(X) = (\lambda + 2 - p - p^2)\theta^2
\end{aligned}$$

$$3^{th} \quad cumulant \quad K_3(X) = (3\lambda + 8 - 3p - 3p^2 - 2p^3)\theta^3$$

$$4^{th} \quad cumulant \quad K_4(X) = (15\lambda + 48 - 15p - 15p^2 - 12p^3 - 6p^4)\theta^4.$$

According to the reciprocal random variable $1/X$ has $CR\left(\lambda, 1/(\lambda^2\theta), 1-p\right)$ distribution, hence we make the substitution $\theta \Rightarrow 1/(\lambda^2\theta)$, and $p \Rightarrow 1-p$. Finally, we double them, in order to get the cumulants of $\tau = X_1^{-1} + X_2^{-1}$. Therefore,

$$K_1(\tau) = 2(\lambda + 1 - (1-p))\Big(\frac{1}{\lambda^2\theta}\Big) = \frac{2}{\lambda^2\theta}(\lambda + p). \tag{9}$$

$$K_2(\tau) = 2\left(\lambda + 2 - (1-p) - (1-p)^2\right)\left(\frac{1}{\lambda^2\theta}\right)^2 = \frac{2}{\lambda^4\theta^2}\left(\lambda + 3p - p^2\right). \tag{10}$$

$$K_3(\tau) = 2\left(3\lambda + 8 - 3(1-p) - 3(1-p)^2 - 2(1-p)^3\right)\left(\frac{1}{\lambda^2\theta}\right)^3$$

$$= \frac{2}{\lambda^6\theta^3}\left(3\lambda + 15p - 9p^2 + 2P^3\right). \tag{11}$$

$$K_4(\tau) = 2\left(15\lambda + 48 - 15(1-p) - 15(1-p)^2 - 12(1-p)^3 - 6(1-p)^4\right)\left(\frac{1}{\lambda^2\theta}\right)^4$$

$$= \frac{2}{\lambda^8\theta^4}\left(15\lambda + 105p - 87p^2 + 36p^3 - 6p^4\right). \tag{12}$$

Theorem 3.2 *Let τ be a random variable of the two-sided crack distribution with parameters λ, θ and p. Then based on theorem 3.1, the first four moments of τ are given by*

$$\mu(\tau) = \frac{2}{\lambda^2\theta}(\lambda + p), \qquad \sigma^2(\tau) = \frac{2}{\lambda^4\theta^2}\left(\lambda + 3p - p^2\right)$$

$$\mu_3(\tau) = \frac{2}{\lambda^6\theta^3}\left(3\lambda + 15p - 9p^2 + 2p^3\right)$$

$$\mu_4(\tau) = \frac{2}{\lambda^8\theta^4}\left(15\lambda + 6\lambda^2 + 36\lambda p - 12\lambda p^2 + 105p - 33p^2\right).$$

Proof. The relations that express the moments as functions of the cumulants are

$$\mu(\tau) = K_1(\tau), \quad \sigma^2(\tau) = K_2(\tau), \quad \mu_3(\tau) = K_3(\tau), \quad \mu_4(\tau) = K_4(\tau) + 3\sigma^4(\tau).$$

Based on theorem 3.1, we make the substitution cumulants into the formulae above in order to get the first four central moments of two-sided crack distribution.

4. Parameter Estimation

Estimation of the parameters λ and θ with fixed the value of p by the method of moments for the two-sided crack distribution can be derived in the following way. Using formulae from the theorem 3.2 we obtain formulae for the expectation and variance

$$E(X) = \mu(X) = \frac{2}{\lambda^2\theta}(\lambda + p), \quad Var(X) = \sigma^2(X) = \frac{2}{\lambda^4\theta^2}\left(\lambda + 3p - p^2\right).$$

Thus, the method of moments estimations are

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{2}{\lambda^2\theta}(\lambda + p) \tag{13}$$

$$S^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 = \frac{2}{\lambda^4\theta^2}\left(\lambda + 3p - p^2\right). \tag{14}$$

Let

$$T = \frac{S^2}{\bar{X}^2} \tag{15}$$

Next, we solve these equations for λ and θ by substituting (13) , and (14) into (15). Hence we get

$$T = \frac{(\lambda + 3p - p^2)}{2\lambda^2 + 4\lambda p + 2p^2}.$$

Therefore,

$$2T\lambda^2 + (4Tp - 1)\lambda + (2Tp^2 - 3p + p^2) = 0 \tag{16}$$

Solving equation (16) for λ we obtain

$$\hat{\lambda}_n = \frac{(1 - 4Tp) + \sqrt{16Tp - 8Tp^2 + 1}}{4T} \tag{17}$$

By equation (13) , and (17) , we get

$$\hat{\theta} = \frac{2(\hat{\lambda}_n - p)}{\bar{X}\hat{\lambda}_n^2} \tag{18}$$

We will notice that the estimate $\hat{\lambda}_n$ is relatively stable whether the value of θ will be changed.

5. Two-Sided Crack Random Number Generation Procedure

First we discuss how to generate random numbers for the one-sided crack distribution. Fix the shape parameter λ, the scale parameter θ and the probability p. According to the Composite Method, then a random number with one-sided crack distribution can be obtained as the following procedure.

1. Generate a random number b from uniform [0, 1].
2. If $b < p$, then generate a random number with $IG(\lambda, \theta)$ distribution. Otherwise, generate a random number with $LB(\lambda, \theta)$ distribution.

For two-sided crack distribution, after we obtain a pair of one-sided crack random numbers τ_1 and τ_2, we calculate $X = \tau_1^{-1} + \tau_2^{-1}$ that is a two-sided crack random numbers.

The following is the same procedure in more details for a $TS - CR(\lambda, \theta, p)$ random number generator procedure.

1. Generate a random number a from uniform [0,1] and an independent a standard normal random number α.

2. Calculate $u = \lambda\theta + \theta\left[\alpha^2 - \sqrt{\alpha^4 + 4\lambda\alpha^2}\right]/2$.
3. If $a < \lambda\theta/(\lambda\theta + u)$, then take $CR = u$, otherwise $CR = \lambda^2\theta^2/u$.
4. Generate a random number b from uniform [0, 1].
5. If $b > p$, then generate a standard normal random number α_1 and take $CR = CR + \theta\alpha_1^2$.
6. Repeat steps 1-5, for generating a pair of one-sided crack random numbers τ_1 and τ_2.
7. Calculate $X = \tau_1^{-1} + \tau_2^{-1}$.

6. Computational Results

In order to appraise the performance of the proposed estimators, we performed a numerical study for different sample sizes and for different parameter values. Also, we calculated the biases and the MSE of proposed estimator from 5,000 simulated values of $(\hat{\lambda}, \hat{\theta})$ by using the R program version 3.2.0 for evaluation.

Note that the proposed estimators are consistent. Therefore, the proposed estimators are asymptotically unbiased. The simulated bias analysis is in agreement with the theoretical result since bias is a decreasing function of n . In other words, as sample size increases, the magnitude of the bias decreases and approaches to 0 as $n \to \infty$. Furthermore, the numerical values of the biases, the MSE of the proposed estimators are reported in appendix.

6.1 Application to Lifetime Data Set

Here, we apply the proposed distribution to a real data set which taken from Folks and Chhikara [7] data, and Rieck's [13] data. The Folks and Chhikara [7] data provides information on the fracture toughnesses of welds. The data are

Table 1 Several data sets obtained from Folk and Chikkara (1977).

fracture									
54.4	62.6	63.2	67.0	70.2	70.5	70.6	71.4	71.8	
74.1	74.1	74.3	78.8	81.8	83.0	84.4	85.3	86.9	87.3

From Table 2, we see that the value of estimate parameters of two-sided crack distribution with fixed $p = 0.5$ and the value of estimate parameters of two-sided BS distribution presented in Lisawadi [12] are similar values. Rieck's [13] data correspond to biaxial fatigue life of metal piece (in cycles) to failure.

Table 2 Point estimates for Folk and Chikkara's [7] data

Estimator	$\hat{\lambda}$	$\hat{\theta}$
Two-sided crack with fixed $p = 0.1$	35.71619	0.0007515509
Two-sided crack with fixed $p = 0.5$	35.86204	0.0007401309
Two-sided crack with fixed $p = 0.9$	35.69111	0.0007351726
Two-sided BS [12]	35.86204	0.000761061

Table 3 Biaxial fatigue life of metal pieces.

fatigue life										
125	127	135	137	185	187	190	190	195	200	212
242	245	255	283	316	327	355	373	386	456	482
552	580	700	736	745	750	804	852	884	977	1040
1066	1093	1114	1125	1300	1536	1583	2208	2266	2834	3280
4707	5046									

Table 4 Point estimates for Rieck's [13] data

Estimator	$\hat{\lambda}$	$\hat{\theta}$
Two-sided crack with fixed $p = 0$	0.3683172	0.005757928
Two-sided crack with fixed $p = 0.1$	0.4064855	0.003933762
Two-sided crack with fixed $p = 0.3$	0.3550685	0.0009263336
Two-sided crack with fixed $p = 0.5$	0.2410718	-0.009448762
Two-sided crack with fixed $p = 0.7$	0.09168065	-0.1534845
Two-sided crack with fixed $p = 0.9$	-0.08453449	-0.2921806
Two-sided crack with fixed $p = 1$	-0.1816241	-0.07596625
Two-sided BS [12]	0.2410718	0.02704306

7. Conclusions

The computational results ensure us that the method of moments estimators works and provides consistent statistics. It also indicate the two-sided crack distribution fits the real data.

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Appendix

Table 5 Simulation Results for $n = 10$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
2	1	0.1	3.1853	0.7551	1.1853	-0.2449	5.4652	0.1965
2	1	0.3	3.1231	0.6188	1.1231	-0.3812	4.8065	0.2093
2	1	0.5	3.0925	0.5027	1.0925	-0.4973	5.0931	0.2781
2	1	0.7	3.1119	0.3824	1.1119	-0.6176	5.0294	0.4392
2	1	0.9	3.1552	0.2232	1.1552	-0.7768	5.6312	1.7164
2	5	0.1	3.1499	3.7765	1.1499	-1.2235	5.5226	4.7662
2	5	0.3	3.0754	3.1056	1.0754	-1.8944	4.7884	5.1840
2	5	0.5	3.1322	2.4721	1.1322	-2.5279	4.8940	7.2343
2	5	0.7	3.1623	1.8692	1.1623	-3.1308	5.8871	13.4059
2	5	0.9	3.1484	1.2059	1.1484	-3.7941	5.5342	53.4954
2	10	0.1	3.1408	7.6539	1.1408	-2.3461	5.0793	19.4169
2	10	0.3	3.1136	6.1943	1.1136	-3.8057	4.8427	21.0399
2	10	0.5	3.1160	4.9880	1.1160	-5.0120	4.9473	28.2904
2	10	0.7	3.1311	3.7966	1.1311	-6.2034	4.8990	52.3647
2	10	0.9	3.1851	2.2925	1.1851	-7.7075	5.6604	176.4887
2	20	0.1	3.1695	15.0085	1.1695	-4.9915	5.1107	76.5150
2	20	0.3	3.1003	12.4336	1.1003	-7.5664	4.8347	83.0775
2	20	0.5	3.0385	10.0848	1.0385	-9.9152	4.3840	110.7879
2	20	0.7	3.1153	7.5331	1.1153	-12.4669	5.1470	200.8392
2	20	0.9	3.1767	5.1259	1.1767	-14.8741	5.7261	421.1890
2	50	0.1	3.1892	37.2545	1.1892	-12.7455	5.1869	480.4597
2	50	0.3	3.1581	30.8248	1.1581	-19.1752	5.1866	532.3039
2	50	0.5	3.1470	24.8072	1.1470	-25.1928	4.9460	721.9675
2	50	0.7	3.1424	18.5758	1.1424	-31.4242	5.4252	1622.0867
2	50	0.9	3.1938	12.5715	1.1938	-37.4285	5.7955	3165.7758
5	1	0.1	7.4269	0.8330	2.4269	-0.1670	28.1120	0.1821
5	1	0.3	7.3873	0.7670	2.3873	-0.2330	24.7694	0.1720
5	1	0.5	7.4781	0.7098	2.4781	-0.2902	31.2836	0.1787
5	1	0.7	7.4378	0.6530	2.4378	-0.3470	27.0892	0.1909
5	1	0.9	7.5874	0.5911	2.5874	-0.4089	28.0402	0.2181
5	5	0.1	7.4172	4.2041	2.4172	-0.7959	26.3466	4.8096
5	5	0.3	7.4133	3.8535	2.4133	-1.1465	27.7301	4.3860
5	5	0.5	7.3630	3.5737	2.3630	-1.4263	25.5022	4.4488
5	5	0.7	7.3839	3.2690	2.3839	-1.7310	26.7426	4.7386
5	5	0.9	7.5298	2.9800	2.5298	-2.0200	28.3959	5.4168
5	10	0.1	7.5500	8.3179	2.5500	-1.6821	29.2963	18.6744
5	10	0.3	7.3173	7.7548	2.3173	-2.2452	25.2256	17.1582
5	10	0.5	7.4017	7.1116	2.4017	-2.8884	26.9563	17.7781
5	10	0.7	7.4341	6.5022	2.4341	-3.4978	26.7063	19.1906
5	10	0.9	7.3322	6.0182	2.3322	-3.9818	25.1540	20.9051
5	20	0.1	7.4555	16.6968	2.4555	-3.3032	26.9712	75.3106
5	20	0.3	7.4886	15.1914	2.4886	-4.8086	26.0789	69.7945
5	20	0.5	7.3293	14.1907	2.3293	-5.8093	22.6684	71.0095
5	20	0.7	7.4307	13.0003	2.4307	-6.9997	26.7710	76.8301
5	20	0.9	7.3649	12.0380	2.3649	-7.9620	26.1624	84.2266
5	50	0.1	7.4736	41.5773	2.4736	-8.4227	28.3511	457.8102
5	50	0.3	7.4341	38.6020	2.4341	-11.3980	28.7482	436.7105
5	50	0.5	7.4270	35.6685	2.4270	-14.3315	30.1333	436.9847
5	50	0.7	7.4530	32.6076	2.4530	-17.3924	27.5745	480.9513
5	50	0.9	7.5116	29.8032	2.5116	-20.1968	29.8025	540.3471
10	1	0.1	14.3047	0.8693	4.3047	-0.1307	87.2673	0.1832
10	1	0.3	14.5297	0.8292	4.5297	-0.1708	98.0161	0.1750
10	1	0.5	14.5206	0.7895	4.5206	-0.2105	90.4070	0.1702
10	1	0.7	14.5701	0.7642	4.5701	-0.2358	104.1126	0.1678
10	1	0.9	14.6867	0.7336	4.6867	-0.2664	106.4748	0.1740
10	5	0.1	14.3217	4.3468	4.3217	-0.6532	85.7740	4.5331

Table 5 (Continued) : Simulation Results for $n = 10$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
10	5	0.3	14.4229	4.1899	4.4229	-0.8101	99.4806	4.4726
10	5	0.5	14.6313	3.9797	4.6313	-1.0203	104.8165	4.3370
10	5	0.7	14.5128	3.8652	4.5128	-1.1348	101.2032	4.3081
10	5	0.9	14.4433	3.7109	4.4433	-1.2891	110.7090	4.1924
10	10	0.1	14.7624	8.5719	4.7624	-1.4281	112.5142	18.8457
10	10	0.3	14.7593	8.2562	4.7593	-1.7438	108.2821	17.5611
10	10	0.5	14.5282	7.9622	4.5282	-2.0378	94.8652	17.2861
10	10	0.7	14.5171	7.6224	4.5171	-2.3776	97.8761	16.7780
10	10	0.9	14.6440	7.3541	4.6440	-2.6459	105.6667	17.2259
10	20	0.1	14.5876	17.3669	4.5876	-2.6331	106.9322	74.6460
10	20	0.3	14.2622	16.8565	4.2622	-3.1435	92.3330	70.3560
10	20	0.5	14.6123	15.8950	4.6123	-4.1050	110.5632	67.5213
10	20	0.7	14.4319	15.3537	4.4319	-4.6463	108.3009	67.2336
10	20	0.9	14.7649	14.6535	4.7649	-5.3465	123.1565	70.1562
10	50	0.1	14.6505	43.6086	4.6505	-6.3914	109.5795	483.2746
10	50	0.3	14.8079	41.3970	4.8079	-8.6030	110.8714	459.3223
10	50	0.5	14.6063	39.8140	4.6063	-10.1860	104.0158	421.4282
10	50	0.7	14.7160	38.1896	4.7160	-11.8104	108.7239	435.2705
10	50	0.9	14.6016	36.9080	4.6016	-13.0920	115.8893	433.0773
20	1	0.1	28.7059	0.8897	8.7059	-0.1103	417.8113	0.1915
20	1	0.3	28.9068	0.8725	8.9068	-0.1275	429.1626	0.1878
20	1	0.5	29.2144	0.8471	9.2144	-0.1529	540.6390	0.1806
20	1	0.7	28.6726	0.8277	8.6726	-0.1723	373.0582	0.1679
20	1	0.9	28.6420	0.8152	8.6420	-0.1848	373.7554	0.1702
20	5	0.1	29.2183	4.3987	9.2183	-0.6013	420.4266	4.9474
20	5	0.3	29.0134	4.2895	9.0134	-0.7105	396.8150	4.4899
20	5	0.5	29.0170	4.1945	9.0170	-0.8055	400.6196	4.3849
20	5	0.7	28.9024	4.1364	8.9024	-0.8636	384.3999	4.3975
20	5	0.9	28.6060	4.0690	8.6060	-0.9310	365.6722	4.1595
20	10	0.1	28.6885	8.8359	8.6885	-1.1641	373.4699	18.8439
20	10	0.3	28.9282	8.6428	8.9282	-1.3572	394.0579	18.8105
20	10	0.5	29.1389	8.3944	9.1389	-1.6056	421.0655	18.0267
20	10	0.7	28.9070	8.3108	8.9070	-1.6892	419.7232	17.2926
20	10	0.9	28.4198	8.2014	8.4198	-1.7986	374.5965	16.7405
20	20	0.1	28.6784	17.7816	8.6784	-2.2184	374.1109	77.1937
20	20	0.3	28.7525	17.2083	8.7525	-2.7917	369.4900	72.2152
20	20	0.5	29.0585	16.9035	9.0585	-3.0965	433.0661	70.2034
20	20	0.7	28.7328	16.5579	8.7328	-3.4421	393.6235	67.3829
20	20	0.9	28.8944	16.3066	8.8944	-3.6934	413.2893	68.6396
20	50	0.1	29.1116	44.3100	9.1116	-5.6900	466.6206	486.9272
20	50	0.3	28.9047	43.3964	8.9047	-6.6036	434.9809	464.6527
20	50	0.5	29.1485	42.0010	9.1485	-7.9990	426.7267	441.9250
20	50	0.7	28.4885	41.4552	8.4885	-8.5448	357.3053	417.9877
20	50	0.9	28.8791	40.6619	8.8791	-9.3381	387.2206	431.4579
50	1	0.1	71.3499	0.9007	21.3499	-0.0993	2560.0353	0.1921
50	1	0.3	71.8436	0.8883	21.8436	-0.1117	2629.3478	0.1903
50	1	0.5	71.7143	0.8723	21.7143	-0.1277	2368.5805	0.1807
50	1	0.7	70.0941	0.8840	20.0941	-0.1160	2076.6215	0.1869
50	1	0.9	72.6119	0.8610	22.6119	-0.1390	2587.6243	0.1864
50	5	0.1	71.6492	4.4838	21.6492	-0.5162	2551.6959	4.8485
50	5	0.3	71.7982	4.4299	21.7982	-0.5701	2450.9821	4.8012
50	5	0.5	71.6386	4.4135	21.6386	-0.5865	2490.5438	4.6713
50	5	0.7	72.2285	4.3349	22.2285	-0.6651	2518.6428	4.6085
50	5	0.9	71.9718	4.3162	21.9718	-0.6838	2571.9202	4.5175
50	10	0.1	71.5407	8.8651	21.5407	-1.1349	2164.8256	18.8049
50	10	0.3	72.0802	8.8537	22.0802	-1.1463	2486.2729	19.1472
50	10	0.5	71.2839	8.9000	21.2839	-1.1000	2698.7333	18.5432

Table 5 (Continued) : Simulation Results for $n = 10$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
50	10	0.7	72.4741	8.7095	22.4741	-1.2905	2667.2910	18.6835
50	10	0.9	71.8814	8.5863	21.8814	-1.4137	2390.7220	17.7402
50	20	0.1	70.8078	18.0272	20.8078	-1.9728	2241.7810	77.7742
50	20	0.3	71.5415	17.7627	21.5415	-2.2373	2758.4477	73.2588
50	20	0.5	70.4998	17.7266	20.4998	-2.2734	2193.5106	72.1906
50	20	0.7	71.7106	17.4746	21.7106	-2.5254	2531.0676	72.6588
50	20	0.9	71.0213	17.5956	21.0213	-2.4044	2373.3667	75.4110
50	50	0.1	71.6998	44.7018	21.6998	-5.2982	2413.4657	477.6310
50	50	0.3	71.2539	44.5701	21.2539	-5.4299	2387.9950	472.0592
50	50	0.5	71.3713	43.6358	21.3713	-6.3642	2317.2949	444.8553
50	50	0.7	71.5842	43.5715	21.5842	-6.4285	2426.0398	449.2774
50	50	0.9	72.0617	43.2875	22.0617	-6.7125	3123.8055	441.5736

Table 6-1 Simulation Results for $n = 50$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
2	1	0.1	2.2008	0.8738	0.2008	-0.1262	0.3464	0.0586
2	1	0.3	2.1980	0.7121	0.1980	-0.2879	0.3315	0.1031
2	1	0.5	2.1942	0.5771	0.1942	-0.4229	0.3438	0.1878
2	1	0.7	2.2098	0.4569	0.2098	-0.5431	0.3851	0.2993
2	1	0.9	2.2025	0.3530	0.2025	-0.6470	0.4027	0.4228
2	5	0.1	2.2030	4.3692	0.2030	-0.6308	0.3449	1.4388
2	5	0.3	2.2027	3.5452	0.2027	-1.4548	0.3217	2.6135
2	5	0.5	2.1890	2.8914	0.1890	-2.1086	0.3417	4.6707
2	5	0.7	2.2011	2.2941	0.2011	-2.7059	0.3672	7.4093
2	5	0.9	2.2093	1.7660	0.2093	-3.2340	0.4159	10.5462
2	10	0.1	2.2045	8.7159	0.2045	-1.2841	0.3467	5.6665
2	10	0.3	2.1989	7.1110	0.1989	-2.8890	0.3298	10.3360
2	10	0.5	2.1923	5.7762	0.1923	-4.2238	0.3356	18.7355
2	10	0.7	2.1827	4.5983	0.1827	-5.4017	0.3482	29.5123
2	10	0.9	2.2038	3.5209	0.2038	-6.4791	0.3977	43.1538
2	20	0.1	2.2096	17.4196	0.2096	-2.5804	0.3530	22.8059
2	20	0.3	2.1870	14.2818	0.1870	-5.7182	0.3183	40.5320
2	20	0.5	2.1894	11.5595	0.1894	-8.4405	0.3233	74.7440
2	20	0.7	2.1807	9.2316	0.1807	-10.7684	0.3384	117.2732
2	20	0.9	2.2033	7.0809	0.2033	-12.9191	0.3853	168.8002
2	50	0.1	2.1955	43.7138	0.1955	-6.2862	0.3325	138.0398
2	50	0.3	2.1984	35.5992	0.1984	-14.4008	0.3298	257.3861
2	50	0.5	2.1936	28.8359	0.1936	-21.1641	0.3342	469.8833
2	50	0.7	2.1947	22.9464	0.1947	-27.0536	0.3492	740.3435
2	50	0.9	2.2155	17.6453	0.2155	-32.3547	0.4220	1055.1268
5	1	0.1	5.3916	0.9369	0.3916	-0.0631	1.6809	0.0459
5	1	0.3	5.3691	0.8666	0.3691	-0.1334	1.6107	0.0495
5	1	0.5	5.3842	0.7962	0.3842	-0.2038	1.6353	0.0657
5	1	0.7	5.4293	0.7293	0.4293	-0.2707	1.7251	0.0916
5	1	0.9	5.3840	0.6769	0.3840	-0.3231	1.7915	0.1191
5	5	0.1	5.4087	4.6655	0.4087	-0.3345	1.6909	1.1281
5	5	0.3	5.4096	4.2998	0.4096	-0.7002	1.6510	1.2521
5	5	0.5	5.3987	3.9773	0.3987	-1.0227	1.6914	1.6633
5	5	0.7	5.3786	3.6731	0.3786	-1.3269	1.6917	2.2314
5	5	0.9	5.4157	3.3700	0.4157	-1.6300	1.8218	3.0201
5	10	0.1	5.3825	9.3841	0.3825	-0.6159	1.6764	4.6011
5	10	0.3	5.3815	8.6451	0.3815	-1.3549	1.6522	5.0605
5	10	0.5	5.3731	7.9719	0.3731	-2.0281	1.5871	6.4794
5	10	0.7	5.4126	7.3122	0.4126	-2.6878	1.7173	9.1363
5	10	0.9	5.4032	6.7376	0.4032	-3.2624	1.7133	12.0526

Table 6-2 (Continued) Simulation Results for $n = 50$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
5	20	0.1	5.3538	18.8744	0.3538	-1.1256	1.6786	18.4845
5	20	0.3	5.3969	17.2353	0.3969	-2.7647	1.6619	20.3410
5	20	0.5	5.3770	15.9406	0.3770	-4.0594	1.6130	26.4450
5	20	0.7	5.3975	14.6448	0.3975	-5.3552	1.6335	35.9495
5	20	0.9	5.4201	13.4692	0.4201	-6.5308	1.8376	48.5060
5	50	0.1	5.3955	46.8047	0.3955	-3.1953	1.7215	113.5838
5	50	0.3	5.3420	43.5195	0.3420	-6.4805	1.6145	125.2276
5	50	0.5	5.4186	39.6107	0.4186	-10.3893	1.7334	168.8810
5	50	0.7	5.3987	36.6944	0.3987	-13.3056	1.7145	224.5247
5	50	0.9	5.4045	33.7322	0.4045	-16.2678	1.8156	300.9989
10	1	0.1	10.6732	0.9618	0.6732	-0.0382	6.0529	0.0430
10	1	0.3	10.7428	0.9175	0.7428	-0.0825	6.1850	0.0426
10	1	0.5	10.7396	0.8797	0.7396	-0.1203	5.9807	0.0442
10	1	0.7	10.7495	0.8466	0.7495	-0.1534	6.1846	0.0521
10	1	0.9	10.6906	0.8168	0.6906	-0.1832	6.0961	0.0590
10	5	0.1	10.6672	4.8109	0.6672	-0.1891	6.0870	1.0785
10	5	0.3	10.6727	4.6119	0.6727	-0.3881	5.9135	1.0211
10	5	0.5	10.7558	4.4011	0.7558	-0.5989	6.1027	1.1381
10	5	0.7	10.6925	4.2410	0.6925	-0.7590	5.7912	1.2411
10	5	0.9	10.7231	4.0806	0.7231	-0.9194	6.5435	1.4905
10	10	0.1	10.7456	9.5418	0.7456	-0.4582	5.9787	4.1950
10	10	0.3	10.7280	9.1889	0.7280	-0.8111	6.0404	4.2572
10	10	0.5	10.7220	8.8412	0.7220	-1.1588	6.2866	4.5851
10	10	0.7	10.6439	8.5311	0.6439	-1.4689	5.6946	4.9077
10	10	0.9	10.6978	8.1597	0.6978	-1.8403	6.1070	5.8777
10	20	0.1	10.7160	19.1910	0.7160	-0.8090	6.2771	17.5934
10	20	0.3	10.6857	18.4615	0.6857	-1.5385	6.1046	17.1862
10	20	0.5	10.7078	17.6697	0.7078	-2.3303	6.0228	17.7686
10	20	0.7	10.7092	16.9564	0.7092	-3.0436	5.8168	20.0799
10	20	0.9	10.7222	16.2822	0.7222	-3.7178	6.2511	23.6884
10	50	0.1	10.7518	47.8117	0.7518	-2.1883	6.3731	110.2551
10	50	0.3	10.7236	45.9336	0.7236	-4.0664	6.0152	104.5120
10	50	0.5	10.7169	44.1663	0.7169	-5.8337	6.0546	112.4843
10	50	0.7	10.7159	42.3889	0.7159	-7.6111	5.9964	125.2522
10	50	0.9	10.6789	40.8635	0.6789	-9.1365	6.0397	146.4234
20	1	0.1	21.3413	0.9692	1.3413	-0.0308	22.8739	0.0406
20	1	0.3	21.2612	0.9539	1.2612	-0.0461	23.0337	0.0401
20	1	0.5	21.3172	0.9316	1.3172	-0.0684	22.8004	0.0397
20	1	0.7	21.4004	0.9101	1.4004	-0.0899	23.5323	0.0415
20	1	0.9	21.3181	0.8959	1.3181	-0.1041	23.1206	0.0426
20	5	0.1	21.4365	4.8276	1.4365	-0.1724	23.3673	1.0340
20	5	0.3	21.4476	4.7316	1.4476	-0.2684	24.0702	1.0182
20	5	0.5	21.2912	4.6669	1.2912	-0.3331	22.8588	1.0065
20	5	0.7	21.4288	4.5413	1.4288	-0.4587	22.7464	1.0243
20	5	0.9	21.3023	4.4738	1.3023	-0.5262	22.5522	1.0359
20	10	0.1	21.4119	9.6544	1.4119	-0.3456	23.2040	3.9854
20	10	0.3	21.3938	9.4662	1.3938	-0.5338	22.5204	3.8720
20	10	0.5	21.3061	9.3113	1.3061	-0.6887	22.1339	3.8653
20	10	0.7	21.3282	9.1379	1.3282	-0.8621	23.1014	4.1226
20	10	0.9	21.3701	8.9273	1.3701	-1.0727	22.6960	4.2765
20	20	0.1	21.3588	19.3935	1.3588	-0.6065	23.6243	16.9321
20	20	0.3	21.3438	18.9689	1.3438	-1.0311	21.9842	15.5735
20	20	0.5	21.4483	18.5283	1.4483	-1.4717	23.5670	16.0770
20	20	0.7	21.3048	18.3023	1.3048	-1.6977	23.1616	16.6587
20	20	0.9	21.4032	17.8481	1.4032	-2.1519	23.9391	17.2430
20	50	0.1	21.3337	48.4567	1.3337	-1.5433	22.5221	101.5575
20	50	0.3	21.3340	47.5087	1.3340	-2.4913	22.5964	98.7841
20	50	0.5	21.3834	46.4313	1.3834	-3.5687	22.8888	100.6110

Table 6-3 (Continued) Simulation Results for $n = 50$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
20	50	0.7	21.2636	45.7408	1.2636	-4.2592	22.4182	100.1175
20	50	0.9	21.3302	44.7740	1.3302	-5.2260	23.4276	106.7061
50	1	0.1	53.2599	0.9763	3.2599	-0.0237	142.1984	0.0409
50	1	0.3	53.2477	0.9681	3.2477	-0.0319	138.1550	0.0397
50	1	0.5	53.2847	0.9593	3.2847	-0.0407	136.7279	0.0392
50	1	0.7	53.1342	0.9549	3.1342	-0.0451	138.7436	0.0395
50	1	0.9	53.1350	0.9463	3.1350	-0.0537	133.3824	0.0388
50	5	0.1	53.3311	4.8782	3.3311	-0.1218	146.8944	1.0203
50	5	0.3	53.0590	4.8524	3.0590	-0.1476	132.6789	0.9765
50	5	0.5	53.3518	4.7879	3.3518	-0.2121	139.6218	0.9579
50	5	0.7	53.3981	4.7464	3.3981	-0.2536	137.5398	0.9651
50	5	0.9	53.1394	4.7298	3.1394	-0.2702	133.6584	0.9535
50	10	0.1	53.1377	9.7834	3.1377	-0.2166	140.0171	4.0517
50	10	0.3	53.1857	9.6873	3.1857	-0.3127	138.9848	3.8785
50	10	0.5	53.0658	9.6209	3.0658	-0.3791	132.1159	3.7926
50	10	0.7	53.2491	9.5156	3.2491	-0.4844	135.7625	3.8624
50	10	0.9	53.2154	9.4573	3.2154	-0.5427	138.2410	3.9399
50	20	0.1	53.1656	19.5635	3.1656	-0.4365	140.1232	16.4211
50	20	0.3	53.3042	19.3540	3.3042	-0.6460	140.4593	16.1576
50	20	0.5	53.3334	19.1712	3.3334	-0.8288	138.4936	15.4834
50	20	0.7	53.2217	19.0650	3.2217	-0.9350	137.9740	15.8858
50	20	0.9	53.6534	18.7669	3.6534	-1.2331	144.7583	15.8091
50	50	0.1	53.0421	49.0120	3.0421	-0.9880	140.5307	101.3126
50	50	0.3	53.3491	48.3058	3.3491	-1.6942	140.8540	97.1496
50	50	0.5	53.5462	47.7937	3.5462	-2.2063	145.4531	100.2113
50	50	0.7	52.9701	47.8554	2.9701	-2.1446	133.8940	97.5666
50	50	0.9	53.2744	47.2293	3.2744	-2.7707	138.9871	99.1340

Table 7-1 Simulation Results for $n = 100$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
2	1	0.1	2.1008	0.8901	0.1008	-0.1099	0.1576	0.0358
2	1	0.3	2.1033	0.7245	0.1033	-0.2755	0.1542	0.0870
2	1	0.5	2.0931	0.5891	0.0931	-0.4109	0.1470	0.1736
2	1	0.7	2.1010	0.4700	0.1010	-0.5300	0.1643	0.2828
2	1	0.9	2.1066	0.3664	0.1066	-0.6336	0.1876	0.4023
2	5	0.1	2.1038	4.4353	0.1038	-0.5647	0.1549	0.8558
2	5	0.3	2.0940	3.6327	0.0940	-1.3673	0.1459	2.1412
2	5	0.5	2.0993	2.9390	0.0993	-2.0610	0.1544	4.3698
2	5	0.7	2.0969	2.3515	0.0969	-2.6485	0.1608	7.0572
2	5	0.9	2.0942	1.8360	0.0942	-3.1640	0.1752	10.0288
2	10	0.1	2.1009	8.8870	0.1009	-1.1130	0.1523	3.4204
2	10	0.3	2.0974	7.2549	0.0974	-2.7451	0.1469	8.6102
2	10	0.5	2.0903	5.8942	0.0903	-4.1058	0.1478	17.3383
2	10	0.7	2.0937	4.7047	0.0937	-5.2953	0.1554	28.2134
2	10	0.9	2.1079	3.6650	0.1079	-6.3350	0.1825	40.2061
2	20	0.1	2.1139	17.6886	0.1139	-2.3114	0.1551	14.1876
2	20	0.3	2.0984	14.5052	0.0984	-5.4948	0.1498	34.5677
2	20	0.5	2.0996	11.7590	0.0996	-8.2410	0.1511	69.8604
2	20	0.7	2.0973	9.3986	0.0973	-10.6014	0.1601	113.0883
2	20	0.9	2.1051	7.3263	0.1051	-12.6737	0.1818	160.9152
2	50	0.1	2.0948	44.5287	0.0948	-5.4713	0.1548	85.5551
2	50	0.3	2.0950	36.3294	0.0950	-13.6706	0.1470	214.7254
2	50	0.5	2.0862	29.4969	0.0862	-20.5031	0.1491	432.6398
2	50	0.7	2.0894	23.5161	0.0894	-26.4839	0.1551	705.7101
2	50	0.9	2.1020	18.3355	0.1020	-31.6645	0.1815	1004.5408

Table 7-2 (Continued) Simulation Results for $n = 100$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
5	1	0.1	5.2004	0.9478	0.2004	-0.0522	0.7670	0.0248
5	1	0.3	5.2156	0.8710	0.2156	-0.1290	0.7274	0.0327
5	1	0.5	5.1801	0.8079	0.1801	-0.1921	0.6936	0.0493
5	1	0.7	5.1900	0.7442	0.1900	-0.2558	0.7265	0.0751
5	1	0.9	5.1859	0.6856	0.1859	-0.3144	0.7609	0.1063
5	5	0.1	5.2172	4.7282	0.2172	-0.2718	0.7745	0.6306
5	5	0.3	5.1941	4.3769	0.1941	-0.6231	0.7258	0.7949
5	5	0.5	5.1781	4.0460	0.1781	-0.9540	0.7238	1.2330
5	5	0.7	5.1845	3.7215	0.1845	-1.2785	0.7057	1.8716
5	5	0.9	5.1946	3.4244	0.1946	-1.5756	0.7641	2.6671
5	10	0.1	5.2072	9.4698	0.2072	-0.5302	0.7628	2.4834
5	10	0.3	5.1916	8.7548	0.1916	-1.2452	0.7095	3.1790
5	10	0.5	5.1656	8.0970	0.1656	-1.9030	0.6784	4.8440
5	10	0.7	5.1719	7.4566	0.1719	-2.5434	0.7195	7.4129
5	10	0.9	5.2036	6.8418	0.2036	-3.1582	0.7922	10.7443
5	20	0.1	5.2148	18.8963	0.2148	-1.1037	0.7242	9.4987
5	20	0.3	5.1926	17.5149	0.1926	-2.4851	0.7469	12.9780
5	20	0.5	5.1911	16.1300	0.1911	-3.8700	0.7124	19.9501
5	20	0.7	5.1863	14.8865	0.1863	-5.1135	0.7036	29.8845
5	20	0.9	5.2000	13.7024	0.2000	-6.2976	0.7782	42.7325
5	50	0.1	5.1804	47.5289	0.1804	-2.4711	0.7215	59.1315
5	50	0.3	5.1968	43.7328	0.1968	-6.2672	0.7213	80.1411
5	50	0.5	5.1949	40.3278	0.1949	-9.6722	0.7277	124.9289
5	50	0.7	5.1975	37.1914	0.1975	-12.8086	0.7726	189.3742
5	50	0.9	5.1850	34.3153	0.1850	-15.6847	0.7930	265.4167
10	1	0.1	10.3230	0.9720	0.3230	-0.0280	2.6500	0.0222
10	1	0.3	10.3382	0.9316	0.3382	-0.0684	2.5993	0.0229
10	1	0.5	10.3337	0.8949	0.3337	-0.1051	2.5950	0.0270
10	1	0.7	10.3410	0.8588	0.3410	-0.1412	2.6318	0.0340
10	1	0.9	10.3456	0.8248	0.3456	-0.1752	2.5939	0.0429
10	5	0.1	10.3511	4.8465	0.3511	-0.1535	2.6979	0.5497
10	5	0.3	10.3467	4.6498	0.3467	-0.3502	2.5521	0.5678
10	5	0.5	10.3210	4.4796	0.3210	-0.5204	2.5742	0.6706
10	5	0.7	10.3619	4.2909	0.3619	-0.7091	2.6685	0.8641
10	5	0.9	10.3914	4.1135	0.3914	-0.8865	2.8304	1.1136
10	10	0.1	10.3240	9.7146	0.3240	-0.2854	2.6228	2.1473
10	10	0.3	10.3325	9.3230	0.3325	-0.6770	2.6174	2.3272
10	10	0.5	10.3572	8.9291	0.3572	-1.0709	2.5875	2.7286
10	10	0.7	10.3517	8.5819	0.3517	-1.4181	2.6749	3.4120
10	10	0.9	10.3356	8.2627	0.3356	-1.7373	2.7084	4.3118
10	20	0.1	10.3250	19.4248	0.3250	-0.5752	2.5398	8.4516
10	20	0.3	10.3016	18.6776	0.3016	-1.3224	2.5046	8.9037
10	20	0.5	10.3821	17.8336	0.3821	-2.1664	2.6649	11.1272
10	20	0.7	10.3446	17.1867	0.3446	-2.8133	2.6185	13.5903
10	20	0.9	10.3814	16.4543	0.3814	-3.5457	2.7660	17.6688
10	50	0.1	10.3438	48.4536	0.3438	-1.5464	2.6115	53.8121
10	50	0.3	10.3560	46.5314	0.3560	-3.4686	2.6879	58.5720
10	50	0.5	10.3227	44.7591	0.3227	-5.2409	2.4452	65.7103
10	50	0.7	10.3314	43.0107	0.3314	-6.9893	2.6477	85.2185
10	50	0.9	10.3530	41.2344	0.3530	-8.7656	2.6697	108.3043
20	1	0.1	20.5949	0.9817	0.5949	-0.0183	9.6383	0.0205
20	1	0.3	20.6387	0.9602	0.6387	-0.0398	9.6639	0.0200
20	1	0.5	20.7258	0.9380	0.7258	-0.0620	10.1197	0.0216
20	1	0.7	20.6183	0.9243	0.6183	-0.0757	10.0460	0.0230
20	1	0.9	20.6503	0.9040	0.6503	-0.0960	9.8970	0.0251
20	5	0.1	20.6822	4.8931	0.6822	-0.1069	10.0698	0.5255
20	5	0.3	20.6657	4.7965	0.6657	-0.2035	9.7866	0.5087
20	5	0.5	20.6839	4.6989	0.6839	-0.3011	9.9385	0.5380

Table 7-3 (Continued) Simulation Results for $n = 100$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
20	5	0.7	20.7005	4.6038	0.7005	-0.3962	10.1801	0.5809
20	5	0.9	20.6710	4.5215	0.6710	-0.4785	10.5055	0.6480
20	10	0.1	20.6875	9.7841	0.6875	-0.2159	10.0491	2.0749
20	10	0.3	20.7642	9.5446	0.7642	-0.4554	9.9334	2.0354
20	10	0.5	20.6232	9.4279	0.6232	-0.5721	10.0398	2.1383
20	10	0.7	20.6222	9.2328	0.6222	-0.7672	10.1015	2.3191
20	10	0.9	20.6322	9.0476	0.6322	-0.9524	9.8935	2.4801
20	20	0.1	20.5731	19.6702	0.5731	-0.3298	9.8212	8.4064
20	20	0.3	20.6576	19.1893	0.6576	-0.8107	9.7402	8.0677
20	20	0.5	20.6245	18.8391	0.6245	-1.1609	9.6318	8.3513
20	20	0.7	20.6472	18.4506	0.6472	-1.5494	9.8212	9.1124
20	20	0.9	20.6377	18.0858	0.6377	-1.9142	9.7831	9.9416
20	50	0.1	20.7151	48.8423	0.7151	-1.1577	10.1605	51.7958
20	50	0.3	20.5457	48.2540	0.5457	-1.7460	9.5970	51.1297
20	50	0.5	20.6249	47.0779	0.6249	-2.9221	9.6399	53.0815
20	50	0.7	20.6751	46.0890	0.6751	-3.9110	10.1358	57.2629
20	50	0.9	20.6413	45.2366	0.6413	-4.7634	10.0629	62.9848
50	1	0.1	51.6628	0.9843	1.6628	-0.0157	60.1677	0.0201
50	1	0.3	51.4497	0.9803	1.4497	-0.0197	59.3034	0.0201
50	1	0.5	51.7160	0.9675	1.7160	-0.0325	59.2288	0.0196
50	1	0.7	51.5228	0.9638	1.5228	-0.0362	61.4466	0.0206
50	1	0.9	51.7746	0.9515	1.7746	-0.0485	62.3935	0.0206
50	5	0.1	51.6224	4.9248	1.6224	-0.0752	59.6085	0.5056
50	5	0.3	51.5961	4.8877	1.5961	-0.1123	59.9215	0.4881
50	5	0.5	51.6380	4.8514	1.6380	-0.1486	64.4005	0.5261
50	5	0.7	51.7268	4.7991	1.7268	-0.2009	61.3706	0.5078
50	5	0.9	51.5271	4.7786	1.5271	-0.2214	60.5759	0.5069
50	10	0.1	51.5224	9.8750	1.5224	-0.1250	60.8840	2.1061
50	10	0.3	51.8382	9.7409	1.8382	-0.2591	64.0959	2.0619
50	10	0.5	51.6256	9.6900	1.6256	-0.3100	59.2610	1.9680
50	10	0.7	51.6332	9.6133	1.6332	-0.3867	60.6028	1.9911
50	10	0.9	51.4472	9.5736	1.4472	-0.4264	60.1952	2.0191
50	20	0.1	51.7134	19.6631	1.7134	-0.3369	59.9332	8.0347
50	20	0.3	51.8343	19.4659	1.8343	-0.5341	61.5694	8.0075
50	20	0.5	51.3610	19.4843	1.3610	-0.5157	59.2228	7.8303
50	20	0.7	51.4763	19.2874	1.4763	-0.7126	59.1351	8.0359
50	20	0.9	51.6404	19.0746	1.6404	-0.9254	59.6829	8.1649
50	50	0.1	51.4926	49.3810	1.4926	-0.6190	59.5726	50.6393
50	50	0.3	51.5176	48.9509	1.5176	-1.0491	58.7331	48.7955
50	50	0.5	51.8381	48.2691	1.8381	-1.7309	61.7338	50.2385
50	50	0.7	51.5955	48.1006	1.5955	-1.8994	59.0513	49.9677
50	50	0.9	51.5815	47.7741	1.5815	-2.2259	61.8902	52.3970

Table 8-1 Simulation Results for $n = 200$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
2	1	0.1	2.0528	0.8967	0.0528	-0.1033	0.0764	0.0229
2	1	0.3	2.0460	0.7331	0.0460	-0.2669	0.0684	0.0768
2	1	0.5	2.0530	0.5929	0.0530	-0.4071	0.0719	0.1684
2	1	0.7	2.0451	0.4761	0.0451	-0.5239	0.0772	0.2754
2	1	0.9	2.0460	0.3735	0.0460	-0.6265	0.0838	0.3928
2	5	0.1	2.0494	4.4858	0.0494	-0.5142	0.0724	0.5521
2	5	0.3	2.0457	3.6648	0.0457	-1.3352	0.0686	1.9227
2	5	0.5	2.0534	2.9661	0.0534	-2.0339	0.0732	4.2025
2	5	0.7	2.0470	2.3789	0.0470	-2.6211	0.0792	6.8944
2	5	0.9	2.0515	1.8643	0.0515	-3.1357	0.0849	9.8394

Table 8-2 (Continued) Simulation Results for $n = 200$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
2	10	0.1	2.0461	8.9794	0.0461	-1.0206	0.0713	2.1718
2	10	0.3	2.0410	7.3399	0.0410	-2.6601	0.0678	7.6381
2	10	0.5	2.0512	5.9376	0.0512	-4.0624	0.0718	16.7639
2	10	0.7	2.0520	4.7543	0.0520	-5.2457	0.0762	27.6085
2	10	0.9	2.0488	3.7303	0.0488	-6.2697	0.0842	39.3367
2	20	0.1	2.0574	17.9032	0.0574	-2.0968	0.0743	9.0530
2	20	0.3	2.0469	14.6602	0.0469	-5.3398	0.0692	30.8039
2	20	0.5	2.0467	11.8913	0.0467	-8.1087	0.0697	66.7856
2	20	0.7	2.0473	9.5188	0.0473	-10.4812	0.0759	110.2216
2	20	0.9	2.0531	7.4577	0.0531	-12.5423	0.0842	157.4129
2	50	0.1	2.0567	44.7545	0.0567	-5.2455	0.0746	56.7380
2	50	0.3	2.0522	36.5533	0.0522	-13.4467	0.0711	194.8925
2	50	0.5	2.0451	29.7416	0.0451	-20.2584	0.0686	416.6594
2	50	0.7	2.0535	23.7693	0.0535	-26.2307	0.0767	690.3681
2	50	0.9	2.0475	18.6479	0.0475	-31.3521	0.0872	983.6510
5	1	0.1	5.0915	0.9558	0.0915	-0.0442	0.3515	0.0131
5	1	0.3	5.1056	0.8786	0.1056	-0.1214	0.3357	0.0229
5	1	0.5	5.0951	0.8129	0.0951	-0.1871	0.3432	0.0416
5	1	0.7	5.0959	0.7486	0.0959	-0.2514	0.3447	0.0681
5	1	0.9	5.1078	0.6893	0.1078	-0.3107	0.3785	0.1005
5	5	0.1	5.0832	4.7851	0.0832	-0.2149	0.3388	0.3181
5	5	0.3	5.1010	4.3974	0.1010	-0.6026	0.3303	0.5647
5	5	0.5	5.0894	4.0684	0.0894	-0.9316	0.3296	1.0277
5	5	0.7	5.0927	3.7467	0.0927	-1.2533	0.3461	1.6947
5	5	0.9	5.1121	3.4432	0.1121	-1.5568	0.3729	2.5202
5	10	0.1	5.0947	9.5465	0.0947	-0.4535	0.3436	1.2878
5	10	0.3	5.0894	8.8173	0.0894	-1.1827	0.3299	2.2203
5	10	0.5	5.0809	8.1377	0.0809	-1.8623	0.3210	4.0892
5	10	0.7	5.0954	7.4882	0.0954	-2.5118	0.3402	6.7926
5	10	0.9	5.0900	6.9060	0.0900	-3.0940	0.3626	9.9636
5	20	0.1	5.1092	19.0570	0.1092	-0.9430	0.3666	5.5418
5	20	0.3	5.0985	17.6123	0.0985	-2.3877	0.3337	8.9918
5	20	0.5	5.1010	16.2395	0.1010	-3.7605	0.3484	16.7635
5	20	0.7	5.0979	14.9741	0.0979	-5.0259	0.3375	27.1905
5	20	0.9	5.0882	13.8074	0.0882	-6.1926	0.3613	39.9078
5	50	0.1	5.0897	47.7911	0.0897	-2.2089	0.3418	32.5515
5	50	0.3	5.0918	44.0843	0.0918	-5.9157	0.3411	56.0588
5	50	0.5	5.0942	40.6377	0.0942	-9.3623	0.3374	103.8521
5	50	0.7	5.0812	37.5244	0.0812	-12.4756	0.3363	167.9578
5	50	0.9	5.0993	34.4986	0.0993	-15.5014	0.3591	249.8315
10	1	0.1	10.1815	0.9735	0.1815	-0.0265	1.2133	0.0111
10	1	0.3	10.1707	0.9363	0.1707	-0.0637	1.2165	0.0133
10	1	0.5	10.1930	0.8976	0.1930	-0.1024	1.2460	0.0186
10	1	0.7	10.1823	0.8638	0.1823	-0.1362	1.2965	0.0260
10	1	0.9	10.1833	0.8291	0.1833	-0.1709	1.2959	0.0358
10	5	0.1	10.1805	4.8696	0.1805	-0.1304	1.2441	0.2811
10	5	0.3	10.1762	4.6781	0.1762	-0.3219	1.2093	0.3292
10	5	0.5	10.1650	4.4984	0.1650	-0.5016	1.1852	0.4460
10	5	0.7	10.1802	4.3161	0.1802	-0.6839	1.2142	0.6436
10	5	0.9	10.1695	4.1512	0.1695	-0.8488	1.2510	0.8821
10	10	0.1	10.1482	9.7705	0.1482	-0.2295	1.2193	1.1080
10	10	0.3	10.1904	9.3475	0.1904	-0.6525	1.2254	1.3460
10	10	0.5	10.1696	8.9978	0.1696	-1.0022	1.2031	1.8060
10	10	0.7	10.1714	8.6380	0.1714	-1.3620	1.2083	2.5578
10	10	0.9	10.1530	8.3110	0.1530	-1.6890	1.2690	3.5045
10	20	0.1	10.1631	19.5106	0.1631	-0.4894	1.2366	4.4552
10	20	0.3	10.1856	18.7095	0.1856	-1.2905	1.2573	5.4148
10	20	0.5	10.1707	17.9850	0.1707	-2.0150	1.1973	7.2523

Table 8-3 (Continued) Simulation Results for $n = 200$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
10	20	0.7	10.1540	17.3050	0.1540	-2.6950	1.2092	10.1224
10	20	0.9	10.1756	16.5846	0.1756	-3.4154	1.2315	14.1685
10	50	0.1	10.1811	48.6943	0.1811	-1.3057	1.2208	27.5285
10	50	0.3	10.1780	46.7920	0.1780	-3.2080	1.2531	33.7373
10	50	0.5	10.1794	44.9438	0.1794	-5.0562	1.2634	46.5861
10	50	0.7	10.1605	43.2348	0.1605	-6.7652	1.2194	63.5904
10	50	0.9	10.1853	41.4600	0.1853	-8.5400	1.2481	88.8323
20	1	0.1	20.3359	0.9844	0.3359	-0.0156	4.7235	0.0108
20	1	0.3	20.2522	0.9685	0.2522	-0.0315	4.4696	0.0105
20	1	0.5	20.3934	0.9426	0.3934	-0.0574	4.6228	0.0121
20	1	0.7	20.3224	0.9272	0.3224	-0.0728	4.5725	0.0137
20	1	0.9	20.3310	0.9087	0.3310	-0.0913	4.7491	0.0165
20	5	0.1	20.3750	4.9123	0.3750	-0.0877	4.7324	0.2665
20	5	0.3	20.3033	4.8299	0.3033	-0.1701	4.5800	0.2698
20	5	0.5	20.3869	4.7154	0.3869	-0.2846	4.6621	0.3031
20	5	0.7	20.3249	4.6352	0.3249	-0.3648	4.5129	0.3419
20	5	0.9	20.2466	4.5602	0.2466	-0.4398	4.5860	0.3981
20	10	0.1	20.3264	9.8451	0.3264	-0.1549	4.5672	1.0180
20	10	0.3	20.3719	9.6303	0.3719	-0.3697	4.6120	1.0900
20	10	0.5	20.3722	9.4434	0.3722	-0.5566	4.8667	1.2479
20	10	0.7	20.2878	9.2919	0.2878	-0.7081	4.6660	1.3716
20	10	0.9	20.2895	9.1066	0.2895	-0.8934	4.8255	1.6389
20	20	0.1	20.3322	19.6848	0.3322	-0.3152	4.4099	3.9817
20	20	0.3	20.3332	19.2935	0.3332	-0.7065	4.6131	4.3481
20	20	0.5	20.3378	18.9089	0.3378	-1.0911	4.5731	4.7535
20	20	0.7	20.3323	18.5392	0.3323	-1.4608	4.6567	5.5488
20	20	0.9	20.3483	18.1613	0.3483	-1.8387	4.6904	6.5863
20	50	0.1	20.3842	49.0906	0.3842	-0.9094	4.6848	26.4050
20	50	0.3	20.3421	48.2101	0.3421	-1.7899	4.5343	27.1846
20	50	0.5	20.3079	47.3069	0.3079	-2.6931	4.3433	28.5582
20	50	0.7	20.3179	46.3771	0.3179	-3.6229	4.6183	34.3898
20	50	0.9	20.3269	45.4425	0.3269	-4.5575	4.6637	40.8439
50	1	0.1	50.7357	0.9915	0.7357	-0.0085	27.1533	0.0099
50	1	0.3	50.6995	0.9841	0.6995	-0.0159	26.8452	0.0099
50	1	0.5	50.9453	0.9721	0.9453	-0.0279	28.4424	0.0104
50	1	0.7	50.9466	0.9641	0.9466	-0.0359	28.0030	0.0105
50	1	0.9	50.8182	0.9590	0.8182	-0.0410	28.3475	0.0110
50	5	0.1	50.7329	4.9586	0.7329	-0.0414	27.2190	0.2490
50	5	0.3	50.6883	4.9247	0.6883	-0.0753	27.8446	0.2530
50	5	0.5	50.8915	4.8637	0.8915	-0.1363	27.5687	0.2534
50	5	0.7	50.7329	4.8402	0.7329	-0.1598	27.2399	0.2556
50	5	0.9	50.7623	4.8002	0.7623	-0.1998	27.8432	0.2701
50	10	0.1	50.6705	9.9275	0.6705	-0.0725	26.4886	0.9837
50	10	0.3	50.9294	9.8005	0.9294	-0.1995	27.5778	0.9975
50	10	0.5	50.9350	9.7220	0.9350	-0.2780	28.5851	1.0428
50	10	0.7	50.8244	9.6654	0.8244	-0.3346	28.0574	1.0465
50	10	0.9	50.7103	9.6108	0.7103	-0.3892	28.1603	1.0851
50	20	0.1	50.7498	19.8305	0.7498	-0.1695	27.7509	4.0412
50	20	0.3	50.7520	19.6817	0.7520	-0.3183	29.2924	4.2310
50	20	0.5	50.8101	19.4883	0.8101	-0.5117	27.3516	3.9978
50	20	0.7	50.9056	19.3005	0.9056	-0.6995	28.0533	4.2337
50	20	0.9	50.7151	19.2185	0.7151	-0.7815	28.2428	4.3480
50	50	0.1	50.6787	49.6386	0.6787	-0.3614	27.0795	24.9514
50	50	0.3	50.7107	49.2158	0.7107	-0.7842	27.2667	25.0151
50	50	0.5	50.7532	48.7634	0.7532	-1.2366	26.8253	24.7421
50	50	0.7	50.7775	48.3614	0.7775	-1.6386	27.4480	25.9053
50	50	0.9	50.7580	48.0026	0.7580	-1.9974	28.0163	26.8914

Table 9-1 Simulation Results for $n = 500$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
2	1	0.1	2.0208	0.9014	0.0208	-0.0986	0.0294	0.0146
2	1	0.3	2.0180	0.7370	0.0180	-0.2630	0.0274	0.0715
2	1	0.5	2.0172	0.5980	0.0172	-0.4020	0.0276	0.1627
2	1	0.7	2.0209	0.4789	0.0209	-0.5211	0.0306	0.2719
2	1	0.9	2.0146	0.3767	0.0146	-0.6233	0.0333	0.3885
2	5	0.1	2.0218	4.5051	0.0218	-0.4949	0.0287	0.3644
2	5	0.3	2.0201	3.6827	0.0201	-1.3173	0.0272	1.7940
2	5	0.5	2.0184	2.9886	0.0184	-2.0114	0.0274	4.0719
2	5	0.7	2.0213	2.3955	0.0213	-2.6045	0.0293	6.7929
2	5	0.9	2.0200	1.8842	0.0200	-3.1158	0.0330	9.7106
2	10	0.1	2.0213	9.0077	0.0213	-0.9923	0.0279	1.4431
2	10	0.3	2.0206	7.3634	0.0206	-2.6366	0.0262	7.1768
2	10	0.5	2.0188	5.9777	0.0188	-4.0223	0.0272	16.2832
2	10	0.7	2.0208	4.7920	0.0208	-5.2080	0.0303	27.1633
2	10	0.9	2.0205	3.7666	0.0205	-6.2334	0.0331	38.8650
2	20	0.1	2.0174	18.0496	0.0174	-1.9504	0.0285	5.6996
2	20	0.3	2.0239	14.7071	0.0239	-5.2929	0.0271	28.9459
2	20	0.5	2.0185	11.9539	0.0185	-8.0461	0.0272	65.1563
2	20	0.7	2.0185	9.5822	0.0185	-10.4178	0.0297	108.6863
2	20	0.9	2.0185	7.5399	0.0185	-12.4601	0.0319	155.2906
2	50	0.1	2.0226	45.0172	0.0226	-4.9828	0.0283	36.6836
2	50	0.3	2.0199	36.7995	0.0199	-13.2005	0.0268	179.9762
2	50	0.5	2.0194	29.8759	0.0194	-20.1241	0.0272	407.6160
2	50	0.7	2.0215	23.9525	0.0215	-26.0475	0.0283	679.4163
2	50	0.9	2.0184	18.8425	0.0184	-31.1575	0.0333	971.0154
5	1	0.1	5.0342	0.9591	0.0342	-0.0409	0.1329	0.0061
5	1	0.3	5.0380	0.8845	0.0380	-0.1155	0.1301	0.0167
5	1	0.5	5.0292	0.8170	0.0292	-0.1830	0.1250	0.0360
5	1	0.7	5.0429	0.7517	0.0429	-0.2483	0.1356	0.0637
5	1	0.9	5.0385	0.6930	0.0385	-0.3070	0.1410	0.0958
5	5	0.1	5.0410	4.7891	0.0410	-0.2109	0.1361	0.1567
5	5	0.3	5.0339	4.4257	0.0339	-0.5743	0.1296	0.4139
5	5	0.5	5.0376	4.0795	0.0376	-0.9205	0.1317	0.9125
5	5	0.7	5.0316	3.7642	0.0316	-1.2358	0.1343	1.5787
5	5	0.9	5.0429	3.4625	0.0429	-1.5375	0.1431	2.4031
5	10	0.1	5.0334	9.5944	0.0334	-0.4056	0.1397	0.6278
5	10	0.3	5.0429	8.8385	0.0429	-1.1615	0.1307	1.6861
5	10	0.5	5.0410	8.1559	0.0410	-1.8441	0.1315	3.6587
5	10	0.7	5.0363	7.5218	0.0363	-2.4782	0.1317	6.3423
5	10	0.9	5.0406	6.9263	0.0406	-3.0737	0.1390	9.6031
5	20	0.1	5.0363	19.1686	0.0363	-0.8314	0.1339	2.4689
5	20	0.3	5.0474	17.6569	0.0474	-2.3431	0.1316	6.8181
5	20	0.5	5.0391	16.3133	0.0391	-3.6867	0.1286	14.6255
5	20	0.7	5.0377	15.0475	0.0377	-4.9525	0.1393	25.3682
5	20	0.9	5.0397	13.8584	0.0397	-6.1416	0.1425	38.3630
5	50	0.1	5.0431	47.8747	0.0431	-2.1253	0.1364	15.7402
5	50	0.3	5.0371	44.2324	0.0371	-5.7676	0.1335	41.9598
5	50	0.5	5.0459	40.7383	0.0459	-9.2617	0.1343	92.3858
5	50	0.7	5.0319	37.6477	0.0319	-12.3523	0.1349	157.7029
5	50	0.9	5.0354	34.6569	0.0354	-15.3431	0.1405	239.2764
10	1	0.1	10.0885	0.9761	0.0885	-0.0239	0.4886	0.0048
10	1	0.3	10.0653	0.9400	0.0653	-0.0600	0.4685	0.0073
10	1	0.5	10.0623	0.9030	0.0623	-0.0970	0.4512	0.0126
10	1	0.7	10.0704	0.8668	0.0704	-0.1332	0.4802	0.0207
10	1	0.9	10.0571	0.8337	0.0571	-0.1663	0.4777	0.0302
10	5	0.1	10.0758	4.8870	0.0758	-0.1130	0.4735	0.1186
10	5	0.3	10.0689	4.6977	0.0689	-0.3023	0.4499	0.1800
10	5	0.5	10.0637	4.5152	0.0637	-0.4848	0.4669	0.3176

Table 9-2 (Continued) Simulation Results for $n = 500$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
10	5	0.7	10.0757	4.3324	0.0757	-0.6676	0.4688	0.5173
10	5	0.9	10.0611	4.1673	0.0611	-0.8327	0.4789	0.7580
10	10	0.1	10.0669	9.7822	0.0669	-0.2178	0.5032	0.4947
10	10	0.3	10.0780	9.3867	0.0780	-0.6133	0.4607	0.7399
10	10	0.5	10.0558	9.0357	0.0558	-0.9643	0.4496	1.2461
10	10	0.7	10.0654	8.6729	0.0654	-1.3271	0.4594	2.0444
10	10	0.9	10.0752	8.3228	0.0752	-1.6772	0.4781	3.0699
10	20	0.1	10.0746	19.5512	0.0746	-0.4488	0.4676	1.8635
10	20	0.3	10.0668	18.7946	0.0668	-1.2054	0.4724	2.9418
10	20	0.5	10.0722	18.0490	0.0722	-1.9510	0.4591	5.0807
10	20	0.7	10.0804	17.3232	0.0804	-2.6768	0.4736	8.3205
10	20	0.9	10.0715	16.6579	0.0715	-3.3421	0.4958	12.2319
10	50	0.1	10.0638	48.9233	0.0638	-1.0767	0.4664	11.6694
10	50	0.3	10.0483	47.0755	0.0483	-2.9245	0.4660	17.8674
10	50	0.5	10.0810	45.0810	0.0810	-4.9190	0.4806	32.5329
10	50	0.7	10.0793	43.3062	0.0793	-6.6938	0.4676	51.9238
10	50	0.9	10.0703	41.6391	0.0703	-8.3609	0.4809	76.3844
20	1	0.1	20.1150	0.9885	0.1150	-0.0115	1.7352	0.0042
20	1	0.3	20.1341	0.9682	0.1341	-0.0318	1.8333	0.0050
20	1	0.5	20.1353	0.9489	0.1353	-0.0511	1.7879	0.0063
20	1	0.7	20.1171	0.9310	0.1171	-0.0690	1.7283	0.0081
20	1	0.9	20.1250	0.9120	0.1250	-0.0880	1.7777	0.0110
20	5	0.1	20.1211	4.9417	0.1211	-0.0583	1.7721	0.1067
20	5	0.3	20.1467	4.8365	0.1467	-0.1635	1.7203	0.1203
20	5	0.5	20.1342	4.7459	0.1342	-0.2541	1.7981	0.1571
20	5	0.7	20.1407	4.6490	0.1407	-0.3510	1.7762	0.2086
20	5	0.9	20.1285	4.5596	0.1285	-0.4404	1.7698	0.2742
20	10	0.1	20.1348	9.8803	0.1348	-0.1197	1.8007	0.4349
20	10	0.3	20.1304	9.6829	0.1304	-0.3171	1.8138	0.4944
20	10	0.5	20.1391	9.4842	0.1391	-0.5158	1.7110	0.6170
20	10	0.7	20.1494	9.2947	0.1494	-0.7053	1.7861	0.8422
20	10	0.9	20.1197	9.1230	0.1197	-0.8770	1.7998	1.0972
20	20	0.1	20.1461	19.7482	0.1461	-0.2518	1.7935	1.7090
20	20	0.3	20.1273	19.3731	0.1273	-0.6269	1.8124	1.9759
20	20	0.5	20.1615	18.9553	0.1615	-1.0447	1.7626	2.5215
20	20	0.7	20.1328	18.6027	0.1328	-1.3973	1.7404	3.2943
20	20	0.9	20.1267	18.2396	0.1267	-1.7604	1.8074	4.4055
20	50	0.1	20.1523	49.3364	0.1523	-0.6636	1.7390	10.4190
20	50	0.3	20.1358	48.4009	0.1358	-1.5991	1.7588	12.1044
20	50	0.5	20.1406	47.4341	0.1406	-2.5659	1.7905	15.7581
20	50	0.7	20.1225	46.5233	0.1225	-3.4767	1.7106	20.4048
20	50	0.9	20.0997	45.6570	0.0997	-4.3430	1.8167	27.1974
50	1	0.1	50.3407	0.9933	0.3407	-0.0067	10.5258	0.0040
50	1	0.3	50.3289	0.9856	0.3289	-0.0144	10.6942	0.0041
50	1	0.5	50.2420	0.9796	0.2420	-0.0204	10.6093	0.0043
50	1	0.7	50.3641	0.9694	0.3641	-0.0306	10.7118	0.0047
50	1	0.9	50.3485	0.9619	0.3485	-0.0381	10.6592	0.0051
50	5	0.1	50.2285	4.9767	0.2285	-0.0233	9.9770	0.0966
50	5	0.3	50.2880	4.9315	0.2880	-0.0685	10.4850	0.1035
50	5	0.5	50.3096	4.8907	0.3096	-0.1093	10.6810	0.1091
50	5	0.7	50.2957	4.8530	0.2957	-0.1470	10.7213	0.1172
50	5	0.9	50.3032	4.8140	0.3032	-0.1860	10.4948	0.1255
50	10	0.1	50.2918	9.9433	0.2918	-0.0567	10.6469	0.4102
50	10	0.3	50.3215	9.8577	0.3215	-0.1423	10.7118	0.4166
50	10	0.5	50.2818	9.7869	0.2818	-0.2131	10.4970	0.4304
50	10	0.7	50.3180	9.7017	0.3180	-0.2983	10.7697	0.4678
50	10	0.9	50.3355	9.6197	0.3355	-0.3803	10.0872	0.4951
50	20	0.1	50.3302	19.8690	0.3302	-0.1310	10.2444	1.5742

Table 9-3 (Continued) Simulation Results for $n = 500$.

λ	θ	p	$\hat{\lambda}$	$\hat{\theta}$	Bias		MSE	
					$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$
50	20	0.3	50.3001	19.7254	0.3001	-0.2746	10.5807	1.6663
50	20	0.5	50.2952	19.5676	0.2952	-0.4324	10.4352	1.7027
50	20	0.7	50.2701	19.4229	0.2701	-0.5771	10.4384	1.8172
50	20	0.9	50.2928	19.2597	0.2928	-0.7403	10.8204	2.0486
50	50	0.1	50.2687	49.7346	0.2687	-0.2654	10.4818	10.0339
50	50	0.3	50.3336	49.2876	0.3336	-0.7124	10.9804	10.6607
50	50	0.5	50.2886	48.9274	0.2886	-1.0726	10.4504	10.5873
50	50	0.7	50.4218	48.4156	0.4218	-1.5844	10.8209	11.8605
50	50	0.9	50.3274	48.1235	0.3274	-1.8765	10.8761	12.8665