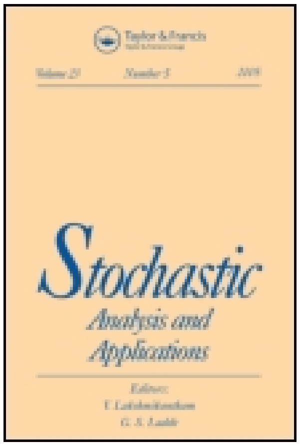

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A Complete Convergence Theorem for Row Sums from Arrays of Rowwise Independent Random Elements in Rademacher Type *p* Banach Spaces

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We extend in several directions a complete convergence theorem for row sums from an array of rowwise independent random variables obtained by Sung, Volodin, and Hu [8] to an array of rowwise independent random elements taking values in a real separable Rademacher type p Banach space. An example is presented which illustrates that our result extends the Sung, Volodin, and Hu result even for the random variable case.

Keywords Array of Banach space valued random elements; Complete convergence; Rate of convergence; Real separable Rademacher type p Banach space; Row sums; Rowwise independent.

Mathematics Subject Classification Primary 60F15, 60B12; Secondary 60B11.

1. Introduction

The concept of complete convergence was introduced by Hsu and Robbins [1] as follows. A sequence of (real valued) random variables $\{U_n, n \geq 1\}$ is said to *converge completely* to constant $c \in \mathbb{R}$ if $\sum_{n=1}^{\infty} P(|U_n - c| > \varepsilon) < \infty$ for all $\varepsilon > 0$. This of course implies by the Borel-Cantelli lemma that $U_n \to c$ almost surely (a.s.). The converse is true if the $U_n, n \geq 1$ are independent. Hsu and Robbins [1] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [2] proved the converse. The Hsu-Robbins-Erdös result is precisely formulated as follows.

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Theorem 1.1 ([1, 2]). *For a sequence of i.i.d. random variables* $\{X_n, n \geq 1\}$, $\sum_{k=1}^{n} X_k/n$ *converges completely to* 0 *if and only if* $EX_1 = 0$ *and* $EX_1^2 < \infty$.

This result has been generalized and extended in several directions by a number of authors; for results up to 1999, see the discussions in Hu, Rosalsky, Szynal, and Volodin [3]. More recent work on complete convergence is that of Hu and Volodin [4], Hu, Li, Rosalsky, and Volodin [5], Hu, Ordóñez Cabrera, Sung, and Volodin [6], Kuczmaszewska [7], Sung, Volodin, and Hu [8], Kruglov, Volodin, and Hu [9], Sung and Volodin [10], Hernández, Urmeneta, and Volodin [11], Sung, Ordóñez Cabrera, and Hu [12], Sung, Urmeneta, and Volodin [13], Chen, Hernández, Urmeneta, and Volodin [14], and Hu, Rosalsky, and Wang [15]. Some of these generalizations and extensions concern a Banach space setting: a sequence of Banach space valued random elements is said to *converge completely* to the 0 element of the Banach space if the corresponding sequence of norms converges completely to 0. Moreover, some of the above extensions and generalizations pertain to the row sums from an array (rather than to only the partial sums from a sequence) of either random variables or Banach space valued random elements; some of these results also indicate the rate of complete convergence.

At the origin of the current investigation is the following complete convergence result of Sung, Volodin, and Hu [8].

Theorem 1.2 ([8]). *Let* $\{X_{nk}, 1 \leq k \leq k_n < \infty, n \geq 1\}$ *be an array of rowwise independent random variables and let* $\{c_n, n \geq 1\}$ *be a sequence of positive constants. Suppose that there exist* $J > 1$ *and* $\delta > 0$ *such that*

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|X_{nk}| > \varepsilon) < \infty \quad \textit{for all } \varepsilon > 0,$$

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E(X_{nk}^2 I(|X_{nk}| \leq \delta)) \right)^J < \infty, \tag{1.1}$$

and

$$\sum_{k=1}^{k_n} E(X_{nk} I(|X_{nk}| \leq \delta)) \to 0. \tag{1.2}$$

Then

$$\sum_{n=1}^{\infty} c_n P\left(\left| \sum_{k=1}^{k_n} X_{nk} \right| > \varepsilon \right) < \infty \quad \textit{for all } \varepsilon > 0.$$

Theorem 1.2 has an interesting development. It was originally formulated by Hu, Szynal, and Volodin [16]. Unfortunately, the "proof" of the above Theorem 1.2 given by Hu, Szynal, and Volodin [16] was not valid as was acknowledged by Hu and Volodin [4] who also pointed out that a minor adjustment to their argument yields a valid proof of Theorem 1.2 if it is assumed in addition that

$$\liminf_{n \to \infty} c_n > 0. \tag{1.3}$$

The condition (1.3) when combined with the other conditions of Theorem 1.2 ensures that

$$\sum_{k=1}^{k_n} X_{nk} \xrightarrow{P} 0 \tag{1.4}$$

which was used by Hu, Szynal, and Volodin [16] in their "proof" of Theorem 1.2. But without the additional condition (1.3), the hypotheses of Theorem 1.2 do not guarantee (1.4) thereby rendering the "proof" of Theorem 1.2 given by Hu, Szynal, and Volodin [16] to be invalid; see the examples presented in [4, Example 1] and [6, Example 1]. But Theorem 1.2 is indeed a correct result exactly as was stated by Hu, Szynal, and Volodin [16] and a valid proof of this was provided by Sung, Volodin, and Hu [8]. Earlier attempts to give a valid proof of Theorem 1.2 were not successful but nevertheless resulted in interesting and similar types of results; see [6, 7].

Actually, Theorem 2.1 was formulated by Sung, Volodin, and Hu [8] with $J \geq 2$ rather than with $J > 1$. However a perusal of their proof reveals that the result is valid for $J > 1$ with no change at all needed in their argument.

Recently, the $J \geq 2$ version of Theorem 1.2 was extended to a Banach space setting by Hu, Rosalsky, and Wang [15]. The underlining Banach space is assumed to be of *Rademacher type p* ($1 \leq p \leq 2$). Without this assumption, a Banach space version of Theorem 1.2 can fail; see Example 4.2 of [3].

In Theorem 3.1, the main result of the current work, we extend Theorem 1.2 in several directions, namely:

(i) The same constant $\delta > 0$ appearing in conditions (1.1) and (1.2) of Theorem 1.2 is taken to be $\delta_1 > 0$ and $\delta_2 > 0$ in the analogous conditions (3.2) and (3.3), respectively, of Theorem 3.1. Since δ_1 and δ_2 play different roles in our proof, there is no reason why they need to the same.
(ii) We assume $J > 0$ rather than $J > 1$.
(iii) We consider an array of rowwise independent random elements $\{V_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ taking values in a real separable Rademacher type p ($1 \leq p \leq 2$) Banach space.
(iv) The component $E(X_{nk}^2 I(|X_{nk}| \leq \delta))$ of (1.1) is replaced by $E(\|V_{nk}\|^p I(\|V_{nk}\| \leq \delta_1))$ in (3.2).

Theorem 3.1 also extends the main result of Hu, Rosalsky, and Wang ([15], Theorem 3.1) which also had a common $\delta > 0$ in its conditions (3.2) and (3.3). Our proof of Theorem 3.1 is different from that of Theorem 1 of [8] (which relied on the Rosenthal [17] inequality) and Theorem 3.1 of [15] (which relied on Theorem 3 of [13]).

This article is organized as follows. For convenience, technical definitions and a lemma are consolidated into Section 2. Theorem 3.1 will be stated and proved in Section 3. In Section 4, we present an illustrative example of an array of random variables satisfying the hypothesis of Theorem 3.1 but not those of Theorem 1.2.

2. Preliminaries

In this section, some definitions will be reviewed and a lemma will be presented.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{X}$ be a real separable Banach space with norm $\|\cdot\|$. It is supposed that $\mathcal{X}$ is equipped with its Borel σ-algebra $\mathcal{B}$; that is, $\mathscr{B}$ is the σ-algebra generated by the class of open subsets of $\mathscr{X}$ determined by $\|\cdot\|$. A *random element* V in $\mathscr{X}$ is an $\mathscr{F}$-measurable transformation from Ω to the measurable space $(\mathscr{X}, \mathscr{B})$.

We define the *expected value* or *mean* of a random element V, denoted EV, to be the *Pettis integral* provided it exists. That is, V has expected value $EV \in \mathscr{X}$ if $f(EV) = E(f(V))$ for every $f \in \mathscr{X}^*$ where $\mathscr{X}^*$ denotes the (*dual*) space of all continuous linear functionals on $\mathscr{X}$. A sufficient condition for EV to exist is that $E\|V\| < \infty$, in which case $\|EV\| \leq E\|V\|$ (see, e.g., [18], pp. 39–40).

Let $\{\varepsilon_n, n \geq 1\}$ be a symmetric *Bernoulli sequence*; that is, $\{\varepsilon_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with $P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = 1/2$. Let $\mathscr{X}^\infty = \mathscr{X} \times \mathscr{X} \times \mathscr{X} \times \cdots$ and define

$$\mathscr{C}(\mathscr{X}) = \left\{(v_1, v_2, \ldots) \in \mathscr{X}^\infty : \sum_{n=1}^{\infty} \varepsilon_n v_n \text{ converges in probability}\right\}.$$

Let $1 \leq p \leq 2$. Then $\mathscr{X}$ is said to be of *Rademacher type p* if there exists a constant $0 < C < \infty$ such that

$$E\left\|\sum_{n=1}^{\infty} \varepsilon_n v_n\right\|^p \leq C \sum_{n=1}^{\infty} \|v_n\|^p \quad \text{for all } (v_1, v_2, \ldots) \in \mathscr{C}(\mathscr{X}).$$

Hoffmann-Jørgensen and Pisier [19] proved for $1 \leq p \leq 2$ that a real separable Banach space is of Rademacher type p if and only if there exists a constant $0 < C < \infty$ such that

$$E\left\|\sum_{j=1}^{n} V_j\right\|^p \leq C \sum_{j=1}^{n} E\|V_j\|^p \tag{2.1}$$

for every finite collection $\{V_1, \ldots, V_n\}$ of independent mean 0 random elements.

If a real separable Banach space is of Rademacher type p for some $1 < p \leq 2$, then it is of Rademacher type q for all $1 \leq q < p$. Every real separable Banach space is of Rademacher type (at least) 1 while the L_p-spaces and l_p-spaces are of Rademacher type $2 \wedge p$ for $p \geq 1$. Every real separable Hilbert space and real separable finite-dimensional Banach space is of Rademacher type 2. In particular, the real line $\mathbb{R}$ is of Rademacher type 2.

For an array of random elements $\{V_{nk}, 1 \leq k \leq k_n < \infty, n \geq 1\}$, if the k_n random elements $\{V_{nk}, 1 \leq k \leq k_n\}$ are independent for all $n \geq 1$, then the array is said to be comprised of *rowwise independent* random elements. Thus, the random elements from the same row are independent but independence is not required to hold between the random elements from different rows.

We now present a version of the famous Hoffmann-Jørgensen [20] inequality; the random elements in Lemma 2.1 do not need to be symmetric. Lemma 2.1 was proved by Hu, Ordóñez Cabrera, Sung, and Volodin [6] in the random variable case but it also holds for random elements as was discussed by Hu, Ordóñez Cabrera, Sung, and Volodin [6] and by Sung, Volodin, and Hu [8].

Lemma 2.1. *Let $V_1, \ldots, V_N$ be independent random elements. Then for every integer $j \geq 1$ and $t > 0$*

$$P\left(\left\|\sum_{k=1}^{N} V_k\right\| > 6^j t\right) \leq C_j P\left(\max_{1\leq k\leq N} \|V_k\| > \frac{t}{4^{j-1}}\right) + D_j \max_{1\leq l\leq N}\left(P\left(\left\|\sum_{k=1}^{l} V_k\right\| > \frac{t}{4^j}\right)\right)^{2^j}$$

where $C_j < \infty$ and $D_j < \infty$ are positive constants depending only on j.

Finally, the symbol C will be used to denote a generic constant ($0 < C < \infty$) whose actual value is unimportant and which is not necessarily the same one in each appearance.

3. Mainstream

With the preliminaries accounted for, the main result may be established. In general, the case where $\liminf_{n\to\infty} k_n < \infty$ is not being precluded although we are mostly interested in the result when $\lim_{n\to\infty} k_n = \infty$.

Theorem 3.1. *Let $\{V_{nk}, 1 \leq k \leq k_n < \infty, n \geq 1\}$ be an array of rowwise independent random elements taking values in a real separable Rademacher type p ($1 \leq p \leq 2$) Banach space $\mathscr{X}$ and let $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose for some $J > 0$ and some $\delta_1, \delta_2 > 0$ that*

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(\|V_{nk}\| > \varepsilon) < \infty \quad \textit{for all } \varepsilon > 0, \tag{3.1}$$

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E(\|V_{nk}\|^p I(\|V_{nk}\| \leq \delta_1))\right)^J < \infty, \tag{3.2}$$

and

$$\sum_{k=1}^{k_n} E(V_{nk} I(\|V_{nk}\| \leq \delta_2)) \to 0. \tag{3.3}$$

Then

$$\sum_{n=1}^{\infty} c_n P\left(\left\|\sum_{k=1}^{k_n} V_{nk}\right\| > \varepsilon\right) < \infty \quad \textit{for all } \varepsilon > 0. \tag{3.4}$$

Proof. Let $\varepsilon > 0$ be arbitrary. Choose a positive integer j and a real number A such that

$$2^j > J \quad \text{and} \quad 0 < A < \min\left\{\frac{\varepsilon}{4^{j-1}\cdot 6\cdot 6^j}, \delta_1, \delta_2\right\}.$$

For $n \geq 1$ and $1 \leq k \leq k_n$, let

$$V_{nk}^{(1)} = V_{nk} I(\|V_{nk}\| \leq A), \quad V_{nk}^{(2)} = V_{nk} I(A < \|V_{nk}\| \leq \delta_2), \quad \text{and} \quad V_{nk}^{(3)} = V_{nk} I(\|V_{nk}\| > \delta_2).$$

Then

$$V_{nk} = V_{nk}^{(1)} - EV_{nk}^{(1)} + V_{nk}^{(2)} - EV_{nk}^{(2)} + V_{nk}^{(3)} + EV_{nk}^{(1)} + EV_{nk}^{(2)}, \quad 1 \le k \le k_n, \ n \ge 1$$

and so

$$\begin{aligned}\sum_{n=1}^{\infty} c_n P\left(\left\| \sum_{k=1}^{k_n} V_{nk} \right\| > \varepsilon\right) &\le \sum_{n=1}^{\infty} c_n P\left(\left\| \sum_{k=1}^{k_n} \left(V_{nk}^{(1)} - EV_{nk}^{(1)}\right) \right\| > \frac{\varepsilon}{3}\right) \\ &\quad + \sum_{n=1}^{\infty} c_n P\left(\left\| \sum_{k=1}^{k_n} \left(V_{nk}^{(2)} - EV_{nk}^{(2)}\right) \right\| > \frac{\varepsilon}{3}\right) \\ &\quad + \sum_{n=1}^{\infty} c_n P\left(\left\| \sum_{k=1}^{k_n} V_{nk}^{(3)} + \sum_{k=1}^{k_n} \left(EV_{nk}^{(1)} + EV_{nk}^{(2)}\right) \right\| > \frac{\varepsilon}{3}\right) \\ &\equiv \mathcal{U} + \mathcal{V} + \mathcal{W} \quad \text{(say)}.\end{aligned}$$

The conclusion (3.4) will be established provided we can show that $\mathcal{U} < \infty$, $\mathcal{V} < \infty$, and $\mathcal{W} < \infty$.

We first show that $\mathcal{U} < \infty$. Note that for all $n \ge 1$,

$$\max_{1 \le k \le k_n} \|V_{nk}^{(1)} - EV_{nk}^{(1)}\| \le 2A < \frac{\varepsilon}{4^{j-1} \cdot 3 \cdot 6^j} \quad \text{a.s.} \tag{3.5}$$

Then applying Lemma 2.1 with $t = \varepsilon/(3 \cdot 6^j)$ to the random elements $V_{nk}^{(1)} - EV_{nk}^{(1)}$, $1 \le k \le k_n$ gives for all $n \ge 1$ that

$$\begin{aligned}&P\left(\left\| \sum_{k=1}^{k_n} (V_{nk}^{(1)} - EV_{nk}^{(1)}) \right\| > \frac{\varepsilon}{3}\right) \\ &\le C_j P\left(\max_{1 \le k \le k_n} \|V_{nk}^{(1)} - EV_{nk}^{(1)}\| > \frac{\varepsilon}{4^{j-1} \cdot 3 \cdot 6^j}\right) \\ &\quad + D_j \max_{1 \le l \le k_n} \left(P\left(\left\| \sum_{k=1}^{l} \left(V_{nk}^{(1)} - EV_{nk}^{(1)}\right) \right\| > \frac{\varepsilon}{4^j \cdot 3 \cdot 6^j}\right)\right)^{2^j} \\ &\le D_j \max_{1 \le l \le k_n} \left(P\left(\left\| \sum_{k=1}^{l} \left(V_{nk}^{(1)} - EV_{nk}^{(1)}\right) \right\| > \frac{\varepsilon}{4^j \cdot 3 \cdot 6^j}\right)\right)^{J} \quad \text{(by (3.5) and } 2^j > J) \\ &\le D_j \left(\frac{4^j \cdot 3 \cdot 6^j}{\varepsilon}\right)^{pJ} \max_{1 \le l \le k_n} \left(E\left\| \sum_{k=1}^{l} (V_{nk}^{(1)} - EV_{nk}^{(1)}) \right\|^p\right)^J \quad \text{(by the Markov inequality)} \\ &\le D_j \left(\frac{4^j \cdot 3 \cdot 6^j}{\varepsilon}\right)^{pJ} \left(C \sum_{k=1}^{k_n} E\|V_{nk}^{(1)} - EV_{nk}^{(1)}\|^p\right)^J \text{(by (2.1))} \\ &\le D_j 2^{pJ} \left(\frac{4^j \cdot 3 \cdot 6^j}{\varepsilon}\right)^{pJ} C^J \left(\sum_{k=1}^{k_n} E\|V_{nk}^{(1)}\|^p\right)^J \\ &\qquad \text{(by the } c_r\text{-inequality and Jensen's inequality)}\end{aligned}$$

$$= D_j 2^{pJ}\left(\frac{4^j \cdot 3 \cdot 6^j}{\varepsilon}\right)^{pJ} C^J\left(\sum_{k=1}^{k_n} E(\|V_{nk}\|^p I(\|V_{nk}\| \leq A))\right)^J$$

$$\leq D_j 2^{pJ}\left(\frac{4^j \cdot 3 \cdot 6^j}{\varepsilon}\right)^{pJ} C^J\left(\sum_{k=1}^{k_n} E(\|V_{nk}\|^p I(\|V_{nk}\| \leq \delta_1))\right)^J.$$

Hence, $\mathscr{U} < \infty$ by the assumption (3.2).

Next, we show that $\mathscr{V} < \infty$. Note that for all $n \geq 1$,

$$\begin{aligned}
&P\left(\left\| \sum_{k=1}^{k_n}(V_{nk}^{(2)} - EV_{nk}^{(2)})\right\| > \frac{\varepsilon}{3}\right) \\
&\leq \frac{3}{\varepsilon}E\left\| \sum_{k=1}^{k_n}(V_{nk}^{(2)} - EV_{nk}^{(2)})\right\| \quad \text{(by the Markov inequality)} \\
&\leq \frac{3}{\varepsilon}\sum_{k=1}^{k_n} E\|V_{nk}^{(2)} - EV_{nk}^{(2)}\| \leq \frac{6}{\varepsilon}\sum_{k=1}^{k_n} E\|V_{nk}^{(2)}\| \\
&= \frac{6}{\varepsilon}\sum_{k=1}^{k_n} E\left(\|V_{nk}\| I(A < \|V_{nk}\| \leq \delta_2)\right) \\
&\leq \frac{6\delta_2}{\varepsilon}\sum_{k=1}^{k_n} E(I(A < \|V_{nk}\| \leq \delta_2)) \\
&\leq \frac{6\delta_2}{\varepsilon}\sum_{k=1}^{k_n} P(\|V_{nk}\| > A).
\end{aligned}$$

Hence, $\mathscr{V} < \infty$ by the assumption (3.1).

Finally, we show that $\mathscr{W} < \infty$. By the assumption (3.3),

$$\sum_{k=1}^{k_n}(EV_{nk}^{(1)} + EV_{nk}^{(2)}) = \sum_{k=1}^{k_n} E(V_{nk} I(\|V_{nk}\| \leq \delta_2)) \to 0$$

and so there exists an integer N such that

$$\left\| \sum_{k=1}^{k_n}(EV_{nk}^{(1)} + EV_{nk}^{(2)})\right\| \leq \frac{\varepsilon}{6} \quad \text{for all } n \geq N. \tag{3.6}$$

Then for $n \geq N$,

$$\begin{aligned}
&P\left(\left\| \sum_{k=1}^{k_n} V_{nk}^{(3)} + \sum_{k=1}^{k_n}(EV_{nk}^{(1)} + EV_{nk}^{(2)})\right\| > \frac{\varepsilon}{3}\right) \\
&\leq P\left(\left\| \sum_{k=1}^{k_n} V_{nk}^{(3)}\right\| + \left\| \sum_{k=1}^{k_n}(EV_{nk}^{(1)} + EV_{nk}^{(2)})\right\| > \frac{\varepsilon}{3}\right) \\
&\leq P\left(\left\| \sum_{k=1}^{k_n} V_{nk} I(\|V_{nk}\| > \delta_2)\right\| > \frac{\varepsilon}{6}\right) \quad \text{(by (3.6))}
\end{aligned}$$

$$\leq P\left(\bigcup_{k=1}^{k_n}[\|V_{nk}\| > \delta_2]\right)$$
$$\leq \sum_{k=1}^{k_n} P(\|V_{nk}\| > \delta_2).$$

Hence, $\mathscr{W} < \infty$ by the assumption (3.1). □

Remark 3.1. Suppose that the hypotheses of Theorem 3.1 are satisfied with $J = 1$. Then (3.2) holds with δ_1 replaced by δ_2. In other words, when $J = 1$, Theorem 3.1 is equivalent to the same theorem but with a common value of $\delta > 0$ taken for δ_1 and δ_2 in (3.2) and (3.3), respectively.

Proof. The assertion is clear if $\delta_2 \leq \delta_1$ so assume that $\delta_2 > \delta_1$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E\left(\|V_{nk}\|^p I(\|V_{nk}\| \leq \delta_2)\right) \\
&= \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\left(\|V_{nk}\|^p I(\|V_{nk}\| \leq \delta_1)\right) + \sum_{k=1}^{k_n} E\left(\|V_{nk}\|^p I(\delta_1 < \|V_{nk}\| \leq \delta_2)\right)\right) \\
&\leq \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} E\left(\|V_{nk}\|^p I(\|V_{nk}\| \leq \delta_1)\right) + \sum_{n=1}^{\infty} c_n \delta_2^p \sum_{k=1}^{k_n} P(\|V_{nk}\| > \delta_1) \\
&< \infty \quad \text{(by (3.2) and (3.1))}.
\end{aligned}$$
□

4. An Interesting Example

In the following example, the conditions of Theorem 3.1 are satisfied, but those of Theorem 1.2 are not.

Example 4.1. Define sequences $\{p_n, n \geq 1\}$, $\{k_n, n \geq 1\}$, and $\{c_n, n \geq 1\}$ as follows. For $n \geq 1$, let

$$p_n = \frac{1}{2^n}, \quad k_n = \begin{cases} \lfloor 2^n n(\log(n+1))^2 \rfloor, & n \text{ odd} \\ \left\lfloor \dfrac{2^n}{n(\log(n+1))^2} \right\rfloor, & n \text{ even} \end{cases}, \quad c_n = \begin{cases} \dfrac{1}{n^2(\log(n+1))^4}, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}.$$

Let $0 < a < \infty$. Let $\{V_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise i.i.d. random variables where

$$P(V_{nk} = -a) = p_n, \quad P\left(V_{nk} = \frac{ap_n}{1-p_n}\right) = 1 - p_n, \quad 1 \leq k \leq k_n, \ n \geq 1.$$

Note that

$$\frac{ap_n}{1-p_n} \downarrow 0. \tag{4.1}$$

We first verify that (3.1) holds. For arbitrary $\varepsilon > 0$, it follows from (4.1) that for all large n,

$$c_n \sum_{k=1}^{k_n} P(|V_{nk}| > \varepsilon) \leq c_n \sum_{k=1}^{k_n} P(V_{nk} = -a) = c_n k_n p_n \leq \frac{1}{n(\log(n+1))^2},$$

which is summable and so (3.1) holds.

Next, we consider the condition (3.2) with $J > 1$ and $p = 2$. If $0 < \delta_1 < a$, then it follows from (4.1) that for all large n

$$E\left(V_{nk}^2 I(|V_{nk}| \leq \delta_1)\right) = \frac{a^2 p_n^2}{(1-p_n)^2}(1-p_n) = \frac{a^2 p_n^2}{1-p_n}, \quad 1 \leq k \leq k_n$$

and so for large n,

$$\begin{aligned} c_n \left(\sum_{k=1}^{k_n} E\left(V_{nk}^2 I(|V_{nk}| \leq \delta_1)\right) \right)^J &= c_n \left(\frac{k_n a^2 p_n^2}{1-p_n} \right)^J \\ &= \frac{a^{2J}}{(1-p_n)^J} c_n (k_n p_n)^J p_n^J \\ &\leq \begin{cases} C \dfrac{n^J (\log(n+1))^{2J}}{n^2 (\log(n+1))^4} \cdot \dfrac{1}{2^{Jn}}, & n \text{ odd} \\ C \dfrac{1}{n^J (\log(n+1))^{2J}} \cdot \dfrac{1}{2^{Jn}}, & n \text{ even} \end{cases} \\ &\leq C \frac{n^J (\log(n+1))^{2J}}{n^2 (\log(n+1))^4} \cdot \frac{1}{2^{Jn}}, \end{aligned}$$

which is summable and, hence, (3.2) holds.

On the other hand, if $\delta_1 \geq a$, then for all $n \geq 1$ and $1 \leq k \leq k_n$,

$$E\left(V_{nk}^2 I(|V_{nk}| \leq \delta_1)\right) = EV_{nk}^2 = a^2 p_n + \frac{a^2 p_n^2}{(1-p_n)^2}(1-p_n) \geq a^2 p_n$$

and so

$$\begin{aligned} \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\left(V_{nk}^2 I(|V_{nk}| \leq \delta_1)\right) \right)^J &\geq \sum_{n=1}^{\infty} c_n (k_n a^2 p_n)^J \\ &\geq \sum_{j=1}^{\infty} c_{2j-1} (k_{2j-1} a^2 p_{2j-1})^J \\ &= \sum_{j=1}^{\infty} \frac{a^{2J} (1+o(1)) (2j-1)^J (\log(2j))^{2J}}{(2j-1)^2 (\log(2j))^4} \\ &= \sum_{j=1}^{\infty} \frac{a^{2J} (1+o(1))}{(2j-1)^{2-J} (\log(2j))^{4-2J}} \\ &= \infty \end{aligned}$$

since $J > 1$. Thus, (3.2) fails. We have shown for $J > 1$ and $p = 2$ that

$$\text{(3.2) holds if } 0 < \delta_1 < a \text{ and (3.2) fails if } \delta_1 \geq a. \tag{4.2}$$

Last, we consider the condition (3.3). If $\delta_2 \geq a$, then for all $n \geq 1$,

$$\sum_{k=1}^{k_n} E\left(V_{nk} I(|V_{nk}| \leq \delta_2)\right) = \sum_{k=1}^{k_n} EV_{nk} = \sum_{k=1}^{k_n} 0 = 0$$

and so (3.3) holds.

On the other hand if $0 < \delta_2 < a$, then it follows from (4.1) that for all large n,

$$\sum_{k=1}^{k_n} E\left(V_{nk} I(|V_{nk}| \leq \delta_2)\right) = \sum_{k=1}^{k_n} \frac{ap_n}{1 - p_n}(1 - p_n) = ak_n p_n \nrightarrow 0$$

since $k_n p_n \sim n(\log n)^2$ for n odd. Thus, (3.3) fails.

We have shown that

$$\text{(3.3) holds if } \delta_2 \geq a \text{ and (3.3) fails if } 0 < \delta_2 < a. \tag{4.3}$$

In summary, for $J > 1$, $p = 2$, and $0 < \delta_1 < a \leq \delta_2 < \infty$, it follows from (4.2) and (4.3) that the hypothesis of Theorem 3.1 are satisfied and hence (3.4) holds. But in view of (4.2) and (4.3), there does not exist a $\delta > 0$ satisfying the hypothesis of Theorem 1.2.

Remark 4.1. It would be interesting to know if an example can be constructed wherein the hypotheses of Theorem 1.2 fail for all $J > 1$ and the hypotheses of Theorem 3.1 hold for some $J > 1$ but (3.2) fails for all $J \in (0, 1]$ and all $\delta_1 > 0$. On the other hand, it would also be interesting to know if an example can be constructed wherein the hypotheses of Theorem 3.1 hold for some $J \in (0, 1]$ but fail for all $J > 1$. We hope that these open problems will be investigated by an interested reader.

Remark 4.2. Hu, Rosalsky, and Wang [15] provided two additional theorems, eight corollaries, three illustrative examples, and several remarks all pertaining to Theorem 3.1 of their article. All of them thus pertain to Theorem 3.1 of the current work.

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