*Stochastic Analysis and Applications*, 25: 281–291, 2007

ISSN 0736-2994 print/1532-9356 online
DOI: 10.1080/07362990601139461



# On Complete Convergence for Arrays of Random Elements and Variables

**Víctor Hernández**
Departamento de Estadística e Investigación Operativa, de la Universidad Nacional de Educación a Distancia, Madrid, Spain

**Henar Urmeneta**
Departamento de Estadística e Investigación Operativa, de la Universidad Pública de Navarra, Pamplona, Spain

**Andrei Volodin**
Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada

**Abstract:** We obtain complete convergence results for arrays of row-wise independent Banach space valued random elements. The main result deals with two cases that usually are considered separately: when no assumptions are made concerning the geometry of the underlying Banach space and when the Banach space is of Rademacher type $p$.

**Keywords:** Array of Banach space valued random elements; Complete convergence; Convergence in probability; Domination conditions; Normed partial sums; Row-wise independence; Type $p$ Banach space.

**AMS Classification:** 60B12; 60F05; 60F25; 60G42.

Accepted November 2, 2006

The authors are grateful to the referee for offering some valuable suggestions which enabled them to improve the overall presentation. The work of A. Volodin is supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

Address correspondence to Henar Urmeneta, Departamento de Estadística e Investigación Operativa, de la Universidad Pública de Navarra, Pamplona, Spain; E-mail: henar@unavarra.es

## I. INTRODUCTION AND PRELIMINARIES

The concept of complete convergence was introduced in [1] as follows. A sequence of random variables $\{U_n, n \geq 1\}$ is said to *converge completely* to a constant $c$ if $\sum_{n=1}^{\infty} P\{|U_n - c| > \varepsilon\} < \infty$ for all $\varepsilon > 0$. By the Borel–Cantelli lemma, this implies $U_n \to c$ almost surely (a.s.) and the converse implication is true if the random variables $\{U_n, n \geq 1\}$ are independent. In [1] it is proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value, if the variance of the summands is finite.

This result has been generalized and extended in several directions (see [2–10] among others). Some of these works deal with a Banach space setting. A sequence of Banach space valued random elements is said to *converge completely* to the 0 element of the Banach space if the corresponding sequence of norms converges completely to 0.

A general result (cf. Theorem 1 below) was presented in [9] establishing complete convergence for the row sums of an array of row-wise independent but not necessarily identically distributed Banach space valued random elements. Their result also specified the corresponding rate of convergence. The result of [9] unifies and extends previously obtained results in the literature in that many of them (for example, results of [1–8] and [10]) follow from it.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and let $B$ be a separable real Banach space with norm $||\cdot||$. A *random element* is defined to be an $\mathscr{A}$-measurable mapping of $\Omega$ into $B$ equipped with the Borel $\sigma$-algebra (that is, the $\sigma$-algebra generated by the open sets determined by $||\cdot||$). A detailed account of basic properties of random elements in separable real Banach spaces can be found in [11].

Let $0 < p \leq 2$ and let $\{\epsilon_n, n \geq 1\}$ be independent and identically distributed Bernoulli random variables with $P\{\epsilon = 1\} = P\{\epsilon = -1\} = 1/2$ for all $n \geq 1$. The real separable Banach space $B$ is said to be of (Rademacher) *type p* if $\sum_{n=1}^{\infty} \epsilon_n v_n$ converges almost surely whenever $\{v_n, n \geq 1\} \subseteq B$ with $\sum_{n=1}^{\infty} ||v_n||^p < \infty$.

Let $\{V_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of random elements with $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ sequences of integers (not necessary positive or finite) such that $u_n \leq v_n$.

The first theorem presented here is a slight modification of the result of [9].

**Theorem 1.** *Let $\{V_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of row-wise independent random elements and $\{\varphi_n, n \geq 0\}$ be an increasing sequence of positive numbers with $\varphi_0 = 0$. Suppose that for all $\varepsilon > 0$ and some $J \geq 2$:*

$$\sum_{n=1}^{\infty} \sum_{k=u_n}^{v_n} P\{||V_{nk}|| > \varepsilon\varphi_n\} < \infty, \tag{1}$$

$$\sum_{n=1}^{\infty}\left(\sum_{k=u_n}^{v_n} E\left(\left\|\frac{V_{nk}}{\varphi_n}\right\|^2 I\{\|V_{nk}\| \leq \varphi_n\}\right)\right)^J < \infty, \tag{2}$$

$$\frac{1}{\varphi_n}\sum_{k=u_n}^{v_n} V_{nk} \to 0 \quad \textit{in probability}. \tag{3}$$

*Then* $\frac{1}{\varphi_n}\sum_{k=u_n}^{v_n} V_{nk}$ *converges completely to zero.*

The second theorem is a slight modification of the result of [12].

**Theorem 2.** *Let* $\{V_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *be an array of row-wise independent random elements taking values in a real separable Banach space of type* $p, 1 \leq p \leq 2$ *and* $\{\varphi_n, n \geq 0\}$ *be an increasing sequence of positive numbers with* $\varphi_0 = 0$. *Suppose that for every* $\varepsilon > 0$ *and some* $J \geq 2$:

(i) $\sum_{n=1}^{\infty}\sum_{k=u_n}^{v_n} P\{\|V_{nk}\| > \varepsilon\varphi_n\} < \infty,$
(ii) $\sum_{n=1}^{\infty}\left(\sum_{k=u_n}^{v_n} E\left(\|\frac{V_{nk}}{\varphi_n}\|^p I\{\|V_{nk}\| \leq \varphi_n\}\right)\right)^J < \infty$
(iii) $\max_{u_n \leq i \leq v_n}\|\sum_{l=u_n}^{i} E\left(\frac{V_{nl}}{\varphi_n} I(\|V_{nl}\| \leq \varphi_n)\right)\| \to 0$ *as* $n \to \infty$.

*Then* $\frac{1}{\varphi_n}\sum_{k=u_n}^{v_n} V_{nk}$ *converges completely to zero.*

Now we present a few domination conditions for arrays.

An array of random elements $\{V_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ is said to be *uniformly dominated* by a random variable $X$ if for some constant $C > 0$ and for all $u_n \leq k \leq v_n$ and $n \geq 1$ we have that

$$P\{||V_{nk}|| > t\} \leq CP\{C|X| > t\} \quad \text{for all } t \geq 0.$$

The following concept was introduced in [5]. An array of random elements $\{V_{nk}, 1 \leq k \leq n, n \geq 1\}$ is said to be *dominated in the Cesàro sense* by a random variable $X$ if some constant $C > 0$ and for all $n \geq 1$ we have that

$$\frac{1}{n}\sum_{k=1}^{n} P\{||V_{nk}|| > t\} \leq CP\{C|X| > t\} \quad \text{for all } t \geq 0.$$

Note that the notion domination in the Cesàro sense is *strictly* weaker than the notion of uniform domination. That means we can provide a simple example of an array that is dominated in the Cesàro sense, but not uniformly dominated; while the uniformly domination assumption implies domination in the Cesàro sense.

Next, [13] introduced the following domination condition. We will say that an array of random elements $\{V_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ is *dominated in the Sung sense* by a random variable $X$ if some constant

$C > 0$, some $p > 0$, and for all $n \geq 1$ we have that

$$\frac{1}{n}\sum_{k=u_n}^{v_n} E||V_{nk}||^p I\{||V_{nk}|| > t\} \leq CE|X|^p I\{|X|^p > t\} \quad \text{for all } t \geq 0.$$

It is also mentioned in [13] that when $u_n = 1, v_n = n$ for $n \geq 1$, then the domination in the Sung sense concept is weaker than the domination in the Cesàro sense, cf. the proof of Corollary 1 from [13]. It will be interesting to establish that the domination in the Sung sense concept is *strictly* weaker than the domination in the Cesàro sense, that is to give an example of an array $\{V_{nk}, 1 \leq k \leq n, n \geq 1\}$ that dominated in the Sung sense but not dominated in the Cesàro sense. We would like to mention here that this question is not as trivial as it seems: domination in the Sung sense has to hold for all $t \geq 0$.

The following theorem is the main result of [13].

**Theorem 3.** *Let $1 \leq p < 2$ and suppose that $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ is an array of row-wise independent random variables with zero mean and dominated in the Sung sense by a random variable $X$ with $E|X|^{2p} < \infty$. Then*

$$\frac{1}{n^{1/p}}\sum_{k=u_n}^{v_n} X_{nk} \quad \textit{converges completely to zero.}$$

The proof of Theorem 3 presented in [13] is rather complicated. The initial objective of the investigation resulting in the present article was only to find a simpler proof using the result of [14]. But it appears that we were able to establish a more general result using Theorem 1 mentioned above. Theorem 3 deals with the random variable case, while our Theorem 4 deals with the Banach space setting and a more general normalization of partial sums.

A novel feature of Theorem 4 is the fact that it provides in a *unified* manner complete convergence result compressing the two cases: (a) a Rademacher type condition is imposed on the Banach space, and (b) no geometric condition is imposed on the Banach space. Typically, in the literature, complete convergence results under (a) and (b) are treated *separately* from each other.

We present the following generalization of domination in the Sung sense. Let $\{\alpha_n, n \geq 1\}$ be a sequence of positive numbers and let $\gamma$: $\mathbf{R}^+ \mapsto \mathbf{R}^+$ be a measurable function. We say that the array $\{V_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ of random elements is $S(\gamma, \alpha)$ *dominated* by a random variable $X$ if there exists a positive constant $C$ such that

$$\sup_{n\geq 1}\frac{1}{\alpha_n}\sum_{k=u_n}^{v_n} E\gamma(||V_{nk}||)I\{|V_{nk}| > t\} \leq CE\gamma(|X|)I\{|X| > t\}$$

for all $t > 0$.

For the special case $\gamma(t) = t^p$ the notion of $S(\gamma, \alpha)$ domination can be rewritten in the following way:

$$\sup_{n\geq 1} \frac{1}{\alpha_n} \sum_{k=u_n}^{v_n} E||V_{nk}||^p I\{|V_{nk}| > t\} \leq CE|X|^p I\{|X| > t\}$$

for all $t > 0$. In this case we will say that the array is $S(p, \alpha)$ *dominated* by a random variable $X$.

**Remark 1.** If we take $\alpha_n = n$ and $\gamma(t) = t^p$ then we obtain the domination in the Sung sense condition.

As we mentioned before, a general normalization of partial sums is considered in Theorem 4. Let $\{\varphi_n, n \geq 0\}$ be an increasing sequence of positive numbers with $\varphi_0 = 0$. We can reformulate the $S(\gamma, \alpha)$ domination condition in the following way, which will be more appropriate for our results:

$$\sup_{n\geq 1} \frac{1}{\alpha_n} \sum_{k=u_n}^{v_n} E\gamma(||V_{nk}||) I\{|V_{nk}| > \varepsilon\varphi_i\} \leq CE\gamma(|X|) I\{|X| > \varepsilon\varphi_i\}$$

for all $\varepsilon > 0$ and $i \geq 0$.

The plan of the article is as follows. In Section II, we formulate and prove a few lemmas pertaining to the current work. In Section III, we apply Theorems 1 and 2 to obtain complete convergence for row sums and derive some consequences.

## II. PRELIMINARY LEMMAS

To prove the main result, we need the following lemmas.

**Lemma 1.** *Let $\phi, \xi, \gamma$: $\mathbf{R}^+ \mapsto \mathbf{R}^+$ be three measurable functions, suppose that $\phi$ is an increasing functions, with $\phi(t) \uparrow \infty$, when $t \uparrow \infty$, and let $X$ be a random variable on $(\Omega, \mathscr{A}, P)$. Define the inverse function $\phi^{-1}(t) = \inf\{s : \phi(s) \geq t\}$.*

*Then*

$$\int_0^\infty \xi(t) \int_{|X|>\phi(t)} \gamma(|X|) dP\, dt \leq \int_\Omega \gamma(|X|) \int_0^{\phi^{-1}(|X|)} \xi(t) dt\, dP.$$

*Proof.* It suffices to take into account that

$$\{(\omega, t) \in (\Omega, R^+); 0 < t < \infty, |X(\omega)| > \phi(t)\}$$
$$\subset \{(\omega, t) \in (\Omega, R^+); \omega \in \Omega, t \leq \phi^{-1}(|X(\omega)|)\}.$$

**Lemma 2.** *Let* $\{\chi_n, n \geq 0\}$ *and* $\{\varphi_n, n \geq 0\}$ *be sequences of nonnegative constants, where* $\{\varphi_n, n \geq 0\}$ *is increasing with* $\chi_0 = 0$ *and* $\varphi_0 = 0$ *and let* $\Delta(k) = \sum_{n=1}^{k} \chi_n$, *with* $k \geq 1$. *We define the inverse function* $\varphi^{-1}(t) = \min\{i : \varphi_i \geq t\}$. *Further, let* $\gamma$: $\mathbf{R}^+ \mapsto \mathbf{R}^+$ *be a measurable function. Then*

$$\sum_{n=1}^{\infty} \chi_n E\gamma(|X|)I\{|X| > \varphi_n\} \leq E\gamma(|X|)\Delta(\varphi^{-1}(|X|)).$$

*Proof.* To prove the lemma, it suffices to apply Lemma 1 with the two following step functions:

$$\xi(t) = \sum_{n=0}^{\infty} \chi_n I\{n \leq t < n+1\},$$

and

$$\phi(t) = \sum_{n=0}^{\infty} \varphi_n I\{n \leq t < n+1\}.$$

Then, $\phi^{-1}(t) = \varphi^{-1}(t)$,

$$\int_0^{\infty} \xi(t) \int_{|X|>\phi(t)} \gamma(|X|)dP\,dt = \sum_{n=1}^{\infty} \chi_n E\gamma(|X|)I\{|X| > \varphi_n\}$$

and

$$\int_{\Omega} \gamma(|X|) \int_0^{\phi^{-1}(|X|)} \xi(t)dt\,dP = E(\gamma(|X|)\Delta(\varphi^{-1}(|X|))).$$

**Lemma 3.** *Let* $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *be an array of random variables* $S(\gamma, \alpha)$ *dominated by a random variable* $X$ *and* $\psi, \gamma$: $\mathbf{R}^+ \mapsto \mathbf{R}^+$ *two measurable functions such that for*

$$\delta_i = \max_{\varphi_{i-1} < t \leq \varphi_i} \frac{\psi(t)}{\gamma(t)}$$

*the sequence* $\{\delta_n,\ n \geq 1\}$ *is increasing. Furthermore, let* $\{\varphi_n, n \geq 0\}$ *and* $\{\alpha_n, n \geq 1\}$ *be two sequences of positive numbers where* $\{\varphi_n, n \geq 0\}$ *is increasing with* $\varphi_0 = 0$.

*Then there exists a positive constant* $\mathbf{C}$ *such that*

$$\frac{1}{\alpha_n} \sum_{k=u_n}^{v_n} E\psi(|X_{nk}|)I\{|X_{nk}| \leq \varphi_n\} \leq \mathbf{C}(E\gamma(|X|) + E\gamma(|X|)\delta_{\varphi^{-1}(|X|)}).$$

*Proof.* For $n \geq 1$,

$$
\begin{aligned}
&\frac{1}{\alpha_n}\sum_{k=u_n}^{v_n} E\psi(|X_{nk}|)I\{|X_{nk}| \leq \varphi_n\}\\
&\quad= \frac{1}{\alpha_n}\sum_{k=u_n}^{v_n}\sum_{i=1}^{n} E\psi(|X_{nk}|)I\{\varphi_{i-1} < |X_{nk}| \leq \varphi_i\}\\
&\quad= \frac{1}{\alpha_n}\sum_{k=u_n}^{v_n}\sum_{i=1}^{n} E\gamma(|X_{nk}|)\frac{\psi(|X_{nk}|)}{\gamma(|X_{nk}|)} I\{\varphi_{i-1} < |X_{nk}| \leq \varphi_i\}\\
&\quad\leq \frac{1}{\alpha_n}\sum_{k=u_n}^{v_n}\sum_{i=1}^{n} E\gamma(|X_{nk}|)\delta_i I\{\varphi_{i-1} < |X_{nk}| \leq \varphi_i\}\\
&\quad= \frac{1}{\alpha_n}\sum_{k=u_n}^{v_n}\sum_{i=1}^{n} E\gamma(|X_{nk}|)\delta_i (I\{|X_{nk}| > \varphi_{i-1}\} - I\{|X_{nk}| > \varphi_i\}\\
&\quad= \frac{1}{\alpha_n}\sum_{k=u_n}^{v_n} E\gamma(|X_{nk}|)\delta_1 I\{|X_{nk}| > \varphi_0\}\\
&\qquad+ \frac{1}{\alpha_n}\sum_{k=u_n}^{v_n}\sum_{i=2}^{n} E\gamma(|X_{nk}|)(\delta_i - \delta_{i-1}) I\{|X_{nk}| > \varphi_{i-1}\}\\
&\qquad- \frac{1}{\alpha_n}\sum_{k=u_n}^{v_n} E\gamma(|X_{nk}|)\delta_n I\{|X_{nk}| > \varphi_n\}\\
&\quad\leq \frac{\delta_1}{\alpha_n}\sum_{k=u_n}^{v_n} E\gamma(|X_{nk}|) I\{|X_{nk}| > \varphi_0\}\\
&\qquad+ \sum_{i=2}^{n}(\delta_i - \delta_{i-1})\frac{1}{\alpha_n}\sum_{k=u_n}^{v_n} E\gamma(|X_{nk}|) I\{|X_{nk}| > \varphi_{i-1}\}\\
&\quad\leq \delta_1 CE\gamma(|X|)I\{|X| > \varphi_0\} + C\sum_{i=2}^{n}(\delta_i - \delta_{i-1})E\gamma(|X|)I\{|X| > \varphi_{i-1}\}
\end{aligned}
$$

where we have used the $S(\gamma, \alpha)$ dominated condition in the last inequality.

Let $\chi_i = \delta_{i+1} - \delta_i$. Then, by Lemma 2 we have

$$
\begin{aligned}
&\frac{1}{\alpha_n}\sum_{k=u_n}^{v_n} E\psi(|X_{nk}|)I\{|X_{nk}| \leq \varphi_n\}\\
&\quad\leq \delta_1 CE\gamma(|X|)I\{|X| > \varphi_0\} + \sum_{i=2}^{n}\chi_{i-1}E\gamma(|X|)I\{|X| > \varphi_{i-1}\}\\
&\quad\leq \delta_1 CE\gamma(|X|) + \sum_{i=1}^{\infty}\chi_i E\gamma(|X|)I\{|X| > \varphi_i\}
\end{aligned}
$$

$$\leq \delta_1 CE\gamma(|X|) + C'E\gamma(|X|) \sum_{k=1}^{\varphi^{-1}(|X|)} \chi_k$$
$$\leq \mathbf{C}(E\gamma(|X|) + E\gamma(|X|)\delta_{\varphi^{-1}(|X|)}).$$

**Lemma 4.** *Let $p \leq q$ and $\{V_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of random elements $S(p, \alpha)$-dominated by a random variable $X$. Further, let $\{\varphi_n, n \geq 0\}$ and $\{\alpha_n, n \geq 1\}$ be two sequences of positive numbers where $\{\varphi_n, n \geq 0\}$ is increasing with $\varphi_0 = 0$.*

*Then there exists a positive constant $\mathbf{C}$ such that*

$$\frac{1}{\alpha_n} \sum_{k=u_n}^{v_n} E\|V_{nk}\|^q I\{\|V_{nk}\| \leq \varphi_n\} \leq \mathbf{C}(E|X|^p + E|X|^p \delta_{\varphi^{-1}(|X|)}).$$

*Proof.* It suffices to apply Lemma 3 with the array $\{\|V_{nk}\|, u_n \leq k \leq v_n, n \geq 1\}$ of random variables, which is $S(p, \alpha)$ dominated by $X$ with $\psi(t) = t^q$ and $\gamma(t) = t^p$.

## III. MAIN RESULTS

With preliminaries accounted for, we can now present our main result.

**Theorem 4.** *Let $1 \leq p \leq q \leq 2$ and $\{V_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of row-wise independent random elements taking values in a real separable Banach space $(B, \|\cdot\|)$ and $S(p, \alpha)$ dominated by a random variable $X$. Suppose that their exists $J \geq 2$ such that $\sum_{n=1}^{\infty}(\alpha_n \varphi_n^{-q})^J < \infty$, $E|X|^q < \infty$ and $E|X|^p \beta(\varphi^{-1}(X/\varepsilon)) < \infty$ for any $\varepsilon > 0$ where $\beta(k) = \sum_{n=1}^{k} \alpha_n \varphi_n^{-p}$, $k \geq 1$ and $\{\varphi_n, n \geq 0\}$ is an increasing sequence of positive constants. Moreover, let one of the following conditions hold:*

(a) *We have the information on a geometry of the Banach space $(B, \|\cdot\|)$: It is of Rademacher type $q$ and $EV_{nk} = 0$, or*
(b) *We do not have information on a geometry of the underlying Banach space, but $\frac{1}{\varphi_n}\sum_{k=u_n}^{v_n} V_{nk} \to 0$ in probability.*

*Then*

$$\frac{1}{\varphi_n} \sum_{k=u_n}^{v_n} V_{nk} \quad \textit{converges completely to zero.}$$

*Proof.* Consider the following three conditions, where $J \geq 2$ and $1 \leq r \leq 2$.

1. $\sum_{n=1}^{\infty} \sum_{k=u_n}^{v_n} P\{\|V_{nk}\| > \varepsilon\varphi_n\} < \infty$.

2. $\sum_{n=1}^{\infty}(\sum_{k=u_n}^{v_n} E(\|\frac{V_{nk}}{\varphi_n}\|^r I\{\|V_{nk}\| \leq \varphi_n\}))^J < \infty$.
3. $\max_{u_n \leq i \leq v_n} \|\sum_{k=u_n}^{i} E(\frac{V_{nk}}{\varphi_n} I\{\|V_{nk}\| \leq \varphi_n\})\| \longrightarrow 0$, as $n \to \infty$.

According to Theorem 2 in case (a) (Banach space is of type $q$), if all three conditions (1), (2), with $r = q$, and (3) hold, then the conclusion of the theorem holds.

According to Theorem 1 in case (b) (general Banach space), if the two conditions (1) and (2) with $r = 2$ hold, then the conclusion of the theorem holds.

So, we need to check that all the conditions (1), (2) with $r = q$ or 2, and (3) follow from the assumptions of Theorem 4.

The first condition:

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=u_n}^{v_n} P\{\|V_{nk}\| > \varepsilon\varphi_n\} &= \sum_{n=1}^{\infty}\sum_{k=u_n}^{v_n} P\{\|V_{nk}\| I\{\|V_{nk}\| > \varepsilon\varphi_n\} > \varepsilon\varphi_n\}\\ &\leq \sum_{n=1}^{\infty}\sum_{k=u_n}^{v_n} \frac{E\|V_{nk}\|^p I\{\|V_{nk}\| > \varepsilon\varphi_n\}}{\varepsilon^p\varphi_n^p}\\ &\leq C\sum_{n=1}^{\infty}\frac{\alpha_n}{\varphi_n^p} E|X|^p I\{|X| > \varepsilon\varphi_n\}\\ &\leq CE|X|^p\beta(\varphi^{-1}(|X|/\varepsilon)) < \infty.\end{aligned}$$

Here we applied Markov's inequality in the first inequality, the $S(p, \alpha)$ domination condition in the second, and Lemma 2 in the last inequality.

For the second condition, let $r = q$. Then

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{k=u_n}^{v_n} E\left\|\frac{V_{nk}}{\varphi_n}\right\|^q I\{\|V_{nk}\| \leq \varphi_n\}\right)^J &\leq C\sum_{n=1}^{\infty}\left(\frac{\alpha_n}{\varphi_n^q}(E|X|^p + E|X|^p\delta_{\varphi^{-1}(|X|)})\right)^J\\ &\leq C(E|X|^p + E|X|^p\delta_{\varphi^{-1}(|X|)})^J < \infty\end{aligned}$$

where Lemma 4 has been used with $\gamma(t) = t^p$, $\phi(t) = t^q$ and $\delta_i = \max_{\varphi_{i-1} < t \leq \varphi_i} t^{q-p} = \varphi_i^{q-p}$ in the last inequality.

And we can deal with the third condition as follows:

$$\begin{aligned}&\max_{u_n \leq i \leq v_n}\left\|\sum_{k=u_n}^{i} E\frac{V_{nk}}{\varphi_n} I\{\|V_{nk}\| \leq \varphi_n\}\right\|\\ &\quad\leq \max_{u_n \leq i \leq v_n}\sum_{k=u_n}^{i}\left\|E\frac{V_{nk}}{\varphi_n} I\{\|V_{nk}\| \leq \varphi_n\}\right\|\\ &\quad= \max_{u_n \leq i \leq v_n}\sum_{k=u_n}^{i}\left\|E\frac{V_{nk}}{\varphi_n} I\{\|V_{nk}\| > \varphi_n\}\right\| \quad (\text{since } EV_{nk} = 0)\end{aligned}$$

$$\leq \max_{u_n \leq i \leq v_n} \sum_{k=u_n}^{i} E\left\|\frac{V_{nk}}{\varphi_n}\right\| I\{\|V_{nk}\| > \varphi_n\}$$

$$= \sum_{k=u_n}^{v_n} E\left\|\frac{V_{nk}}{\varphi_n}\right\| I\{\|V_{nk}\| > \varphi_n\}$$

$$\leq \sum_{k=u_n}^{v_n} E\left\|\frac{V_{nk}}{\varphi_n}\right\|^p I\{\|V_{nk}\| > \varphi_n\}$$

$$\leq C\frac{\alpha_n}{\varphi_n^p} E|X|^p I\{|X| > \varphi_n\} \quad \text{(by the } S(p,\alpha) \text{ domination condition)}$$

$$\leq C\frac{\alpha_n}{\varphi_n^p \beta(n)} E|X|^p \beta(\varphi^{-1}(|X|)) I\{|X| > \varphi_n\}$$

$$\leq CE|X|^p \beta(\varphi^{-1}(|X|)) I\{|X| > \varphi_n\} \longrightarrow 0 \quad \text{as } n \to \infty.$$

The random variables case is of course of special interest. We can formulate the following result for this special case.

**Corollary.** *Let $1 \leq p < 2$ and $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of row-wise independent random variables satisfying $EX_{nk} = 0$ and $S(\alpha, p)$ dominated by a random variable $X$. Suppose that there exists $q$ in $[p, 2]$ and $J \geq 2$ such that $\sum_{n=1}^{\infty}(\alpha_n \varphi_n^{-q})^J < \infty$. Furthermore, let $E|X|^q < \infty$ and $E|X|^p \beta(\varphi^{-1}(|X|)) < \infty$ where $\beta(k) = \sum_{n=1}^{k} \alpha_n \varphi_n^{-p}$, with $k \geq 1$. Then*

$$\frac{1}{\varphi_n} \sum_{k=u_n}^{v_n} X_{nk} \quad \textit{converges completely to zero.}$$

*Proof.* The real line is of type $q$ for any $q \leq 2$ and we can apply Theorem 3(a). Simple details are left to a reader. Alternately, we can use the result of [14] and repeat all derivations by analogy with the proof of Theorem 4.

**Remark 2.** Take $\alpha_n = n$ and $\varphi_n = n^{1/p}$ to obtain Theorem 1 of [13].

## REFERENCES


1. Hsu, P.L., and Robbins, H. 1947. Complete convergence and the law of large numbers. *Proc. Nat. Acad. Sci. USA* 33:25–31.
2. Pruitt, W.E. 1966. Summability of independent random variables. *J. Math. Mech.* 15:769–776.
3. Rohatgi, V.K. 1971. Convergence of weighted sums of independent random variables. *Proc. Cambridge Philos. Soc.* 69:305–307.
4. Hu, T.-C., Móricz, F., and Taylor, R.L. 1989. Strong laws of large numbers for arrays of row-wise independent random variables. *Acta Math. Acad. Sci. Hungar.* 54:153–162.

5. Gut, A. 1992. Complete convergence for arrays. *Periodica Math. Hungarica* 25:51–75.
6. Wang, X., Rao, M.B., and Yang, X. 1993. Convergence rates on strong laws of large numbers for arrays of row-wise independent elements. *Stochastic Anal. Appl.* 11:115–132.
7. Kuczmaszewska, A., and Szynal, D. 1994. On complete convergence in a Banach space. *Internat. J. Math. Math. Sci.* 17:1–14.
8. Sung, S.H. 1997. Complete convergence for weighted sums of arrays of row-wise independent *B*-valued random variables. *Stochastic Anal. Appl.* 15:255–267.
9. Hu, T.-C., Rosalsky, A., Szynal, D., and Volodin, A. 1999. On complete convergence for arrays of row-wise independent random elements in Banach spaces. *Stochastic Anal. Appl.* 17:963–992.
10. Hu, T.-C., Li, D., Rosalsky, A., and Volodin, A. 2003. On the rate of complete convergence for weighted sums of Banach space valued random elements. *Theory Probab. Appl.* 47:455–468.
11. Taylor, R.L. 1978. *Stochastic Convergence of Weighted Sums of Random Elements in Linear Spaces*. Vol. 672. Lect. Notes Math. Springer-Verlag, Berlin.
12. Hu, T.-C., Ordóñez Cabrera M., Sung S.H., and Volodin A.I. 2003. Complete convergence for arrays of row-wise independent random variables. *Commun. Korean Math. Soc.* 18:375–383.
13. Sung S.H. 2000. Complete convergence for arrays of row-wise independent random variables (II). *Bull Korean Math. Soc.* 37:255–263.
14. Hu T.-C., Szynal D., and Volodin A.I. 1998. A note on complete convergence for arrays. *Statist. Probab. Lett.* 38:27–31.