THEORY PROBAB. APPL.
Vol. 65, No. 3, pp. 470–481

ON THE SUB-GAUSSIANITY OF THE r-CORRELOGRAMS*

R. GIULIANO†, M. ORDÓÑEZ CABRERA‡, AND A. VOLODIN§

Abstract. In this article, we show that the centered relay correlation function is a sub-Gaussian random variable. This is done by a careful analysis of its Laplace transform and by estimating the sub-Gaussian standard of the r-correlograms.

Key words. relay correlogram function, sub-Gaussian random variables, sub-Gaussian standard, Laplace transform, stationary Gaussian process

DOI. 10.1137/S0040585X97T99006X

1. Introduction. This paper concerns the classical problem of establishing properties of estimators for the correlation function of a stationary Gaussian process. Stationary Gaussian processes are the main object and bases for a large variety of probability and statistical models (see, for example, [1] and [2]). It is a well-known fact that a stationary Gaussian process is characterized by its autocorrelation function (see definitions in what follows). This is why an estimation of the autocorrelation function from the observations of some realization of the stationary Gaussian process plays a crucial role in the construction of an appropriate model for data.

To be more precise, consider a real-valued standard stationary Gaussian process $Z = \{Z(t),\, t \geqslant 0\}$ such that $\mathbf{E}Z(t) = 0$ and $\operatorname{Var} Z(t) = 1$ for all $t \geqslant 0$. This process is characterized by its autocorrelation function

$$\rho(h) = \frac{\mathbf{E}[(Z(t) - \mathbf{E}Z(t))(Z(t+h) - \mathbf{E}Z(t+h))]}{\sqrt{\operatorname{Var} Z(t) \operatorname{Var} Z(t+h)}}, \qquad h \geqslant 0, \quad t \geqslant 0.$$

Note that in our case, $\rho(h) = \mathbf{E}Z(t)Z(t+h)$, $h \geqslant 0$, $t \geqslant 0$.

In the time series analysis, a discrete time $i \geqslant 0$ is considered (see, for example, [2]), and the following estimator of $\rho(h)$ is usually applied:

$$\widehat{\rho}(h) = n^{-1} \sum_{i=0}^{n-h} Z_i Z_{i+h}, \qquad h = 0, 1, 2, \dots. \tag{1.1}$$

In [3], a continuous time estimator of $\rho(h)$ is introduced. More precisely, let $T > 0$ be the time horizon; then the estimator studied in [3] is the following autocorrelation

*Received by the editors September 26, 2018; revised February 23, 2020. This research was supported by the subsidy allocated to Kazan Federal University for the state assignment in the sphere of scientific activities (project 1.13556.2019/13.1). The first author was supported by the University of Pisa (Fondi di Ateneo). The second author was partially supported by the Plan Andaluz de Investigacion de la Junta de Andalucia FQM-127 grant P08-FQM-03543, and by MEC grant MTM2015-65242-C2-1-P. A part of the research of the third author was done during a visit to the University of Pisa in December 2016. The authors express their appreciation to the INDAM for supporting this visit. Originally published in the Russian journal *Teoriya Veroyatnostei i ee Primeneniya*, 65 (2020), pp. 602–616.
https://doi.org/10.1137/S0040585X97T99006X

†Dipartimento di Matematica, Università di Pisa, Italy (giuliano@dm.unipi.it).
‡Departamento Análisis Matemático, Universidad de Sevilla, Spain (cabrera@us.es).
§Department of Mathematics and Statistics, University of Regina, SK S4S 0A2, Canada (andrei.volodin@uregina.ca).

process $R = \{R(h),\, h \in [0,T]\}$:

$$R(h) = \frac{1}{T}\int_0^T Z_s Z_{s+h}\, ds. \tag{1.2}$$

We refer the reader to [3] for properties of the autocorrelation process and some sharp exponential bounds for its deviation probabilities.

In this paper, we do not consider an autocorrelation function. Instead, we consider the relay correlation function (defined in what follows), which is a modification of the autocorrelation function. It is introduced mainly for computational advantages. The relay correlation function is mostly used in engineering applications (see, for example, [4]).

Now we define the relay correlogram function. This is a process similar in some sense to the autocorrelation process (1.2).

DEFINITION 1.1. *The* relay correlation function *of a standard stationary Gaussian process* $\{Z(t),\, t \geqslant 0\}$ *is the function*

$$\rho^r(h) = \mathbf{E}[Z(0)\operatorname{sign} Z(h)],$$

where $\operatorname{sign} Z(h)$ *is defined as*

$$\operatorname{sign} Z(h) = \mathbf{1}_{\{Z(h)>0\}} - \mathbf{1}_{\{Z(h)<0\}},$$

and $\mathbf{1}_A$ *denotes the indicator function of the event* A.

Clearly, $\rho^r(h) = \mathbf{E}[Z(s)\operatorname{sign} Z(s+h)]$ for any $s \geqslant 0$, since the process Z is stationary.

Notice that $\rho^r(h)$ takes in account only the sign, not the whole value, of the second random variable (r.v.). This is the main advantage of the relay correlation function over the usual autocorrelation function. For instance, consider a statistical problem of a plug-in estimation of the autocorrelation function based on (1.1) having a dataset of the process realizations $\{z_i,\, 1 \leqslant i \leqslant n\}$ in hand. Obviously, the calculation of $\sum_i^{n-h} z_i z_{i+h}$ involves a product for each term of the sum and the sum itself. But the calculation for the relay correlation function, that is, $\sum_i^{n-h} z_i \operatorname{sign} z_{i+h}$, involves only a control of the sign for each term and the sum. From the computational point of view, a product is more expensive than a sign control, and therefore the second calculation is cheaper than the first one.

Also, we do not lose any information if we consider the relay correlation function over the autocorrelation function. Namely, the following identity is true:

$$\begin{aligned}\rho^r(h) &= \mathbf{E}[Z(0)\operatorname{sign} Z(h)] = \mathbf{E}\big[\operatorname{sign} Z(h)\mathbf{E}[Z(0)\,|\,Z(h)]\big]\\ &= \mathbf{E}[\rho(h)Z(h)\operatorname{sign} Z(h)] = \rho(h)\mathbf{E}|Z(h)| = \sqrt{\frac{2}{\pi}}\rho(h).\end{aligned}$$

Here we used the fact that the r.v.'s $Z(0)$ and $Z(h)$ are standard normal variables and jointly normally distributed; hence $\mathbf{E}[Z(0)\,|\,Z(h)] = \rho(h)Z(h)$.

In order to estimate the relay correlation function in a way similar to how (1.2) estimates the autocorrelation function, we introduce the following definition.

Definition 1.2. *The process* $\{\widehat{R}^r(h),\, h \geqslant 0\}$ *is called an* r-correlogram process (*or a* sample relay correlation process) *of* $(Z(t))$ *if*

$$\widehat{R}^r(h) = \frac{1}{T}\int_0^T Z(t)\,\mathrm{sign}\,Z(t+h)\,dt, \qquad h > 0, \tag{1.3}$$

and the integral is interpreted as a mean square Riemann integral.

Since for every h we have $\mathbf{E}\widehat{R}^r(h) = \rho^r(h)$, the r-correlogram process is an unbiased estimator for the relay correlation function.

In the following, we consider the centered r-correlogram process

$$\widehat{U}^r(h) = \widehat{R}^r(h) - \mathbf{E}\widehat{R}^r(h) = \widehat{R}^r(h) - \rho^r(h), \qquad h \geqslant 0.$$

The main result of this paper shows that the centered relay correlation process is a sub-Gaussian random process.

Now we review the notion of sub-Gaussianity. To the best of our knowledge, the notion of a sub-Gaussian r.v. was introduced by Kahane [5] as *sous-gaussienne variables aléatoires*. A detailed discussion and proofs of all results that we present in what follows may be found in [3, section 1.1]. We collect only the most important definitions and results here.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space on which all r.v.'s that we consider are defined. An r.v. X is called *sub-Gaussian* if its moment generating function (Laplace transform) $\mathbf{E}e^{tX}$ is defined for all t and there exists a positive constant a such that for every t,

$$\mathbf{E}e^{tX} \leqslant e^{a^2t^2/2}.$$

The smallest a that satisfies the previous inequality is called the *sub-Gaussian standard* of the r.v. X and is denoted by $\tau(X)$, that is,

$$\tau(X) = \inf\{a > 0 \colon \mathbf{E}e^{tX} \leqslant e^{a^2t^2/2} \text{ for all } t\}.$$

The space of all sub-Gaussian r.v.'s is denoted by $\mathrm{Sub}(\Omega)$. It can be proved that $\mathrm{Sub}(\Omega)$ is a Banach space with the norm $\tau(\,\cdot\,)$. Moreover, if $\{X_n,\, n \geqslant 1\}$ is a sequence of r.v.'s in $\mathrm{Sub}(\Omega)$ which converges in probability to an r.v. X and such that $\sup_{n\geqslant 1}\tau(X_n) < \infty$, then X is a sub-Gaussian r.v., and $\tau(X) \leqslant \sup_{n\geqslant 1}\tau(X_n)$.

Sub-Gaussian r.v.'s are interesting because they possess the following exponential upper bound for the tail of the distribution. If X is a sub-Gaussian r.v., then

$$\mathbf{P}(|X| \geqslant x) \leqslant 2e^{-x^2/(2\tau^2(X))}. \tag{1.4}$$

The problem of finding a sub-Gaussian standard for an r-correlogram process was first considered in the Ph.D. thesis (unpublished but available online) of Castellucci [6], a former student of Dr. R. Giuliano, a coauthor of this paper. In [6], the following result has been proved.

Theorem 1.1. *Let* $\{\widehat{U}^r(h),\, h \geqslant 0\}$ *be a centered* r*-correlogram process. For any* $h \geqslant 0$, *the r.v.* $\widehat{U}^r(h)$ *is sub-Gaussian with the sub-Gaussian standard*

$$\tau(\widehat{U}^r(h)) \leqslant \sqrt[4]{3.1}\left(3 + \frac{\rho^2(h)}{\pi}\right). \tag{1.5}$$

The calculations of the sub-Gaussian standard for the centered r-correlogram process $\widehat{U}^r(h)$ presented in [6] are quite cumbersome and not straightforward. In this paper, we were able to obtain a much more precise estimation of the sub-Gaussian standard using a much simpler technique.

2. Main result. In order to formulate our main results, we introduce the following notation. Let X and Y be two standard normal r.v.'s (with correlation coefficient ρ). That is, $X \sim \mathbf{N}(0,1)$, $Y \sim \mathbf{N}(0,1)$, $\operatorname{cor}(X,Y) = \rho$. Further, we consider the r.v.'s $\widetilde{U} = X\mathbf{1}_{\{Y>0\}}$ and $U = \widetilde{U} - \mathbf{E}\widetilde{U}$. It appears that the main "technical" question we need to solve in order to estimate the sub-Gaussian standard of the centered r-correlogram process $\widehat{U}^r(h)$ is finding the sub-Gaussian standard τ of the r.v. U. Namely, the following result can be presented.

Proposition 2.1. *Let $\{\widehat{U}^r(h),\, h \geqslant 0\}$ be a centered r-correlogram process. For any $h \geqslant 0$, the r.v. $\widehat{U}^r(h)$ is sub-Gaussian with the sub-Gaussian standard*

$$\tau(\widehat{U}^r(h)) \leqslant 2\tau(U), \tag{2.1}$$

where $U = Z(0)\mathbf{1}_{\{Z(h)>0\}} - \mathbf{E}[Z(0)\mathbf{1}_{\{Z(h)>0\}}]$.

Proof. Fix any $h > 0$. We consider the left Riemann sums of the integral in (1.3).

Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of the interval $[0,T]$, and let $\Delta t_k = t_k - t_{k-1}$, $1 \leqslant k \leqslant n$. Further, let

$$S = \sum_{k=1}^{n} Z(t_k) \operatorname{sign} Z(t_k + h) \Delta t_k$$

be a left Reimann sum of the integral in (1.3), and consider the centered variable

$$S - \mathbf{E}S = \sum_{k=1}^{n} \{Z(t_k) \operatorname{sign} Z(t_k+h) - \mathbf{E}[Z(t_k) \operatorname{sign} Z(t_k+h)]\}\Delta t_k.$$

The r.v. $Z(t_k) \operatorname{sign} Z(t_k + h)$ can be written in the form

$$Z(t_k)(\mathbf{1}_{\{Z(t_k+h)>0\}} - \mathbf{1}_{\{Z(t_k+h)<0\}}),$$

and hence,

$$\begin{aligned}
Z(t_k) &\operatorname{sign} Z(t_k+h) - \mathbf{E}[Z(t_k) \operatorname{sign} Z(t_k+h)] \\
&= \big(Z(t_k)\mathbf{1}_{\{Z(t_k+h)>0\}} - \mathbf{E}[Z(t_k)\mathbf{1}_{\{Z(t_k+h)>0\}}]\big) \\
&\quad - \big(Z(t_k)\mathbf{1}_{\{Z(t_k+h)<0\}} - \mathbf{E}[Z(t_k)\mathbf{1}_{\{Z(t_k+h)<0\}}]\big).
\end{aligned}$$

Consider two sequences of r.v.'s

$$A_k^+(h) = Z(t_k)\mathbf{1}_{\{Z(t_k+h)>0\}} - \mathbf{E}[Z(t_k)\mathbf{1}_{\{Z(t_k+h)>0\}}]$$

and

$$A_k^-(h) = -\big\{Z(t_k)\mathbf{1}_{\{Z(t_k+h)<0\}} - \mathbf{E}[Z(t_k)\mathbf{1}_{\{Z(t_k+h)<0\}}]\big\}.$$

Then

$$S = \sum_{k=1}^{n} \big(A_k^+(h) + A_k^-(h)\big)\Delta t_k.$$

The underlying process $\{Z(t),\, t \geqslant 0\}$ is stationary, and hence the r.v.'s $A_k^+(h)$, $k \geqslant 1$, have the same distribution as $U = Z(0)\mathbf{1}_{\{Z(h)>0\}} - \mathbf{E}[Z(0)\mathbf{1}_{\{Z(h)>0\}}]$. Similarly, the r.v.'s $A_k^-(h)$, $k \geqslant 1$, have the same distribution as $V = -(Z(0)\mathbf{1}_{\{Z(h)<0\}} - \mathbf{E}[Z(0)\mathbf{1}_{\{Z(h)<0\}}])$.

Moreover, using symmetry arguments we prove Proposition 3.1 in the next section to the effect that the r.v.'s U and V are identically distributed.

In Proposition 4.1 in what follows, we prove that the r.v. U is sub-Gaussian. Hence, since $\sum_{k=1}^n \Delta t_k = T$ and τ is a norm, we have

$$\tau\biggl(\frac{1}{T}(S - \mathbf{E}S)\biggr) = \frac{1}{T}\tau\biggl(\sum_{k=1}^{n}\bigl(A_k^+(h) + A_k^-(h)\bigr)\Delta t_k\biggr)$$
$$\leqslant \frac{1}{T}\sum_{k=1}^{n}\Delta t_k\bigl(\tau(A_k^+(h)) + \tau(A_k^-(h))\bigr) = \frac{1}{T}\sum_{k=1}^{n}\Delta t_k 2\tau(U) = 2\tau(U).$$

Now as a Reimann integral, $\widehat{U}^r(h) = \lim_{\max \Delta t_k \to 0}(1/T)(S - \mathbf{E}S)$ in mean squares and hence in probability. Since the space $\mathrm{Sub}(\Omega)$ is closed under convergence in probability (see above), we have

$$\tau(\widehat{U}^r(h)) \leqslant 2\tau(U).$$

Proposition 2.1 is proved.

Moreover, in Proposition 4.1 it is shown that $\tau(U) = 1$. Hence the following result is true.

THEOREM 2.1. *Let $\{\widehat{U}^r(h), h \geqslant 0\}$ be a centered r-correlogram process. For any $h \geqslant 0$, the r.v. $\widehat{U}^r(h)$ is sub-Gaussian with the sub-Gaussian standard*

$$\tau(\widehat{U}^r(h)) \leqslant 2. \tag{2.2}$$

A simple comparison shows that estimate (2.2) is much sharper than (1.5).

Theorem 2.1 allows us to construct a pointwise confidence interval for $\widehat{R}^r(h)$.

COROLLARY 2.1. *For every h and for every positive x,*

$$\mathbf{P}\bigl(|\widehat{R}^r(h) - \rho^r(h)| > x\bigr) \leqslant 2e^{-\pi^2 x^2/8}.$$

Proof. By (1.4) and since $|\rho(h)| \leqslant 1$, we easily get

$$\mathbf{P}\bigl(|\widehat{R}^r(h) - \rho^r(h)| > x\bigr) \leqslant 2e^{-x^2/(2\tau^2(U))} \leqslant 2e^{-\pi^2 x^2/8}.$$

Corollary 2.1 is proved.

3. Technicalities. Let X and Y be two standard normal r.v.'s with the correlation coefficient ρ. That is, $X \sim \mathbf{N}(0,1)$, $Y \sim \mathbf{N}(0,1)$, $\mathrm{cor}(X,Y) = \rho$. It is simple to write the joint density function of X and Y:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\biggl\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\biggr\}, \qquad -\infty < x, y < \infty.$$

Consider the r.v. $\widetilde{U} = X\mathbf{1}_{\{Y>0\}}$. Note that the r.v. X is continuous and the r.v. $\mathbf{1}_{\{Y>0\}}$ is discrete.

Observe that the r.v. $U = Z(0)\mathbf{1}_{\{Zh>0\}} - \mathbf{E}[Z(0)\mathbf{1}_{\{Z(h)>0\}}]$ from the proof of Proposition 2.1 is of the form

$$U = X\mathbf{1}_{\{Y>0\}} - \mathbf{E}[X\mathbf{1}_{\{Y>0\}}],$$

where clearly we have put $X = Z(0)$, $Y = Z(h)$, $\rho = \rho(h)$, $|\rho| \leqslant 1$.

In what follows, in Proposition 3.1 we show that the r.v.'s U and V, $V = -(X\mathbf{1}_{\{Y<0\}} - \mathbf{E}[X\mathbf{1}_{\{Y>0\}}])$, are identically distributed, and hence it is enough to consider only the r.v. U.

We can derive the d.f. of the r.v. $\widetilde{U} = X\mathbf{1}_{\{Y>0\}}$. The r.v. $\widetilde{U}$ takes value zero with probability 1/2, and the remaining 1/2 is spread over by the normal distribution:

$$F_{\widetilde{U}}(t) = \mathbf{P}(\widetilde{U} < t) = \begin{cases} \displaystyle\int_{-\infty}^{t} dx \int_{0}^{+\infty} f_{X,Y}(x,y)\,dy & \text{for } t \leqslant 0, \\ \displaystyle\frac{1}{2} + \int_{-\infty}^{t} dx \int_{0}^{+\infty} f_{X,Y}(x,y)\,dy & \text{for } t > 0. \end{cases}$$

Therefore, the d.f. of $\widetilde{U}$ has a gap 1/2 at $t = 0$. For example, for $\rho = 0$ the d.f.'s $F_{\widetilde{U}}(0-) = 1/4$ and $F_{\widetilde{U}}(0+) = 3/4$.

First we prove that r.v.'s $U = X\mathbf{1}_{\{Y\geqslant 0\}}$ and $V = -X\mathbf{1}_{\{Y\leqslant 0\}}$ are identically distributed. For this we need the following lemma.

LEMMA 3.1. *Let (X,Y) be a bivariate random vector having a density f such that $f(x,y) = f(-x,-y)$. Let $\phi(x,y)$ be a measurable function, and denote $U = \phi(X,Y)$ and $V = \phi(-X,-Y)$. Then U and V have the same distribution.*

The proof of this lemma is straightforward and omitted.

Applying the preceding lemma to a bivariate Gaussian law with mean 0 and to the function $\phi(x,y) = x\mathbf{1}_{[0,+\infty)}(y)$, we get the following result.

PROPOSITION 3.1. *Let (X,Y) be a bivariate random vector having Gaussian law with mean 0. Then the two r.v.'s $U = X\mathbf{1}_{\{Y\geqslant 0\}}$; $V = -X\mathbf{1}_{\{Y\leqslant 0\}}$ have the same distribution.*

Proof. Since

$$\{Y \neq 0\} \subseteq \{-X\mathbf{1}_{\{Y\leqslant 0\}} = -X\mathbf{1}_{\{Y<0\}}\},$$

we have

$$1 = \mathbf{P}(Y \neq 0) \leqslant \mathbf{P}(-X\mathbf{1}_{\{Y\leqslant 0\}} = -X\mathbf{1}_{\{Y<0\}}) \leqslant 1.$$

Hence

$$\mathbf{P}(-X\mathbf{1}_{\{Y\leqslant 0\}} = -X\mathbf{1}_{\{Y<0\}}) = 1.$$

This implies that U and $V = -X\mathbf{1}_{\{Y<0\}}$ have the same distribution. Proposition 3.1 is proved.

In order to prove the sub-Gaussianity of the r.v. U we need to know the moment generating functions (Laplace transforms) of $\widetilde{U}$ and U.

PROPOSITION 3.2. *The moment generating function of $\widetilde{U}$ is given by the formula*

$$\widetilde{\mathcal{M}}(t) = e^{t^2/2}\Phi(\rho t) + \frac{1}{2}, \tag{3.1}$$

where $\Phi(t) = (2\pi)^{-1/2}\int_{-\infty}^{t} e^{x^2/2}\,dx$ denotes the d.f. of the standard Gaussian law $\mathbf{N}(0,1)$.

Proof. The moment generating function of the r.v. $\widetilde{U}$ can be evaluated as

$$\begin{aligned} \mathbf{E}e^{t\widetilde{U}} &= \int_{-\infty}^{+\infty} e^{tx}\,dx \int_{0}^{+\infty} f_{X,Y}(x,y)\,dy + \int_{-\infty}^{+\infty} dx \int_{-\infty}^{0} f_{X,Y}(x,y)\,dy \\ &= \int_{-\infty}^{+\infty} e^{tx}\,dx \int_{0}^{+\infty} f_{X,Y}(x,y)\,dy + \frac{1}{2}; \end{aligned}$$

however, the following approach is probably more elegant.

Since

$$e^{t\widetilde{U}} = e^{tX}\mathbf{1}_{\{Y>0\}} + \mathbf{1}_{\{Y\leqslant 0\}},$$

we can write

$$\mathbf{E}e^{t\widetilde{U}} = \mathbf{P}(Y\leqslant 0) + \mathbf{E}[e^{tX}\mathbf{1}_{\{Y>0\}}] = \frac{1}{2} + \mathbf{E}\big[\mathbf{1}_{\{Y>0\}}\mathbf{E}[e^{tX}\,|\,Y]\big].$$

Now X, conditioned to Y, has the normal distribution with mean ρY and variance $1-\rho^2$. Hence

$$\mathbf{E}[e^{tX}\,|\,Y] = \exp\biggl\{t\rho Y + (1-\rho^2)\frac{t^2}{2}\biggr\}.$$

From this we obtain

$$\begin{aligned}\mathbf{E}\big[\mathbf{1}_{\{Y>0\}}\mathbf{E}[e^{tX}\,|\,Y]\big] &= \int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\biggl\{-\frac{y^2}{2} + t\rho y + (1-\rho^2)\frac{t^2}{2}\biggr\}\,dy\\ &= e^{t^2/2}\int_0^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\biggl\{-\frac{(y-t\rho)^2}{2}\biggr\}\,dy = e^{t^2/2}\Phi(\rho t).\end{aligned}$$

Proposition 3.2 is proved.

PROPOSITION 3.3. *The moment generating function of U is given by the formula*

$$\mathcal{M}(t) = e^{-t\rho/\sqrt{2\pi}}\biggl(e^{t^2/2}\Phi(\rho t) + \frac{1}{2}\biggr). \tag{3.2}$$

Proof. We can write the direct formula for expectation of the r.v. $\widetilde{U}$:

$$\mathbf{E}\widetilde{U} = \int_{-\infty}^{\infty} x\,dx\int_0^\infty f_{X,Y}(x,y)\,dy;$$

however, it is probably simpler to differentiate the moment generating function at zero.

As an immediate consequence of formula (3.1), we get

$$\mathbf{E}\widetilde{U} = \widetilde{\mathcal{M}}'(0) = \frac{\rho}{\sqrt{2\pi}}.$$

Hence the moment generating function $\mathcal{M}$ of $U = \widetilde{U} - \mathbf{E}\widetilde{U}$ is $e^{-\rho t/(\sqrt{2\pi})}\widetilde{\mathcal{M}}(t)$. Proposition 3.3 is proved.

Remark 3.1. Knowing the moment generating function $\widetilde{\mathcal{M}}(t)$ and taking the second derivative at zero, we obtain that $\mathbf{E}\widetilde{U}^2 = 1/2$. Hence the variance of the r.v. U is

$$\operatorname{Var} U = \frac{1}{2}\biggl(1 - \frac{\rho^2}{\pi}\biggr). \tag{3.3}$$

Note that from here we can state that the sub-Gaussian standard $\tau(U) \geqslant (1-\rho^2/\pi)/2$. It follows from the fact that for any sub-Gaussian r.v. X, the sub-Gaussian standard $\tau(X) \geqslant \operatorname{Var} X$ (see [3, Lemma 1.2]).

4. Proof of the main result.

PROPOSITION 4.1. *The r.v. U is sub-Gaussian, and $\tau(U) = 1$.*

Proof. First we show that $\tau(U) \leqslant 1$. By definition of the sub-Gaussian standard, in order to show that $\tau(U) \leqslant 1$ we need to prove that the Laplace transform (moment generating function) of the r.v. U satisfies the inequality $\mathbf{E}e^{tU} \leqslant e^{t^2/2}$. According to formula (3.2), this is equivalent to saying that

$$e^{-\rho t/\sqrt{2\pi}}\left(e^{t^2/2}\Phi(\rho t) + \frac{1}{2}\right) \leqslant e^{t^2/2} \tag{4.1}$$

for all $-\infty < t < \infty$ and all $-1 \leqslant \rho \leqslant 1$.

Simple calculations show that (4.1) is equivalent to the statement $f(\rho, t) \geqslant 0$, where

$$f(\rho, t) = \exp\left\{\frac{t^2}{2} + \frac{\rho}{\sqrt{2\pi}}t\right\} - \exp\left\{\frac{t^2}{2}\right\}\Phi(\rho t) - 0.5,$$

and $-\infty < t < \infty$ and $-1 \leqslant \rho \leqslant 1$.

Note that the function $f(\rho, t)$ is "even," that is, $f(\rho, t) = f(-\rho, -t)$, and hence it is enough to consider only nonnegative values of t. So in the following we assume that $t \geqslant 0$.

Now we investigate the behavior of the function $f(\rho, t)$ in the region $-1 \leqslant \rho \leqslant 1$, $t \geqslant 0$ and show that it is always nonnegative.

The derivation of the function $f(\rho, t)$ by t is quite cumbersome, so we take the derivative with respect to ρ,

$$\frac{\partial f(\rho, t)}{\partial \rho} = \frac{e^{t^2/2}t}{\sqrt{2\pi}}\left(e^{\rho t/\sqrt{2\pi}} - e^{-\rho^2 t^2/2}\right).$$

Letting $\partial f(\rho, t)/\partial \rho = 0$, we obtain the following three solutions.

1. The first solution is $t = 0$. It is a boundary point (we consider only $t \geqslant 0$) and gives the global minimum. Note that $f(\rho, 0) = 0$ for all ρ.

2. The second solution is $\rho = 0$. This is a local minimum because

$$\frac{\partial^2 f(\rho, t)}{\partial \rho^2} = \frac{e^{t^2/2}t^2}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{2\pi}}e^{\rho t/\sqrt{2\pi}} + \rho t e^{-\rho^2 t^2/2}\right)$$

for the second derivative and since

$$\frac{\partial^2 f(0, t)}{\partial \rho^2} = \frac{e^{t^2/2}t^2}{2\pi}$$

is positive at this point (the case $t = 0$ is considered above).

Note that $f(0, t) = (1/2)(e^{t^2/2} - 1) \geqslant 0$ for all t.

3. The third solution is $\rho = -\sqrt{2/\pi}(1/t)$. This is a local maximum because the second derivative

$$\frac{\partial^2 f(-\sqrt{2/\pi}(1/t), t)}{\partial \rho^2} = -\frac{e^{t^2/2}t^2}{2\pi}e^{-1/\pi}$$

is negative (again, the case $t = 0$ was considered above).

The typical behavior of the function $f(\rho, t)$, when t is fixed, is presented in Figure 1.

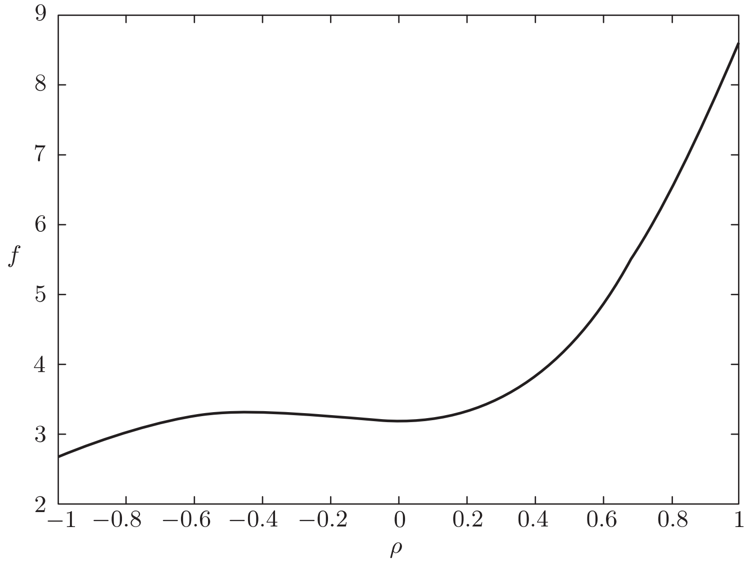

FIG. 1. *Graph of the function* $f(\rho,t), -1 \leqslant \rho \leqslant 1$, *when* t *is fixed.*

From our investigation of the derivative and Figure 1 we see that the function $f(\rho,t)$ is positive at the point of the local minimum $\rho = 0$. The function is increasing on the interval $\rho \in (0,1]$, so it is nonnegative.

Further, $f(\rho,t)$ is decreasing between the local maximum $\rho = -\sqrt{2/\pi}(1/t)$ and the local minimum $\rho = 0$, and because it is positive at the point of the local minimum $\rho = 0$, it is nonnegative in the interval $\rho \in [-\sqrt{2/\pi}(1/t), 0]$ too.

The above function is increasing in the interval $\rho \in [-1, -\sqrt{2/\pi}(1/t)]$, and hence it is left to show that at the point $\rho = -1$ the value is nonnegative.

Notice that

$$\begin{aligned} f(-1,t) &= \exp\left\{\frac{t^2}{2} - \frac{t}{\sqrt{2\pi}}\right\} - \exp\left\{\frac{t^2}{2}\right\}\Phi(-t) - 0.5 \\ &= \exp\left\{\frac{t^2}{2}\right\}\left(\Phi(t) + \exp\left\{-\frac{t}{\sqrt{2\pi}}\right\} - \frac{1}{2}\exp\left\{-\frac{t^2}{2}\right\} - 1\right), \end{aligned} \tag{4.2}$$

where $t > 0$.

Figure 2 shows the behavior of the function $f(-1,t)$, $0 \leqslant t \leqslant 2$.

Our goal is to show that $f(-1,t) \geqslant 0$ for $t > 0$. Canceling out $\exp\{t^2/2\}$ in (4.2), it is necessary to show that

$$G(t) = \Phi(t) + \exp\left\{-\frac{t}{\sqrt{2\pi}}\right\} - \frac{1}{2}\exp\left\{-\frac{t^2}{2}\right\} - 1 \geqslant 0. \tag{4.3}$$

Figure 3 shows the behavior of the function $G(t)$, $0 \leqslant t \leqslant 6$.

First notice that

$$\lim_{t\to+\infty} G(t) = 1 + 0 - 0 - 1 = 0.$$

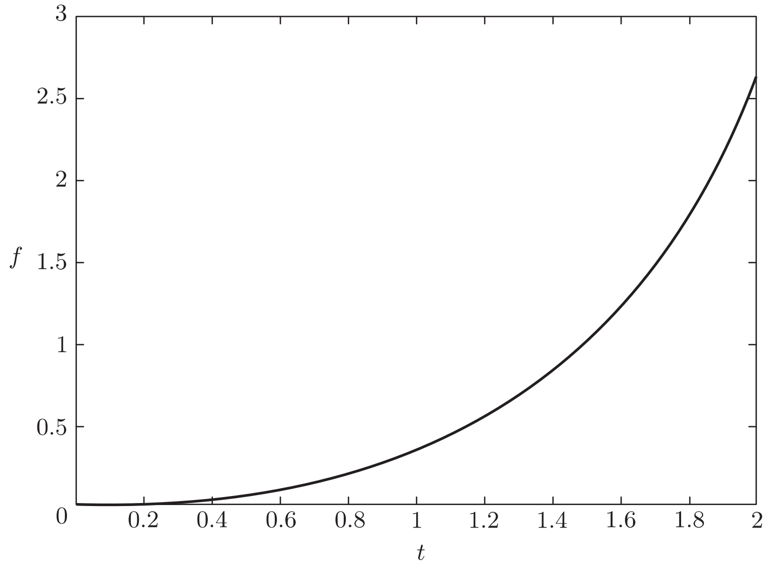

FIG. 2. *Graph of the function* $f(-1,t)$, $1 \leqslant t \leqslant 2$.

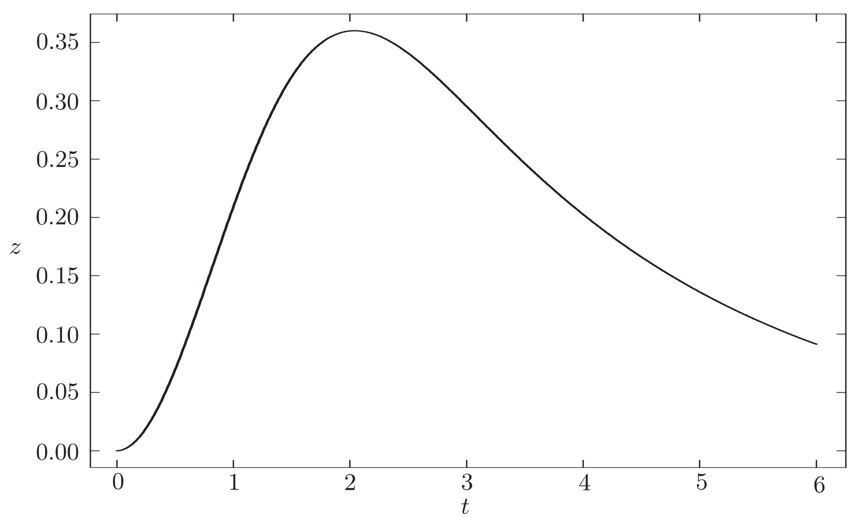

FIG. 3. *Graph of the function* $G(t)$, $0 \leqslant t \leqslant 6$.

Taking the derivative, we get

$$\begin{aligned} G'(t) &= \frac{1}{\sqrt{2\pi}} \exp\biggl\{-\frac{t^2}{2}\biggr\} - \frac{1}{\sqrt{2\pi}} \exp\biggl\{-\frac{t}{\sqrt{2\pi}}\biggr\} + \frac{t}{2} \exp\biggl\{-\frac{t^2}{2}\biggr\} \\ &= \frac{1}{\sqrt{2\pi}} \exp\biggl\{-\frac{t^2}{2}\biggr\} \biggl[1 - \exp\biggl\{\frac{t^2}{2} - \frac{t}{\sqrt{2\pi}}\biggr\} + \sqrt{\frac{\pi}{2}}\, t\biggr]. \end{aligned}$$

Hence $G'(t) \geqslant 0$ if and only if

$$1 - \exp\biggl\{\frac{t^2}{2} - \frac{t}{\sqrt{2\pi}}\biggr\} + \sqrt{\frac{\pi}{2}}\, t \geqslant 0,$$

which is equivalent to

$$\exp\left\{\frac{t^2}{2} - \frac{t}{\sqrt{2\pi}}\right\} \leqslant 1 + \sqrt{\frac{\pi}{2}}\, t$$

and, in turn, to

$$\frac{t^2}{2} - \frac{t}{\sqrt{2\pi}} - \ln\left(1 + \sqrt{\frac{\pi}{2}}\, t\right) \leqslant 0.$$

We study the function

$$h(t) = \frac{t^2}{2} - \frac{t}{\sqrt{2\pi}} - \ln\left(1 + \sqrt{\frac{\pi}{2}}\, t\right).$$

Figure 4 shows the behavior of the function $h(t)$, $0 \leqslant t \leqslant 2.5$.

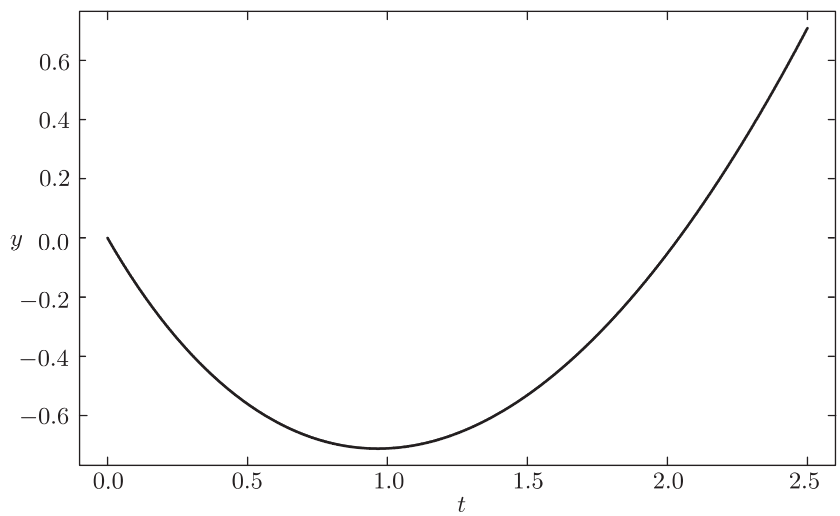

FIG. 4. *Graph of the function* $h(t)$, $0 \leqslant t \leqslant 2.5$.

We have $h(0) = 0$, and h is negative in the neighborhood of 0 since, as $t \to 0^+$, the principal part of h is

$$-\frac{t}{\sqrt{2\pi}} - \sqrt{\frac{\pi}{2}}\, t \leqslant 0.$$

On the other hand,

$$\lim_{t\to+\infty} h(t) = +\infty.$$

Hence there exists t_0 such that $h(t_0) = 0$. We prove that t_0 is unique.

We have

$$h'(t) = \frac{\sqrt{\pi/2}\, t^2 + t/2 - (1/\sqrt{2\pi} + \sqrt{\pi/2})}{1 + \sqrt{\pi/2}\, t}.$$

Letting $h'(t) = 0$, we find two solutions,

$$t_{1,2} = \frac{-1/2 \pm \sqrt{9/4 + 2\pi}}{\sqrt{2\pi}}.$$

One of these solutions is negative, so we consider only the positive solution,

$$t^* = \frac{-1/2 + \sqrt{9/4 + 2\pi}}{\sqrt{2\pi}} \approx 0.965903745.$$

Obviously, $h'(t) > 0$ for $t > t^*$ and $h'(t) < 0$ for $t < t^*$. Hence t^* is the only minimum point for h in the region $t \geqslant 0$, and this implies that t_0 is unique. Therefore, $G'(t) > 0$ for $t < t_0$ and $G'(t) < 0$ for $t > t_0$; i.e., t_0 is a unique maximum point for $G(t)$, $t \geqslant 0$.

Rough evaluation of t_0 can be done numerically. We found that $t_0 \approx 2.0414$.

Finally, we show that $\tau(U) \geqslant 1$. From Proposition 3.3 we know that the moment generating function of U is given by formula (3.2), that is,

$$\mathbf{E}e^{tU} = e^{-t\rho/\sqrt{2\pi}}\left(e^{t^2/2}\Phi(\rho t) + \frac{1}{2}\right).$$

Obviously, if we drop $1/2$ in (3.2), we obtain the inequality

$$\mathbf{E}e^{tU} \geqslant e^{-t\rho/\sqrt{2\pi}}e^{t^2/2}\Phi(\rho t).$$

Let t be such that $\rho t \geqslant 0$; then $\Phi(\rho t) \geqslant 1/2$ and

$$\mathbf{E}e^{tU} \geqslant \frac{1}{2}e^{-t\rho/\sqrt{2\pi}}e^{t^2/2} = \frac{1}{2}\exp\left\{\frac{t^2}{2} - t\frac{\rho}{\sqrt{2\pi}}\right\}.$$

Now, for any $\varepsilon > 0$, choose t such that $\rho t \geqslant 0$ and

$$\frac{\varepsilon}{2}t^2 - \frac{\rho}{\sqrt{2\pi}} - \ln 2 \geqslant 0.$$

Then

$$\mathbf{E}e^{tU} \geqslant \frac{1}{2}\exp\left\{\frac{t^2}{2} - t\frac{\rho}{\sqrt{2\pi}}\right\} \geqslant \exp\left\{(1-\varepsilon)\frac{t^2}{2}\right\}.$$

By definition of the sub-Gaussian standard and by arbitrariness of $\varepsilon > 0$, we obtain that $\tau(U) \geqslant 1$. Proposition 4.1 is proved.

Acknowledgments. The authors are grateful to the referee not only for carefully reading the manuscript and offering substantial comments and suggestions which improved the presentation, but also for an exceptionally elegant proof of the fact that $\tau(U) \geqslant 1$ that we were happy to add to the formulation and proof of Proposition 4.1. Without it, our presentation would not be complete.

REFERENCES

[1] S. M. Ross, *Introduction to Probability Models*, 11th ed., Elsevier/Academic Press, Amsterdam, 2014.

[2] P. J. Brockwell and R. A. Davis, *Time Series: Theory and Methods*, reprint of the 2nd ed., Springer Ser. Statist., Springer, New York, 2013, https://doi.org/10.1007/978-1-4419-0320-4.

[3] V. V. Buldygin and Yu. V. Kozachenko, *Metric Characterization of Random Variables and Random Processes*, Transl. Math. Monogr. 188, Amer. Math. Soc., Providence, RI, 2000.

[4] A. M. Yaglom, *Correlation Theory of Stationary and Related Random Functions*, Vols. 1, 2, Springer Ser. Statist., Springer-Verlag, New York, 1987.

[5] J. P. Kahane, *Propriétés locales des fonctions à séries de Fourier aléatoires*, Studia Math., 19 (1960), pp. 1–25, https://doi.org/10.4064/sm-19-1-1-25.

[6] A. Castellucci, *ϕ-Subgaussian Random Processes: Properties and Their Applications*, Ph.D. thesis, Univ. Pisa, 2008; available online from https://etd.adm.unipi.it/theses/available/etd-02092008-223901/.