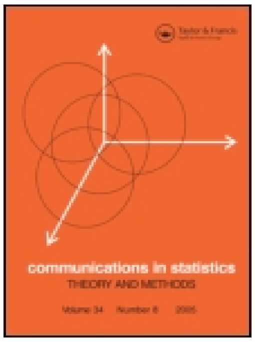

Communications in Statistics - Theory and Methods

ISSN: 0361-0926 (Print) 1532-415X (Online) Journal homepage: http://www.tandfonline.com/loi/lsta20

General results of complete convergence and complete moment convergence for weighted sums of some class of random variables

Xuejun Wang, Shuhe Hu & Andrei Volodin

To cite this article: Xuejun Wang, Shuhe Hu & Andrei Volodin (2016) General results of complete convergence and complete moment convergence for weighted sums of some class of random variables, Communications in Statistics - Theory and Methods, 45:15, 4494-4508, DOI: 10.1080/03610926.2014.921308

To link to this article: http://dx.doi.org/10.1080/03610926.2014.921308

Accepted author version posted online: 30 Nov 2015.
Published online: 30 Nov 2015.

COMMUNICATIONS IN STATISTICS—THEORY AND METHODS
2016, VOL. 45, NO. 15, 4494–4508
http://dx.doi.org/10.1080/03610926.2014.921308

General results of complete convergence and complete moment convergence for weighted sums of some class of random variables

Xuejun Wang[a], Shuhe Hu[a], and Andrei Volodin[b]

[a]School of Mathematical Science, Anhui University, Hefei, Anhui, P.R. China; [b]Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada

ABSTRACT
In the article, the complete convergence and complete moment convergence for weighted sums of sequences of random variables satisfying a maximal Rosenthal type inequality are studied. As an application, the Marcinkiewicz–Zygmund type strong law of large numbers is obtained. Our partial results generalize and improve the corresponding ones of Shen (2013).

ARTICLE HISTORY
Received 28 December 2013
Accepted 29 April 2014

KEYWORDS
Complete convergence; Complete moment convergence; Rosenthal type maximal inequality; Marcinkiewicz–Zygmund type strong law of large numbers.

MATHEMATICS SUBJECT CLASSIFICATION
60F15

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ an array of constants. The strong convergence results for weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ have been studied by many authors, see for example, Choi and Sung (1987), Cuzick (1995), Wu (1999), Bai and Cheng (2000), Chen and Gan (2007), Cai (2008), Sung (2001, 2011), Shen (2011, 2013), Shen and Wu (2013), Wang et al. (2011a, b, 2012a, b, c), Zhou et al. (2011), Wu (2010, 2012a, b), Xu and Tang (2012), and so forth. Many useful linear statistics are these weighted sums. Examples include least-squares estimators, non parametric regression function estimators, and jackknife estimates among others. Bai and Cheng (2000) proved the strong law of large numbers for weighted sums

$$\frac{1}{b_n}\sum_{i=1}^{n} a_{ni}X_i \to 0, \quad a.s.$$

when $\{X, X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with $EX = 0$ and $E\exp(h|X|^{\gamma}) < \infty$ for some $h > 0$ and $\gamma > 0$, and $\{a_{ni}, 1 \leq i \leq$

CONTACT Xuejun Wang wxjahdx@126.com School of Mathematical Science, Anhui University, Hefei 230601, P.R. China.

$n, n \geq 1\}$ is an array of constants satisfying

$$A_\alpha \doteq \limsup_{n\to\infty} A_{\alpha,n} < \infty,\ A^\alpha_{\alpha,n} \doteq \frac{1}{n}\sum_{i=1}^{n} |a_{ni}|^\alpha \tag{1}$$

for some $1 < \alpha < 2$, where $b_n = n^{1/\alpha}(\log n)^{1/\gamma+\gamma(\alpha-1)/\alpha(1+\gamma)}$.

Sung (2011) generalized and improved the result of Bai and Cheng (2000) for independent and identically distributed random variables to the case of identically distributed negatively associated (NA, in short) random variables under much weaker conditions and obtained the following complete convergence result for weighted sums of identically distributed NA random variables.

Theorem A. *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NA random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying (1) for some $0 < \alpha \leq 2$. Let $b_n = n^{1/\alpha}\log^{1/\gamma} n$ for some $\gamma > 0$. Furthermore, suppose that $EX = 0$ when $1 < \alpha \leq 2$. Then the following statements hold:*

(i) If $\alpha > \gamma$, then $E|X|^\alpha < \infty$ implies

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon b_n\right) < \infty \text{ for all } \varepsilon > 0. \tag{2}$$

(ii) If $\alpha = \gamma$, then $E|X|^\alpha \log|X| < \infty$ implies (2).

(iii) If $\alpha < \gamma$, then $E|X|^\gamma < \infty$ implies (2).

The technique used in Sung (2011) is the result of Chen et al. (2007) for NA random variables, which is not proved for many sequences of random variables, such as ρ-mixing random variables, φ-mixing random variables, α-mixing random variables, ρ^*-mixing random variables, ρ^--mixing random variables, asymptotically almost negatively associated random variables, and so forth. So it is very challenging to generalize the result of Theorem A to the case of the sequences above. Recently, Shen (2013) made a great contribution to this work. She further studied the strong convergence for a sequence of random variables satisfying a Rosenthal type maximal inequality by using different method from that of Sung (2011) and obtained the following result.

Theorem B. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed random variables. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ for some $0 < \alpha \leq 2$. $EX_n = 0$ when $1 < \alpha \leq 2$. Let $b_n = n^{1/\alpha}\log^{1/\gamma} n$ for some $\gamma > 0$. Assume that for any $q \geq 2$, there exists a positive constant C_q depending only on q such that*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y_{ni} - EY_{ni})\right|^q\right) \leq C_q\left[\sum_{i=1}^{n} E|Y_{ni}|^q + \left(\sum_{i=1}^{n} EY_{ni}^2\right)^{q/2}\right], \tag{3}$$

where $Y_{ni} = -b_n I(a_{ni}X_i < -b_n) + a_{ni}X_i I(|a_{ni}X_i| \leq b_n) + b_n I(a_{ni}X_i > b_n)$ or $Y_{ni} = a_{ni}X_i I(|a_{ni}X_i| \leq b_n)$. Furthermore, suppose that

$$\sum_{n=1}^{\infty} n^{-1}\left[\sum_{i=1}^{n} P(|a_{ni}X_i| > b_n)\right]^{q/2} < \infty \tag{4}$$

for $Y_{ni} = -b_n I(a_{ni}X_i < -b_n) + a_{ni}X_i I(|a_{ni}X_i| \le b_n) + b_n I(a_{ni}X_i > b_n)$. *If*

$$\begin{cases} E|X_1|^\alpha < \infty, & for\ \alpha > \gamma, \\ E|X_1|^\alpha \log|X_1| < \infty, & for\ \alpha = \gamma, \\ E|X_1|^\gamma < \infty, & for\ \alpha < \gamma, \end{cases} \tag{5}$$

then (2) *holds.*

Theorem B is a general result of complete convergence for sequences of random variables satisfying a Rosenthal type maximal inequality. For many sequences, such as ρ-mixing random variables, φ-mixing random variables, α-mixing random variables, ρ^*-mixing random variables, and so on, condition (3) is satisfied. Unfortunately, Theorem B does not generalize Theorem A for identically distributed NA random variables to the case of sequences of random variables satisfying a Rosenthal type maximal inequality completely, since condition (4) is needed additionally for $Y_{ni} = -b_n I(a_{ni}X_i < -b_n) + a_{ni}X_i I(|a_{ni}X_i| \le b_n) + b_n I(a_{ni}X_i > b_n)$.

Inspired by Sung (2011) and Shen (2013), we will generalize Theorem A for identically distributed NA random variables to the case of sequences of random variables satisfying a Rosenthal type maximal inequality completely. Since the condition (4) is not needed, our results improve the corresponding one of Shen (2013). In addition, the condition of identical distribution is not needed, but the condition of stochastic domination is needed. The concept of stochastic domination is as follows.

Definition 1.1. A sequence $\{X_n, n \ge 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \le CP(|X| > x) \tag{6}$$

for all $x \ge 0$ and $n \ge 1$.

The following basic property for stochastic domination will be used frequently throughout the article. For the proof, one can refer to Wu (2006), Shen et al. (2013), or Tang (2013).

Lemma 1.1. *Let* $\{X_n, n \ge 1\}$ *be a sequence of random variables which is stochastically dominated by a random variable* X. *Then for any* $\alpha > 0$ *and* $b > 0$,

$$E|X_n|^\alpha I(|X_n| \le b) \le C_1 \left[E|X|^\alpha I(|X| \le b) + b^\alpha P(|X| > b)\right], \tag{7}$$

$$E|X_n|^\alpha I(|X_n| > b) \le C_2 E|X|^\alpha I(|X| > b), \tag{8}$$

where C_1 *and* C_2 *are positive constants.*

Throughout the article, let $I(A)$ be the indicator function of the set A. C denotes a positive constant which may be different in various places and $a_n = O(b_n)$ stands for $a_n \le Cb_n$. Denote $\log x = \ln\max(x, e)$ for $x \in \mathbf{R}$.

2. Main results

In this section, we will study the complete convergence for weighted sums of a class of random variables.

Inspired by Sung (2011) and Shen (2013), we can get the following general result of complete convergence for sequences of random variables satisfying a maximal Rosenthal type inequality. The method for the proof is similar to that of Shen (2013), which is different from that of Sung (2011).

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ for some $0 < \alpha \leq 2$. $EX_n = 0$ when $1 < \alpha \leq 2$. Let $b_n = n^{1/\alpha} \log^{1/\gamma} n$ for some $\gamma > 0$. Assume that for any $q \geq 2$, there exists a positive constant C_q depending only on q such that*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right|^q\right) \leq C_q\left[\sum_{i=1}^{n}E|Y_{ni}|^q + \left(\sum_{i=1}^{n}EY_{ni}^2\right)^{q/2}\right], \tag{9}$$

where $Y_{ni} = -b_n I(a_{ni}X_i < -b_n) + a_{ni}X_i I(|a_{ni}X_i| \leq b_n) + b_n I(a_{ni}X_i > b_n)$ or $Y_{ni} = a_{ni}X_i I(|a_{ni}X_i| \leq b_n)$. If

$$\begin{cases} E|X|^\alpha < \infty, & for\ \alpha > \gamma, \\ E|X|^\alpha \log|X| < \infty, & for\ \alpha = \gamma, \\ E|X|^\gamma < \infty, & for\ \alpha < \gamma, \end{cases} \tag{10}$$

then (2) holds.

Proof. We only need to prove that (2) holds for

$$Y_{ni} = -b_n I(a_{ni}X_i < -b_n) + a_{ni}X_i I(|a_{ni}X_i| \leq b_n) + b_n I(a_{ni}X_i > b_n).$$

The proof for $Y_{ni} = a_{ni}X_i I(|a_{ni}X_i| \leq b_n)$ is analogous.

Without loss of generality, we may assume that $\sum_{i=1}^{n} |a_{ni}|^\alpha \leq n$. It is easy to check that for any $\varepsilon > 0$,

$$\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}a_{ni}X_i\right| > \varepsilon b_n\right) \subset \left(\max_{1\leq i\leq n}|a_{ni}X_i| > b_n\right)\bigcup\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}Y_{ni}\right| > \varepsilon b_n\right),$$

which implies that

$$\begin{aligned} &P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}a_{ni}X_i\right| > \varepsilon b_n\right) \\ &\leq P\left(\max_{1\leq i\leq n}|a_{ni}X_i| > b_n\right) + P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}Y_{ni}\right| > \varepsilon b_n\right) \\ &\leq C\sum_{i=1}^{n}P(|a_{ni}X| > b_n) + P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right| > \varepsilon b_n - \max_{1\leq j\leq n}\left|\sum_{i=1}^{j}EY_{ni}\right|\right). \end{aligned}$$

First, we will show that

$$b_n^{-1}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}EY_{ni}\right| \to 0 \text{ as } n\to\infty. \tag{11}$$

When $1 < \alpha \leq 2$, we have by $EX_n = 0$, Lemma 1.1, Markov's inequality and (10) that

$$\begin{aligned} b_n^{-1}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}EY_{ni}\right| &\leq C\sum_{i=1}^{n}P(|a_{ni}X| > b_n) + b_n^{-1}\sum_{i=1}^{n}E|a_{ni}X_i|I(|a_{ni}X_i| > b_n) \\ &\leq Cb_n^{-\alpha}\sum_{i=1}^{n}|a_{ni}|^\alpha E|X|^\alpha + Cb_n^{-\alpha}\sum_{i=1}^{n}|a_{ni}|^\alpha E|X|^\alpha \\ &\leq CE|X|^\alpha(\log n)^{-\alpha/\gamma} \to 0 \text{ as } n\to\infty. \end{aligned} \tag{12}$$

When $0 < \alpha \leq 1$, we have by Lemma 1.1, Markov's inequality and (10) again that

$$\begin{aligned} b_n^{-1} \max_{1\leq j\leq n} \left| \sum_{i=1}^{j} EX_i^{(n)} \right| &\leq \sum_{i=1}^{n} P(|a_{ni}X_i| > b_n) + b_n^{-1} \sum_{i=1}^{n} E|a_{ni}X_i| I(|a_{ni}X_i| \leq b_n) \\ &\leq C \sum_{i=1}^{n} P(|a_{ni}X| > b_n) + Cb_n^{-1} \sum_{i=1}^{n} E|a_{ni}X| I(|a_{ni}X| \leq b_n) \\ &\leq Cb_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X|^{\alpha} + Cb_n^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X|^{\alpha} I(|a_{ni}X| \leq b_n) \\ &\leq Cb_n^{-\alpha} nE|X|^{\alpha} + Cb_n^{-\alpha} nE|X|^{\alpha} \\ &= 2CE|X|^{\alpha} (\log n)^{-\alpha/\gamma} \to 0 \text{ as } n \to \infty. \end{aligned} \tag{13}$$

By (12) and (13), we can get (11) immediately. Hence, for n large enough,

$$P\left(\max_{1\leq j\leq n} \left| \sum_{i=1}^{j} a_{ni}X_i \right| > \varepsilon b_n \right) \leq C \sum_{i=1}^{n} P\left(|a_{ni}X| > b_n\right) + P\left(\max_{1\leq j\leq n} \left| \sum_{i=1}^{j} (Y_{ni} - EY_{ni}) \right| > \frac{\varepsilon}{2} b_n \right).$$

To prove (2), we only need to show that

$$I \doteq \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P\left(|a_{ni}X| > b_n\right) < \infty, \tag{14}$$

$$J \doteq \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq j\leq n} \left| \sum_{i=1}^{j} (Y_{ni} - EY_{ni}) \right| > \frac{\varepsilon}{2} b_n \right) < \infty. \tag{15}$$

First, we will prove (14). By $\sum_{i=1}^{n} |a_{ni}|^{\alpha} \leq n$ and (10), we can get that

$$\begin{aligned} I &\leq \sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} E|a_{ni}X|^{\alpha} I\left(|a_{ni}X| > b_n\right) \\ &\leq \sum_{n=1}^{\infty} n^{-2} (\log n)^{-\alpha/\gamma} \sum_{i=1}^{n} E|a_{ni}X|^{\alpha} I\left(\sum_{i=1}^{n} |a_{ni}X|^{\alpha} > n(\log n)^{\alpha/\gamma} \right) \\ &\leq \sum_{n=1}^{\infty} n^{-2} (\log n)^{-\alpha/\gamma} \sum_{i=1}^{n} E|a_{ni}X|^{\alpha} I\left(|X|^{\alpha} > (\log n)^{\alpha/\gamma}\right) \\ &\leq \sum_{n=1}^{\infty} n^{-1} (\log n)^{-\alpha/\gamma} E|X|^{\alpha} I\left(|X|^{\gamma} > \log n\right) \\ &= \sum_{m=1}^{\infty} E|X|^{\alpha} I\left(\log m < |X|^{\gamma} \leq \log(m+1)\right) \sum_{n=1}^{m} n^{-1} (\log n)^{-\alpha/\gamma} \\ &\leq \begin{cases} C \sum_{m=1}^{\infty} E|X|^{\alpha} I\left(\log m < |X|^{\gamma} \leq \log(m+1)\right), & \textit{for } \alpha > \gamma \\ C \sum_{m=1}^{\infty} E|X|^{\alpha} I\left(\log m < |X|^{\gamma} \leq \log(m+1)\right) \log\log m, & \textit{for } \alpha = \gamma \\ C \sum_{m=1}^{\infty} E|X|^{\alpha} I\left(\log m < |X|^{\gamma} \leq \log(m+1)\right) (\log m)^{1-\alpha/\gamma}, & \textit{for } \alpha < \gamma \end{cases} \end{aligned}$$

$$
\leq \begin{cases} CE|X|^{\alpha}, & for\ \alpha > \gamma \\ CE|X|^{\alpha}\log|X|, & for\ \alpha = \gamma \\ CE|X|^{\gamma}, & for\ \alpha < \gamma \end{cases}
$$
$$
< \infty,
$$

which implies (14).

In the following, we will prove (15). Let $q > \max\{2, \alpha, \gamma, \frac{2\gamma}{\alpha}\}$. By Markov's inequality and condition (9), we have

$$
\begin{aligned}
J &\leq C\sum_{n=1}^{\infty} n^{-1}b_n^{-q}\sum_{i=1}^{n} E\,|Y_{ni}|^q + C\sum_{n=1}^{\infty} n^{-1}b_n^{-q}\left(\sum_{i=1}^{n} EY_{ni}^2\right)^{q/2} \\
&\doteq J_1 + J_2.
\end{aligned} \tag{16}
$$

To prove (15), it suffices to show that $J_1 < \infty$ and $J_2 < \infty$.

For $j \geq 1$ and $n \geq 2$, denote

$$
I_{nj} = \left\{1 \leq i \leq n : \frac{n}{j+1} < |a_{ni}|^{\alpha} \leq \frac{n}{j}\right\}. \tag{17}
$$

In view of $\sum_{i=1}^{n} |a_{ni}|^{\alpha} \leq n$, it is easy to see that $\{I_{nj}, j \geq 1\}$ are disjoint and $\bigcup_{j=1}^{\infty} I_{nj} = \{1 \leq i \leq n : a_{ni} \neq 0\}$. Hence, we have for all $m \geq 1$ that

$$
\begin{aligned}
n &\geq \sum_{i=1}^{n} |a_{ni}|^{\alpha} = \sum_{\{1\leq i\leq n: a_{ni}\neq 0\}} |a_{ni}|^{\alpha} = \sum_{j=1}^{\infty}\sum_{i\in I_{nj}} |a_{ni}|^{\alpha} \\
&\geq n\sum_{j=1}^{\infty}(j+1)^{-1}\sharp I_{nj} \geq n\sum_{j=m}^{\infty}(j+1)^{-q/\alpha}(j+1)^{q/\alpha-1}\sharp I_{nj} \\
&\geq n\sum_{j=m}^{\infty}(j+1)^{-q/\alpha}(m+1)^{q/\alpha-1}\sharp I_{nj},
\end{aligned}
$$

which implies that for all $m \geq 1$,

$$
\sum_{j=m}^{\infty}(j+1)^{-q/\alpha}\sharp I_{nj} \leq Cm^{1-q/\alpha},\ \ n \geq 2. \tag{18}
$$

By C_r inequality, Lemma 1.1, (14) and (17), we can get that

$$
\begin{aligned}
J_1 &\leq C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P\left(|a_{ni}X_i| > b_n\right) + C\sum_{n=1}^{\infty} n^{-1}b_n^{-q}\sum_{i=1}^{n} E|a_{ni}X_i|^q I\left(|a_{ni}X_i| \leq b_n\right) \\
&\leq C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P\left(|a_{ni}X| > b_n\right) + C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\sum_{i=1}^{n} E|a_{ni}X|^q I\left(|a_{ni}X| \leq b_n\right) \\
&\leq C\sum_{n=2}^{\infty} n^{-1-q/\alpha}(\log n)^{-q/\gamma}\sum_{j=1}^{\infty}\sum_{i\in I_{nj}} E|a_{ni}X|^q I\left(|X| \leq (j+1)^{1/\alpha}(\log n)^{1/\gamma}\right) \\
&\leq C\sum_{n=2}^{\infty} n^{-1-q/\alpha}(\log n)^{-q/\gamma}\sum_{j=1}^{\infty} n^{q/\alpha}j^{-q/\alpha}E|X|^q I\left(|X| \leq (j+1)^{1/\alpha}(\log n)^{1/\gamma}\right)\sharp I_{nj}
\end{aligned}
$$

$$\leq C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-q/\gamma}\sum_{j=1}^{\infty}j^{-q/\alpha}E|X|^{q}I\left(|X|\leq(\log n)^{1/\gamma}\right)\sharp I_{nj}$$
$$+C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-q/\gamma}\sum_{j=1}^{\infty}j^{-q/\alpha}\sum_{k=1}^{j}E|X|^{q}I\left(k^{1/\alpha}(\log n)^{1/\gamma}<|X|\right.$$
$$\left.\leq (k+1)^{1/\alpha}(\log n)^{1/\gamma}\right)\sharp I_{nj}\doteq J_{11}+J_{12}.$$

If $\alpha>\gamma$, we have by (18) and $E|X|^{\alpha}<\infty$ that

$$\begin{aligned}J_{11}&\leq C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-q/\gamma}E|X|^{q}I\left(|X|\leq(\log n)^{1/\gamma}\right)\\&\leq C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-\alpha/\gamma}E|X|^{\alpha}I\left(|X|\leq(\log n)^{1/\gamma}\right)\\&\leq C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-\alpha/\gamma}<\infty.\end{aligned}\tag{19}$$

If $\alpha\leq\gamma$, we have by (10) and (18) that

$$\begin{aligned}J_{11}&\leq C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-q/\gamma}E|X|^{q}I\left(|X|\leq(\log n)^{1/\gamma}\right)\\&\leq C\sum_{m=2}^{\infty}E|X|^{q}I\left((\log(m-1))^{1/\gamma}<|X|\leq(\log m)^{1/\gamma}\right)\sum_{n=m}^{\infty}n^{-1}(\log n)^{-q/\gamma}\\&\leq C\sum_{m=2}^{\infty}(\log m)^{1-q/\gamma}E|X|^{q}I\left((\log(m-1))^{1/\gamma}<|X|\leq(\log m)^{1/\gamma}\right)\\&\leq CE|X|^{\gamma}<\infty.\end{aligned}\tag{20}$$

By (19) and (20), we can get that $J_{11}<\infty$. Next, we will prove that $J_{12}<\infty$.

It follows by (10) and (18) again that

$$J_{12}=C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-q/\gamma}\sum_{k=1}^{\infty}E|X|^{q}I\left(k^{1/\alpha}(\log n)^{1/\gamma}<|X|\right.$$
$$\left.\leq (k+1)^{1/\alpha}(\log n)^{1/\gamma}\right)\sum_{j=k}^{\infty}j^{-q/\alpha}\sharp I_{nj}$$
$$\leq C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-q/\gamma}\sum_{k=1}^{\infty}k^{1-q/\alpha}E|X|^{q}I\left(k^{1/\alpha}(\log n)^{1/\gamma}<|X|\leq(k+1)^{1/\alpha}(\log n)^{1/\gamma}\right)$$
$$\leq C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-\alpha/\gamma}E|X|^{\alpha}I\left(|X|>(\log n)^{1/\gamma}\right)$$
$$=C\sum_{m=2}^{\infty}E|X|^{\alpha}I\left((\log m)^{1/\gamma}<|X|\leq(\log(m+1))^{1/\gamma}\right)\sum_{n=2}^{m}n^{-1}(\log n)^{-\alpha/\gamma}$$

$$\leq \begin{cases} C\sum_{m=1}^{\infty} E|X|^{\alpha} I\left(\log m < |X|^{\gamma} \leq \log(m+1)\right), & for\ \alpha > \gamma \\ C\sum_{m=1}^{\infty} E|X|^{\alpha} I\left(\log m < |X|^{\gamma} \leq \log(m+1)\right) \log\log m, & for\ \alpha = \gamma \\ C\sum_{m=1}^{\infty} E|X|^{\alpha} I\left(\log m < |X|^{\gamma} \leq \log(m+1)\right) (\log m)^{1-\alpha/\gamma}, & for\ \alpha < \gamma \end{cases}$$

$$\leq \begin{cases} CE|X|^{\alpha}, & for\ \alpha > \gamma \\ CE|X|^{\alpha}\log|X|, & for\ \alpha = \gamma \\ CE|X|^{\gamma}, & for\ \alpha < \gamma \end{cases}$$

$$< \infty.$$

By $J_{11} < \infty$ and $J_{12} < \infty$, we can get that $J_1 < \infty$.

To prove (15), it suffices to show that $J_2 < \infty$.

By Markov's inequality, $\sum_{i=1}^{n} |a_{ni}|^{\alpha} \leq n$ and condition (10), we have

$$\begin{aligned} \sum_{i=1}^{n} P\left(|a_{ni}X_i| > b_n\right) &\leq C\sum_{i=1}^{n} P\left(|a_{ni}X| > b_n\right) \\ &\leq Cb_n^{-\alpha}\sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X|^{\alpha} \\ &\leq CE|X|^{\alpha}(\log n)^{-\alpha/\gamma} \to 0 \text{ as } n \to \infty. \end{aligned} \tag{21}$$

Hence, by C_r inequality, Lemma 1.1, (21), (14), and (10), we can get that

$$\begin{aligned} J_2 &\doteq C\sum_{n=1}^{\infty} n^{-1} b_n^{-q}\left(\sum_{i=1}^{n} EY_{ni}^2\right)^{q/2} \\ &\leq C\sum_{n=1}^{\infty} n^{-1} b_n^{-q}\left[\sum_{i=1}^{n} b_n^2 P\left(|a_{ni}X_i| > b_n\right) + \sum_{i=1}^{n} E|a_{ni}X_i|^2 I\left(|a_{ni}X_i| \leq b_n\right)\right]^{q/2} \\ &\leq C\sum_{n=1}^{\infty} n^{-1} b_n^{-q}\left[\sum_{i=1}^{n} b_n^2 P\left(|a_{ni}X| > b_n\right) + \sum_{i=1}^{n} E|a_{ni}X|^2 I\left(|a_{ni}X| \leq b_n\right)\right]^{q/2} \\ &\leq C\sum_{n=1}^{\infty} n^{-1}\left[\sum_{i=1}^{n} P\left(|a_{ni}X| > b_n\right)\right]^{q/2} + C\sum_{n=1}^{\infty} n^{-1} b_n^{-q}\left[\sum_{i=1}^{n} E|a_{ni}X|^2 I\left(|a_{ni}X| \leq b_n\right)\right]^{q/2} \\ &\leq C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P\left(|a_{ni}X| > b_n\right) + C\sum_{n=1}^{\infty} n^{-1}\left[\sum_{i=1}^{n} b_n^{-\alpha} E|a_{ni}X|^{\alpha} I\left(|a_{ni}X| \leq b_n\right)\right]^{q/2} \\ &\leq C + C\left(E|X|^{\alpha}\right)^{q/2}\sum_{n=1}^{\infty} n^{-1}(\log n)^{-\frac{\alpha q}{2\gamma}} < \infty. \end{aligned}$$

Therefore, (15) follows from (16) and $J_1 < \infty, J_2 < \infty$ immediately. This completes the proof of the theorem. □

If the Rosenthal type inequality for the maximal partial sum is replaced by the partial sum, then we can get the following complete convergence result for a class of random variables. The proof is similar to that of Theorem 2.1. So the details are omitted.

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ for some $0 < \alpha \leq 2$. $EX_n = 0$ when $1 < \alpha \leq 2$. Let $b_n = n^{1/\alpha}\log^{1/\gamma} n$ for* *some* $\gamma > 0$. *Assume that for any* $q \geq 2$, *there exists a positive constant* C_q *depending only on* q *such that*

$$E\left(\left|\sum_{i=1}^{n}(Y_{ni}-EY_{ni})\right|^{q}\right) \leq C_q\left[\sum_{i=1}^{n}E|Y_{ni}|^{q}+\left(\sum_{i=1}^{n}EY_{ni}^{2}\right)^{q/2}\right], \tag{22}$$

where $Y_{ni} = -b_n I(a_{ni}X_i < -b_n) + a_{ni}X_i I(|a_{ni}X_i| \leq b_n) + b_n I(a_{ni}X_i > b_n)$ *or* $Y_{ni} = a_{ni}X_i I(|a_{ni}X_i| \leq b_n)$. *If* (10) *satisfies, then*

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\left|\sum_{i=1}^{n}a_{ni}X_i\right| > \varepsilon b_n\right) < \infty \text{ for all } \varepsilon > 0. \tag{23}$$

The following result provides the Marcinkiewicz–Zygmund type strong law of large numbers for weighted sums $\sum_{i=1}^{n} a_iX_i$ of a class of random variables satisfying the Rosenthal type maximal inequality. The proof is similar to that of Theorem 2.2 in Shen (2013). So we omit the details.

Theorem 2.3. *Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables which is stochastically dominated by a random variable* X. *Let* $\{a_n, n \geq 1\}$ *be a sequence of constants satisfying* $\sum_{i=1}^{n}|a_i|^{\alpha} = O(n)$ *for some* $0 < \alpha \leq 2$. $EX_n = 0$ *when* $1 < \alpha \leq 2$. *Let* $b_n = n^{1/\alpha}\log^{1/\gamma} n$ *for some* $\gamma > 0$. *Assume that for any* $q \geq 2$, *there exists a positive constant* C_q *depending only on* q *such that* (9) *holds, where* $Y_{ni} = -b_n I(a_iX_i < -b_n) + a_iX_i I(|a_iX_i| \leq b_n) + b_n I(a_iX_i > b_n)$ *or* $Y_{ni} = a_iX_i I(|a_iX_i| \leq b_n)$. *If* (10) *holds, then*

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}a_iX_i\right| > \varepsilon b_n\right) < \infty \text{ for all } \varepsilon > 0 \tag{24}$$

and

$$\frac{1}{b_n}\sum_{i=1}^{n}a_iX_i \to 0 \ \ a.s., \quad as\ n \to \infty. \tag{25}$$

Remark 2.1. Just as Shen (2013) stated that there are many sequences of dependent random variables satisfying (9) for all $q \geq 2$. Examples include sequences of NA random variables (see Shao, 2000), ρ^*-mixing random variables (see Utev and Peligrad, 2003), φ-mixing random variables with the mixing coefficients satisfying certain conditions (see Wang et al., 2010), ρ^--mixing random variables with the mixing coefficients satisfying certain conditions (see Wang and Lu, 2006), asymptotically almost negatively associated random variables (see Yuan and An, 2009), and negatively superadditive dependent random variables (see Wang et al., 2014). There are also many sequences of dependent random variables satisfying (22) for all $q \geq 2$. Examples not only include the sequences of above, but also include sequences of NOD random variables (see Asadian et al., 2006) and extended negatively dependent random variables (see Shen, 2011).

Remark 2.2. Theorem 2.1 generalizes the corresponding one of Sung (2011) and improves the corresponding one of Shen (2013), respectively. In addition, Theorem 2.3 improves the corresponding one of Theorem 2.2 in Shen (2013).

The following theorem, which is inspired by Kuczmaszewska (2009), is a very general result on complete convergence for weighted sums of a class of random variables.

Theorem 2.4. *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise random variables and $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of constants. Let $\{b_n, n \geq 1\}$ be an increasing sequence of positive integers and $\{c_n, n \geq 1\}$ be a sequence of positive real numbers. Assume that for any $q \geq 2$, there exists a positive constant C_q depending only on q such that*

$$E\left(\max_{1\leq j\leq b_n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right|^q\right) \leq C_q\left[\sum_{i=1}^{b_n}E|Y_{ni}|^q + \left(\sum_{i=1}^{b_n}EY_{ni}^2\right)^{q/2}\right], \tag{26}$$

where $1 \leq i \leq b_n$, $n \geq 1$ and

$$Y_{ni} = -\varepsilon b_n^{\frac{1}{t}} I\left(a_{ni}X_{ni} \leq -\varepsilon b_n^{\frac{1}{t}}\right) + a_{ni}X_{ni}I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right) + \varepsilon b_n^{\frac{1}{t}} I\left(a_{ni}X_{ni} \geq \varepsilon b_n^{\frac{1}{t}}\right).$$

If for some $p \geq 2$, $t > 0$ and any $\varepsilon > 0$, the following conditions are fulfilled:

$$(a) \sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\left(|a_{ni}X_{ni}| \geq \varepsilon b_n^{\frac{1}{t}}\right) < \infty,$$

$$(b) \sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{b_n} P\left(|a_{ni}X_{ni}| \geq \varepsilon b_n^{\frac{1}{t}}\right)\right)^{\frac{p}{2}} < \infty,$$

$$(c) \sum_{n=1}^{\infty} c_n b_n^{-\frac{p}{t}} \sum_{i=1}^{b_n} |a_{ni}|^p E|X_{ni}|^p I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right) < \infty,$$

$$(d) \sum_{n=1}^{\infty} c_n b_n^{-\frac{p}{t}} \left(\sum_{i=1}^{b_n} a_{ni}^2 EX_{ni}^2 I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right)\right)^{\frac{p}{2}} < \infty,$$

then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\leq j\leq b_n}\left|\sum_{i=1}^{j}\left(a_{ni}X_{ni} - a_{ni}EX_{ni}I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right)\right)\right| \geq \varepsilon b_n^{\frac{1}{t}}\right\} < \infty. \tag{27}$$

Proof. It is easily seen that,

$$\begin{aligned}
&P\left\{\max_{1\leq j\leq b_n}\left|\sum_{i=1}^{j}\left[a_{ni}X_{ni} - a_{ni}EX_{ni}I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right)\right]\right| > \varepsilon b_n^{\frac{1}{t}}\right\} \\
&\leq P\left\{\max_{1\leq j\leq b_n}\left|\sum_{i=1}^{j}\left[a_{ni}X_{ni}I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right) - a_{ni}EX_{ni}I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right)\right]\right| > \varepsilon b_n^{\frac{1}{t}}\right\} \\
&\quad + P\left(\max_{1\leq j\leq b_n}\left|\sum_{i=1}^{j} a_{ni}X_{ni}I\left(|a_{ni}X_{ni}| \geq \varepsilon b_n^{\frac{1}{t}}\right)\right| > 0\right) \\
&\leq \sum_{i=1}^{b_n} P\left(|a_{ni}X_{ni}| \geq \varepsilon b_n^{\frac{1}{t}}\right) + P\left(\max_{1\leq j\leq b_n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right| > \frac{1}{2}\varepsilon b_n^{\frac{1}{t}}\right) \\
&\quad + P\left(\max_{1\leq j\leq b_n} \left|\sum_{i=1}^{j} \varepsilon b_n^{\frac{1}{t}}\left[I\left(a_{ni}X_{ni} \geq \varepsilon b_n^{\frac{1}{t}}\right) - I\left(a_{ni}X_{ni} \leq -\varepsilon b_n^{\frac{1}{t}}\right)\right.\right.\right.
\end{aligned}$$

$$-P\left(a_{ni}X_{ni}\geq\varepsilon b_n^{\frac{1}{t}}\right)+P\left(a_{ni}X_{ni}\leq-\varepsilon b_n^{\frac{1}{t}}\right)\Big]|>\frac{1}{2}\varepsilon b_n^{\frac{1}{t}}\right)$$

$$=\sum_{i=1}^{b_n}P\left(|a_{ni}X_{ni}|\geq\varepsilon b_n^{\frac{1}{t}}\right)+P\left(\max_{1\leq j\leq b_n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right|>\frac{1}{2}\varepsilon b_n^{\frac{1}{t}}\right)$$

$$+P\left(\max_{1\leq j\leq b_n}|\sum_{i=1}^{j}\Big[I\left(a_{ni}X_{ni}\geq\varepsilon b_n^{\frac{1}{t}}\right)-I\left(a_{ni}X_{ni}\leq-\varepsilon b_n^{\frac{1}{t}}\right)\right.$$

$$\left.-P\left(a_{ni}X_{ni}\geq\varepsilon b_n^{\frac{1}{t}}\right)+P\left(a_{ni}X_{ni}\leq-\varepsilon b_n^{\frac{1}{t}}\right)\Big]|>\frac{1}{2}\right)$$

$$\leq\sum_{i=1}^{b_n}P\left(|a_{ni}X_{ni}|\geq\varepsilon b_n^{\frac{1}{t}}\right)+\left(\frac{\varepsilon}{2}\right)^{-p}b_n^{-\frac{p}{t}}E\left(\max_{1\leq j\leq b_n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right|^p\right)$$

$$+2E\left\{\max_{1\leq j\leq b_n}\left(\sum_{i=1}^{j}\left[I\left(|a_{ni}X_{ni}|\geq\varepsilon b_n^{\frac{1}{t}}\right)+P\left(|a_{ni}X_{ni}|\geq\varepsilon b_n^{\frac{1}{t}}\right)\right]\right)\right\}$$

$$=5\sum_{i=1}^{b_n}P\left(|a_{ni}X_{ni}|\geq\varepsilon b_n^{\frac{1}{t}}\right)+\left(\frac{\varepsilon}{2}\right)^{-p}b_n^{-\frac{p}{t}}E\left(\max_{1\leq j\leq b_n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right|^p\right).\tag{28}$$

For $p\geq 2$, we have by C_r-inequality that

$$E|Y_{ni}|^p\leq C|a_{ni}|^pE|X_{ni}|^pI\left(|a_{ni}X_{ni}|<\varepsilon b_n^{\frac{1}{t}}\right)+Cb_n^{\frac{p}{t}}P\left(|a_{ni}X_{ni}|\geq\varepsilon b_n^{\frac{1}{t}}\right).\tag{29}$$

It follows by (26), (28), and (29) that

$$P\left\{\max_{1\leq j\leq b_n}\left|\sum_{i=1}^{j}\left[a_{ni}X_{ni}-a_{ni}EX_{ni}I\left(|a_{ni}X_{ni}|<\varepsilon b_n^{\frac{1}{t}}\right)\right]\right|>\varepsilon b_n^{\frac{1}{t}}\right\}$$

$$\leq C\sum_{i=1}^{b_n}P\left(|a_{ni}X_{ni}|\geq\varepsilon b_n^{\frac{1}{t}}\right)+Cb_n^{-\frac{p}{t}}\sum_{i=1}^{b_n}|a_{ni}|^pE|X_{ni}|^pI\left(|a_{ni}X_{ni}|<\varepsilon b_n^{\frac{1}{t}}\right)$$

$$+C\left(\sum_{i=1}^{b_n}P\left(|a_{ni}X_{ni}|\geq\varepsilon b_n^{\frac{1}{t}}\right)\right)^{\frac{p}{2}}+Cb_n^{-\frac{p}{t}}\left(\sum_{i=1}^{b_n}a_{ni}^2EX_{ni}^2I\left(|a_{ni}X_{ni}|<\varepsilon b_n^{\frac{1}{t}}\right)\right)^{\frac{p}{2}}.\tag{30}$$

The desired result (27) follows by (30) and conditions (a)–(d) immediately. This completes the proof of the theorem. □

Remark 2.3. According to the proof of Theorem 2.4, we can see that if (26) holds for

$$Y_{ni}=a_{ni}X_{ni}I\left(|a_{ni}X_{ni}|<\varepsilon b_n^{\frac{1}{t}}\right),\ 1\leq i\leq b_n,\ n\geq 1,$$

then we can still get (27) for all $\varepsilon>0$ and the condition (b) in Theorem 2.4 is not needed.

Remark 2.4. We pointed out that Kuczmaszewska (2009) obtained Theorem 2.4 for arrays of rowwise negatively associated random variables under the conditions (a), (c), and (d). We checked the inequality (10) of Theorem 2.1 in Kuczmaszewska (2009) carefully and found that the condition (b) should be added in Theorem 2.1 in Kuczmaszewska (2009). In addition, the ranges $p>2$ and $0<t<2$ in Theorem 2.1 in Kuczmaszewska (2009) are extended to $p\geq 2$ and $t>0$, respectively. So our results of Theorem 2.4 and Remark 2.3 generalize the corresponding one of Theorem 2.1 in Kuczmaszewska (2009).

If condition (a) in Theorem 2.4 is replaced by the following stronger condition (a'):

$$(a')\ \sum_{n=1}^{\infty} c_n b_n^{-\frac{1}{t}} \sum_{i=1}^{b_n} E|a_{ni}X_{ni}| I\left(|a_{ni}X_{ni}| \geq \varepsilon b_n^{\frac{1}{t}}\right) < \infty,$$

then we can get the following result on complete moment convergence for weighted sums of a class of random variables.

Theorem 2.5. *Let the conditions of Theorem 2.4 hold and the condition (a) is replaced by (a'), then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} c_n E\left\{\max_{1\leq j\leq b_n} b_n^{-\frac{1}{t}} \left|\sum_{i=1}^{j}\left(a_{ni}X_{ni} - a_{ni}EX_{ni}I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right)\right)\right| - \varepsilon\right\}^{+} < \infty \tag{31}$$

and

$$\sum_{n=1}^{\infty} c_n E\left\{\max_{1\leq j\leq b_n} b_n^{-\frac{1}{t}} \left|\sum_{i=1}^{j}(a_{ni}X_{ni} - a_{ni}EX_{ni})\right| - \varepsilon\right\}^{+} < \infty. \tag{32}$$

Proof. We use the same notations as those in Theorem 2.4. It is easily checked that

$$\begin{aligned} E\left|Y_{ni} - a_{ni}X_{ni}\right| &= E\left(|a_{ni}X_{ni}| - \varepsilon b_n^{\frac{1}{t}}\right) I\left(|a_{ni}X_{ni}| \geq \varepsilon b_n^{\frac{1}{t}}\right) \\ &\leq E|a_{ni}X_{ni}| I\left(|a_{ni}X_{ni}| \geq \varepsilon b_n^{\frac{1}{t}}\right). \end{aligned} \tag{33}$$

Hence, we have by (26), (29), and (33) that for all $\varepsilon > 0$,

$$\begin{aligned} &\sum_{n=1}^{\infty} c_n E\left\{\max_{1\leq j\leq b_n} b_n^{-\frac{1}{t}} \left|\sum_{i=1}^{j}\left(a_{ni}X_{ni} - a_{ni}EX_{ni}I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right)\right)\right| - \varepsilon\right\}^{+} \\ &\leq \sum_{n=1}^{\infty} c_n E\left\{\max_{1\leq j\leq b_n} b_n^{-\frac{1}{t}} \left|\sum_{i=1}^{j}(Y_{ni} - EY_{ni})\right| - \varepsilon\right\}^{+} \\ &\quad + \sum_{n=1}^{\infty} c_n E\left\{\max_{1\leq j\leq b_n} b_n^{-\frac{1}{t}} \left|\sum_{i=1}^{j}\left(a_{ni}X_{ni} - Y_{ni} - E(a_{ni}X_{ni} - Y_{ni})\right.\right.\right. \\ &\quad \left.\left.\left. + a_{ni}EX_{ni}I\left(|a_{ni}X_{ni}| \geq \varepsilon b_n^{\frac{1}{t}}\right)\right)\right|\right\} \\ &\leq \sum_{n=1}^{\infty} c_n \int_0^{\infty} P\left\{\max_{1\leq j\leq b_n} b_n^{-\frac{1}{t}} \left|\sum_{i=1}^{j}(Y_{ni} - EY_{ni})\right| > x + \varepsilon\right\} dx \\ &\quad + 3\sum_{n=1}^{\infty} c_n b_n^{-\frac{1}{t}} \sum_{i=1}^{b_n} E|a_{ni}X_{ni}| I\left(|a_{ni}X_{ni}| \geq \varepsilon b_n^{\frac{1}{t}}\right) \\ &\leq \sum_{n=1}^{\infty} c_n b_n^{-\frac{p}{t}} \int_0^{\infty} \frac{1}{(x+\varepsilon)^p} E\left(\max_{1\leq j\leq b_n} \left|\sum_{i=1}^{j}(Y_{ni} - EY_{ni})\right|^p\right) dx \\ &\quad + 3\sum_{n=1}^{\infty} c_n b_n^{-\frac{1}{t}} \sum_{i=1}^{b_n} E|a_{ni}X_{ni}| I\left(|a_{ni}X_{ni}| \geq \varepsilon b_n^{\frac{1}{t}}\right) \end{aligned}$$

$$\leq C\sum_{n=1}^{\infty} c_n b_n^{-\frac{p}{t}} \sum_{i=1}^{b_n} |a_{ni}|^p E|X_{ni}|^p I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right)$$
$$+C\sum_{n=1}^{\infty} c_n b_n^{-\frac{1}{t}} \sum_{i=1}^{b_n} E|a_{ni}X_{ni}| I\left(|a_{ni}X_{ni}| \geq \varepsilon b_n^{\frac{1}{t}}\right)$$
$$+C\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{b_n} P\left(|a_{ni}X_{ni}| \geq \varepsilon b_n^{\frac{1}{t}}\right)\right)^{\frac{p}{2}}$$
$$+C\sum_{n=1}^{\infty} c_n b_n^{-\frac{p}{t}} \left(\sum_{i=1}^{b_n} a_{ni}^2 E X_{ni}^2 I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right)\right)^{\frac{p}{2}}.$$

The desired result (31) follows by the inequality above and the conditions (a'), (b), (c), and (d) immediately.

Similar to the proof of (31), we can also get (32). This completes the proof of the theorem. □

Remark 2.5. It is easily seen that

$$\sum_{n=1}^{\infty} c_n E\left\{\max_{1\leq j\leq b_n} b_n^{-\frac{1}{t}} \left|\sum_{i=1}^{j}\left(a_{ni}X_{ni} - a_{ni}EX_{ni}I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right)\right)\right| - \varepsilon\right\}^{+}$$
$$= \sum_{n=1}^{\infty} c_n \int_0^{\infty} P\left\{\max_{1\leq j\leq b_n} b_n^{-\frac{1}{t}} \left|\sum_{i=1}^{j}\left(a_{ni}X_{ni} - a_{ni}EX_{ni}I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right)\right)\right| - \varepsilon > t\right\} dt$$
$$\geq \sum_{n=1}^{\infty} c_n \int_0^{\varepsilon} P\left\{\max_{1\leq j\leq b_n} b_n^{-\frac{1}{t}} \left|\sum_{i=1}^{j}\left(a_{ni}X_{ni} - a_{ni}EX_{ni}I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right)\right)\right| - \varepsilon > t\right\} dt$$
$$\geq \varepsilon \sum_{n=1}^{\infty} c_n P\left\{\max_{1\leq j\leq b_n} \left|\sum_{i=1}^{j}\left(a_{ni}X_{ni} - a_{ni}EX_{ni}I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right)\right)\right| > 2\varepsilon b_n^{\frac{1}{t}}\right\}$$

and

$$\sum_{n=1}^{\infty} c_n E\left\{\max_{1\leq j\leq b_n} b_n^{-\frac{1}{t}} \left|\sum_{i=1}^{j}(a_{ni}X_{ni} - a_{ni}EX_{ni})\right| - \varepsilon\right\}^{+}$$
$$= \sum_{n=1}^{\infty} c_n \int_0^{\infty} P\left\{\max_{1\leq j\leq b_n} b_n^{-\frac{1}{t}} \left|\sum_{i=1}^{j}(a_{ni}X_{ni} - a_{ni}EX_{ni})\right| - \varepsilon > t\right\} dt$$
$$\geq \sum_{n=1}^{\infty} c_n \int_0^{\varepsilon} P\left\{\max_{1\leq j\leq b_n} b_n^{-\frac{1}{t}} \left|\sum_{i=1}^{j}(a_{ni}X_{ni} - a_{ni}EX_{ni})\right| - \varepsilon > t\right\} dt$$
$$\geq \varepsilon \sum_{n=1}^{\infty} c_n P\left\{\max_{1\leq j\leq b_n} \left|\sum_{i=1}^{j}(a_{ni}X_{ni} - a_{ni}EX_{ni})\right| > 2\varepsilon b_n^{\frac{1}{t}}\right\}.$$

By (31) and (32), we have that for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\leq j\leq b_n} \left|\sum_{i=1}^{j}\left(a_{ni}X_{ni} - a_{ni}EX_{ni}I\left(|a_{ni}X_{ni}| < \varepsilon b_n^{\frac{1}{t}}\right)\right)\right| \geq \varepsilon b_n^{\frac{1}{t}}\right\} < \infty \tag{34}$$

and

$$\sum_{n=1}^{\infty} c_n P \left\{ \max_{1 \le j \le b_n} \left| \sum_{i=1}^{j} (a_{ni} X_{ni} - a_{ni} E X_{ni}) \right| \ge \varepsilon b_n^{\frac{1}{t}} \right\} < \infty \tag{35}$$

respectively. Hence, the complete moment convergence implies the complete convergence.

Acknowledgments

The authors are most grateful to the Editor-in-Chief N. Balakrishnan and two anonymous referees for careful reading of the manuscript and valuable suggestions which helped in improving an earlier version of this article.

Funding

This work was supported by the National Natural Science Foundation of China (11201001, 11501004, 11501005), the Natural Science Foundation of Anhui Province (1508085J06, 1608085QA02), the Provincial Natural Science Research Project of Anhui Colleges (KJ2015A018) and the Research Teaching Model Curriculum of Anhui University (xjyjkc1407).

References

Asadian, N., Fakoor, V., Bozorgnia, A. (2006). Rosenthal's type inequalities for negatively orthant dependent random variables. *J. Iran. Stat. Soc.* 5(1–2):66–75.

Bai, Z.D., Cheng, P.E. (2000). Marcinkiewicz strong laws for linear statistics. *Stat. Probab. Lett.* 46:105–112.

Cai, G.H. (2008). Strong laws for weighted sums of NA random variables. *Metrika* 68:323–331.

Chen, P., Gan, S. (2007). Limiting behavior of weighted sums of i.i.d. random variables. *Stat. Probab. Lett.* 77:1589–1599.

Chen, P.Y., Hu, T.C., Liu, X., Volodin, A. (2007). On complete convergence for arrays of rowwise negatively associated random variables. *Theory Probab. Appl.* 52:1–5.

Choi, B.D., Sung, S.H. (1987). Almost sure convergence theorems of weighted sums of random variables. *Stochastic Anal. Appl.* 5:365–377.

Cuzick, J. (1995). A strong law for weighted sums of i.i.d. random variables. *J. Theor. Probab.* 8:625–641.

Kuczmaszewska, A. (2009). On complete convergence for arrays of rowwise negatively associated random variables. *Stat. Probab. Lett.* 79:116–124.

Shao, Q.M. (2000). A comparison theorem on moment inequalities between negatively associated and independent random variables. *J. Theor. Probab.* 13(2):343–356.

Shen, A.T. (2011). Probability inequalities for END sequence and their applications. *J. Inequal. Appl.* 2011:98.

Shen, A.T. (2011). Some strong limit theorems for arrays of rowwise negatively orthant dependent random variables. *J. Inequal. Appl.* 2011:93.

Shen, A.T. (2013). On strong convergence for weighted sums of a class of random variables. *Abstr. Appl. Anal.*2013, Article ID 216236: 1–7.

Shen, A.T., Wang, X.H., Zhu, H.Y. (2013). On complete convergence for weighted sums of ρ^*-mixing random variables. *Abstr. Appl. Anal.*2013, Article ID 947487: 1–7.

Shen, A.T., Wu, R.C. (2013). Strong convergence results for weighted sums of $\tilde{\rho}$-mixing random variables. *J. Inequal. Appl.* 2013:327.

Sung, S.H. (2001). Strong laws for weighted sums of i.i.d. random variables. *Stat. Probab. Lett.* 52:413–419.

Sung, S.H. (2011). On the strong convergence for weighted sums of random variables. *Stat. Pap.* 52:447–454.

Tang, X.F. (2013). Strong convergence results for arrays of rowwise pairwise NQD random variables. *J. Inequal. Appl.* 2013:102.
Utev, S., Peligrad, M. (2003). Maximal inequalities and an invariance principle for a class of weakly dependent random variables. *J. Theor. Probab.* 16(1):101–115.
Wang, J.F., Lu, F.B. (2006). Inequalities of maximum of partial sums and weak convergence for a class of weak dependent random variables. *Acta Math. Sin., English Ser.* 22(3):693–700.
Wang, X.J., Deng, X., Zheng, L.L., Hu, S.H. (2014). Complete convergence for arrays of rowwise negatively superadditive-dependent random variables and its applications. *Stat.: J. Theor. Appl. Stat.* 48(4):834–850.
Wang, X.J., Hu, S.H., Volodin, A. (2011a). Strong limit theorems for weighted sums of NOD sequence and exponential inequalities. *Bull. Kor. Math. Soc.* 48(5):923–938.
Wang, X.J., Hu, S.H., Yang, W.Z. (2011b). Complete convergence for arrays of rowwise asymptotically almost negatively associated random variables. *Discr. Dyn. Nat. Soc.*2011, Article ID 717126:1–11.
Wang, X.J., Hu, S.H., Yang, W.Z. (2012c). Complete convergence for arrays of rowwise negatively orthant dependent random variables. *RACSAM* 106(2):235–245.
Wang, X.J., Hu, S.H., Yang, W.Z., Shen, Y. (2010). On complete convergence for weighted sums of φ-mixing random variables. *J. Inequal. Appl.*2010, Article ID 372390:1–13.
Wang, X.J., Hu, S.H., Yang, W.Z., Wang, X.H. (2012a). On complete convergence of weighted sums for arrays of rowwise asymptotically almost negatively associated random variables. *Abstr. Appl. Anal.*2012, Article ID 315138:1–15.
Wang, X.J., Li, X.Q., Yang, W.Z., Hu, S.H. (2012b). On complete convergence for arrays of rowwise weakly dependent random variables. *Appl. Math. Lett.* 25:1916–1920.
Wu, Q.Y. (2006). *Probability Limit Theory for Mixing Sequences*. Beijing: Science Press of China.
Wu, Q.Y. (2010). A strong limit theorem for weighted sums of sequences of negatively dependent random variables. *J. Inequal. Appl.*2010, Article ID 383805:1–8.
Wu, Q.Y. (2012a). Sufficient and necessary conditions of complete convergence for weighted sums of PNQD random variables. *J. Appl. Math.*2012, Article ID 104390:1–10.
Wu, Q.Y. (2012b). A complete convergence theorem for weighted sums of arrays of rowwise negatively dependent random variables. *J. Inequal. Appl.* 2012:50.
Wu, W.B. (1999). On the strong convergence of a weighted sum. *Stat. Probab. Lett.* 44:19–22.
Xu, H., Tang, L. (2012). Some convergence properties for weighted sums of pairwise NQD sequences. *J. Inequal. Appl.* 2012:255.
Yuan, D.M., An, J. (2009). Rosenthal type inequalities for asymptotically almost negatively associated random variables and applications. *Sci. China Ser. A: Math.* 52(9):1887–1904.
Zhou, X.C., Tan, C.C., Lin, J.G. (2011). On the strong laws for weighted sums of ρ^*-mixing random variables. *J. Inequal. Appl.*2011, Article ID 157816:1–8.