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A note on the exponential inequality for a class of dependent random variables

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ABSTRACT

An exponential inequality is established for a random variable with the finite Laplace transform. Using this inequality, we obtain an exponential inequality for identically distributed acceptable random variables (a class of random variables introduced in Giuliano Antonini, Kozachenko, and Volodin (2008) which includes negatively dependent random variables). Our result improves the corresponding ones in Kim and Kim (2007), Nooghabi and Azarnoosh (2009), Sung (2009), Xing (2009), Xing and Yang (2010) and Xing, Yang, Liu, and Wang (2009). Our method is much simpler than those in the literature.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. The exponential inequality for the partial sum $\sum_{i=1}^{n}(X_i - EX_i)$ is very useful in many probabilistic derivations. In particular, it provides a measure of the convergence rate for the strong law of large numbers. There exist several versions available in the literature for independent random variables with assumptions of uniform boundedness or some, quite relaxed, control on their moments. If the independent case is classical in the literature, the treatment of dependent variables is more recent.

One of the dependence structure that has attracted the interest of probabilists and statisticians is negative association. The concept of negatively associated random variables was introduced by Alam and Saxena (1981) and carefully studied by Joag-Dev and Proschan (1983).

A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated if for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0$$

whenever f_1 and f_2 are coordinatewise increasing (or coordinatewise decreasing) and the covariance exists. An infinite family of random variables is negatively associated if every finite subfamily is negatively associated.

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As pointed out and proved by Joag-Dev and Proschan (1983), a number of well known multivariate distributions possesses the negative association property, such as multinomial, convolution of unlike multinomial, multivariate hypergeometric, Dirichlet, permutation distribution, negatively correlated normal distribution, random sampling without replacement, and joint distribution of ranks.

The counterpart of the negative association is positive association. The concept of positively associated random variables was introduced by Esary, Proschan, and Walkup (1967). The exponential inequalities for positively associated random variables were obtained by Devroye (1991), Ioannides and Roussas (1999), Oliveira (2005), Sung (2007), Xing and Yang (2008); Xing, Yang, and Liu (2008). On the other hand, Kim and Kim (2007), Nooghabi and Azarnoosh (2009), Roussas (1996), Sung (2009), Xing (2009), Xing and Yang (2010), and Xing et al. (2009) obtained exponential inequalities for negatively associated random variables.

The next dependence notion important for this paper is the notion of negatively dependent random variables. Joag-Dev and Proschan (1983) pointed out that negative association property implies negative dependence, but negative dependence does not imply negative association. They gave an example of a collection of random variables that are negatively dependent, but not negatively associated. Negative association is much more restrictive and a stronger property than negative dependence.

The concept of negatively dependent random variables was introduced by Lehmann (1966) as follows.

A finite family of random variables $\{X_1, \ldots, X_n\}$ is said to be *negatively dependent* if the following two inequalities hold:

$$P(X_1 \le x_1, \ldots, X_n \le x_n) \le \prod_{i=1}^{n} P(X_i \le x_i)$$

and

$$P(X_1 > x_1, \ldots, X_n > x_n) \le \prod_{i=1}^{n} P(X_i > x_i)$$

for all real numbers $x_1, \ldots, x_n$. An infinite family of random variables is negatively dependent if every finite subfamily is negatively dependent.

Finally, the following notion is slightly weaker than which was introduced in Giuliano Antonini et al. (2008).

We say that a finite family of random variables $\{X_1, \ldots, X_n\}$ is *acceptable* if there exists $\delta > 0$ such that for any real λ such that $|\lambda| \le \delta$,

$$E\exp\left(\lambda \sum_{i=1}^{n} X_i\right) \le \prod_{i=1}^{n} E\exp(\lambda X_i).$$

Note that in this definition we implicitly assume that the expectations are finite (otherwise the definition has no sense). Hence we assume that the random variables have the finite Laplace transform (or moment generating function) near zero. A sequence of random variables $\{X_n, n \ge 1\}$ is *acceptable* if every finite subfamily is acceptable.

We say that this definition of acceptability is weaker than which was introduced in Giuliano Antonini et al. (2008), because in Giuliano Antonini et al. (2008) it is required that the inequality holds for all λ, while here we require only for $|\lambda| \le \delta$.

As is mentioned in Giuliano Antonini et al. (2008), a sequence of negatively dependent random variables with a finite Laplace transform or finite moment generating function near zero (and hence a sequence of negatively associated random variables with finite Laplace transform, too) provides us an example of acceptable random variables. For example, Xing et al. (2009) consider a strictly stationary negatively associated sequence of random variables. According to the sentence above, a sequence of strictly stationary and negatively associated random variables is acceptable. Another interesting example of a sequence $\{Z_n, n \ge 1\}$ of acceptable random variables can be constructed in the following way. Feller (1971, Problem III.1) (cf. also Romano and Siegel (1986), Section 4.30) provides an example of two random variables X and Y such that the density of their sum is the convolution of their densities, yet they are not independent. It is simple to see that X and Y are not negatively dependent either. Since they are bounded, their Laplace transforms $E\exp(\lambda X)$ and $E\exp(\lambda Y)$ are finite for any λ. Next, since the density of their sum is the convolution of their densities, we have

$$E\exp(\lambda(X+Y)) = E\exp(\lambda X)\, E\exp(\lambda Y).$$

The announced sequence of acceptable random variables $\{Z_n, n \ge 1\}$ can be now constructed in the following way. Let (X_k, Y_k) be independent copies of the random vector (X, Y), $k \ge 1$. For any $n \ge 1$ set $Z_n = X_k$ if $n = 2k+1$ and $Z_n = Y_k$ if $n = 2k$. Hence, the model of acceptable random variables that we consider in this paper is more general than models considered in the previous papers (such as a sequence of negatively associated random variables considered in Xing et al. (2009)).

In this paper, we establish an exponential inequality for a random variable with the finite Laplace transform. Using this inequality, we obtain an exponential inequality for identically distributed acceptable random variables which have the finite Laplace transforms. For the special case of negatively associated random variables, our result improves the corresponding ones in Kim and Kim (2007), Nooghabi and Azarnoosh (2009), Sung (2009), Xing (2009), Xing and Yang (2010), and Xing et al. (2009). The technique used in the literature mentioned above is based on so-called monotone truncation method. Our technique does not use the truncation method and so that our results are much more general and simpler, and their proofs are much simpler.

In Giuliano Antonini et al. (2008) a notion of m-acceptable random variables is introduced. We would like to mention that the results of this paper can be generalized to the case of m-acceptable random variables, too. We decided not to provide these generalizations since they make formulations of the statements very cumbersome.

2. Main results

The following lemma is an exponential inequality for a random variables which is not necessarily bounded. It plays an essential role in our main results.

Lemma 2.1. *Let X be a random variable with $Ee^{\delta|X|} < \infty$ for some $\delta > 0$. Then for any $0 < \lambda \le \delta/2$,*

$$Ee^{\lambda(X-EX)} \le \exp(K\lambda^2),$$

where K is defined as $K = 2(E|X|^4)^{1/2}Ee^{\delta|X|}$.

Proof. From the inequality $e^x \le 1 + x + \frac{x^2}{2}e^{|x|}$ for all $x \in R$, we have by the Hölder inequality, the c_r-inequality, and the Jensen inequality that for any $0 < \lambda \le \delta/2$,

$$\begin{aligned}
Ee^{\lambda(X-EX)} &\le 1 + \lambda E(X-EX) + \frac{\lambda^2}{2}E\left((X-EX)^2e^{\lambda|X-EX|}\right)\\
&= 1 + \frac{\lambda^2}{2}E\left((X-EX)^2e^{\lambda|X-EX|}\right)\\
&\le 1 + \frac{\lambda^2}{2}\left(E(X-EX)^4\right)^{1/2}\left(Ee^{2\lambda|X-EX|}\right)^{1/2} \quad \text{(by the Hölder inequality)}\\
&\le 1 + \frac{\lambda^2}{2}\left(2^3(E|X|^4 + |EX|^4)\right)^{1/2}\left(Ee^{2\lambda|X|}e^{2\lambda|EX|}\right)^{1/2} \quad \text{(by the } c_r\text{-inequality)}\\
&\le 1 + 2\lambda^2\left(E|X|^4\right)^{1/2}\left(Ee^{2\lambda|X|}Ee^{2\lambda|X|}\right)^{1/2} \quad \text{(by the Jensen inequality)}\\
&= 1 + 2\lambda^2\left(E|X|^4\right)^{1/2}Ee^{2\lambda|X|}\\
&\le 1 + 2\lambda^2\left(E|X|^4\right)^{1/2}Ee^{\delta|X|}\\
&= 1 + \lambda^2K\\
&\le \exp(K\lambda^2),
\end{aligned}$$

since $1 + x \le e^x$ for all $x \in R$. Here $K = 2(E|X|^4)^{1/2}Ee^{\delta|X|}$. Hence the result is proved. □

Remark 2.1. There exist several exponential inequalities for a bounded random variable. Hoeffding (1963) proved that if $a \le X \le b$, then for any $\lambda > 0$,

$$Ee^{\lambda(X-EX)} \le \exp(\lambda^2(b-a)^2/8).$$

Chow (1966) obtained that if $EX = 0$ and $|X| \le 1$, then for any real number λ

$$Ee^{\lambda X} \le \exp(\lambda^2).$$

Our exponential inequality holds for not only a bounded random variable but also an unbounded random variable with the finite Laplace transform.

Now we state and prove one of our main results.

Theorem 2.1. *Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed acceptable random variables with $Ee^{\delta|X_1|} < \infty$ for some $\delta > 0$. Then for any $0 < \epsilon \le K\delta$,*

$$P\left(\left|\sum_{i=1}^{n}(X_i - EX_i)\right| > n\epsilon\right) \le 2\exp\left(-\frac{n\epsilon^2}{4K}\right),$$

where $K = 2(E|X_1|^4)^{1/2}Ee^{\delta|X_1|}$.

Proof. Let $0 < \epsilon \le K\delta$. Then we have by Markov's inequality, the definition of acceptable random variables and Lemma 2.1 that for any $0 < \lambda \le \delta/2$,

$$\begin{aligned} P\left(\sum_{i=1}^{n}(X_i - EX_i) > n\epsilon\right) &= P\left(\exp\left(\lambda\sum_{i=1}^{n}(X_i - EX_i)\right) > \exp(\lambda n\epsilon)\right) \\ &\le \exp(-\lambda n\epsilon)E\exp\left(\lambda\sum_{i=1}^{n}(X_i - EX_i)\right) \\ &\le \exp(-\lambda n\epsilon)\prod_{i=1}^{n}E\exp(\lambda(X_i - EX_i)) \\ &\le \exp(-\lambda n\epsilon)\prod_{i=1}^{n}\exp(K\lambda^2) \\ &= \exp(-\lambda n\epsilon + K\lambda^2 n). \end{aligned}$$

Optimizing the exponent in the term of this upper bound, we find $\lambda = \epsilon/(2K)$. Note that $\epsilon/(2K) \le \delta/2$, since $0 < \epsilon \le K\delta$. Putting $\lambda = \epsilon/(2K)$, we get

$$P\left(\sum_{i=1}^{n}(X_i - EX_i) > n\epsilon\right) \le \exp\left(-\frac{n\epsilon^2}{4K}\right). \tag{2.1}$$

Since $\{-X_n, n \ge 1\}$ are also acceptable random variables, we can replace X_i by $-X_i$ in the above statement. That is,

$$P\left(-\sum_{i=1}^{n}(X_i - EX_i) > n\epsilon\right) \le \exp\left(-\frac{n\epsilon^2}{4K}\right). \tag{2.2}$$

Observing that

$$P\left(\left|\sum_{i=1}^{n}(X_i - EX_i)\right| > n\epsilon\right) = P\left(\sum_{i=1}^{n}(X_i - EX_i) > n\epsilon\right) + P\left(-\sum_{i=1}^{n}(X_i - EX_i) > n\epsilon\right),$$

the result follows by (2.1) and (2.2). □

Example 2.1. Consider a stationary sequence $\{X_n, n \ge 1\}$ of Gaussian random variables such that their autocovariance function is negative. Then it is a negative associated sequence according to Joag-Dev and Proschan (1983, p. 293) and hence an acceptable sequence because the Laplace transform for a Gaussian random variable is finite. Therefore all assumptions of Theorem 2.1 are satisfied and hence the conclusion of the theorem is true.

Remark 2.2. For the special case of negatively associated random variables, when $\epsilon = \left(2\delta eE|X_1|^2 c_n/n\right)^{1/2}$ and $0 < c_n \le \left(eE|X_1|^2 n/(8\delta)\right)^{1/3}$, Sung (2009) obtained the upper bound

$$2\left(1 + \frac{Ee^{\delta|X_1|}}{\delta^3 eE|X_1|^2 c_n}\right)\exp(-\delta c_n). \tag{2.3}$$

To compare our upper bound, we choose $\epsilon = \left(2\delta eE|X_1|^2 c_n/n\right)^{1/2}$ and $0 < c_n \le K^2\delta n/(2eE|X_1|^2)$. Noting that $\epsilon/(K\delta) \le 1$, we have, by Theorem 2.1, the following upper bound

$$2\exp\left(-\frac{\delta eE|X_1|^2 c_n}{2K}\right). \tag{2.4}$$

The convergence rates of (2.3) and (2.4) are the same. However, the restriction on c_n in Sung (2009) is stronger than $0 < c_n \le K^2\delta n/(2eE|X_1|^2)$. Hence our result generalizes the result of Sung (2009).

By choosing $\epsilon = 2(K\alpha\log n/n)^{1/2}$ in Theorem 2.1, we have the following result.

Theorem 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed acceptable random variables with $Ee^{\delta|X_1|} < \infty$ for some $\delta > 0$. Set $\epsilon_n = 2(K\alpha\log n/n)^{1/2}$, where $\alpha > 0$ and $K = 2(E|X_1|^4)^{1/2}Ee^{\delta|X_1|}$. Then for all large n,*

$$P\left(\left|\sum_{i=1}^{n}(X_i - EX_i)\right| > n\epsilon_n\right) \le 2\exp(-\alpha\log n).$$

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Proof. Let $\epsilon_n = 2(K\alpha \log n/n)^{1/2}$, where $\alpha > 0$ and $K = 2(E|X_1|^4)^{1/2}Ee^{\delta|X_1|}$. Then $\epsilon_n/(K\delta) \le 1$ for all large n. Hence the result follows directly from Theorem 2.1. □

Remark 2.3. For the special case of negatively associated random variables, let us compare our result with the results of Kim and Kim (2007), Nooghabi and Azarnoosh (2009), Xing (2009), Xing et al. (2009), and Xing and Yang (2010).

(1) When $\epsilon_n = O(1)(p_n \log^3 n/n)^{1/2}$ and $1 \le p_n < n/2$, Kim and Kim (2007) obtained the upper bound

$$\left(4 + \frac{Ee^{\delta|X_1|}n^2}{9\alpha^3 p_n \log^3 n}\right)\exp(-\alpha \log n),$$

where $0 < \alpha < \delta$.

(2) When $\epsilon_n = O(1)(p_n \log^3 n/n)^{1/2}$ and $1 \le p_n < n/2$, Nooghabi and Azarnoosh (2009) obtained the upper bound

$$\left(2\left(1 + \frac{\alpha C_0}{4}\right) + \frac{2Ee^{\delta|X_1|}n^2}{9\alpha^2 p_n \log^3 n}\right)\exp(-\alpha \log n)$$

under the additional assumption on the covariance structure

$$\frac{\log n}{p_n \log^2 n}\exp\left\{\left(\frac{\alpha n \log n}{2p_n}\right)^{1/2}\right\}\sum_{j=p_n+2}^{\infty}|\mathrm{Cov}(X_1, X_j)| \le C_0 < \infty,$$

where $0 < \alpha < \delta$.

(3) When $\epsilon_n = O(1)(p_n \log^3 n/n)^{1/2}$ and $n/(\alpha \log n) \ge p_n \to \infty$, Xing (2009) obtained the upper bound

$$\left(C_1 + \frac{9C_2Ee^{\delta|X_1|}}{25\alpha^3 p_n \log^3 n}\right)\exp(-\alpha \log n)$$

under the additional assumption on the covariance structure

$$\frac{1}{n^{2\alpha/3}p_n \log n}\exp\left\{\left(\frac{\alpha n \log n}{p_n}\right)^{1/2}\right\}\sum_{j=p_n+1}^{\infty}|\mathrm{Cov}(X_1, X_j)| < \infty,$$

where $C_1 > 0$ and $C_2 > 0$ are positive constants and $0 < \alpha \le \delta$.

(4) When $\epsilon_n = O(1)(\log^3 n/n)^{1/2}$ and $1 \le p_n \le O(1)(n/\log n)^{1/2}$, Xing et al. (2009) obtained the upper bound

$$\left(4 + \frac{CEe^{\delta|X_1|}}{4\alpha^3 \log^3 n}\right)\exp(-\alpha \log n),$$

where $C > 0$ is a positive constant and $0 < \alpha \le \delta$.

(5) When $\epsilon_n = O(1)(\log n/n)^{1/2}$ and $1 \le p_n \le O(1)(n/\log n)^{1/2}$, Xing and Yang (2010) obtained the upper bound

$$\left(4 + \frac{C_1Ee^{\delta|X_1|}}{4C_2\delta^3 \log n}\right)\exp(-\delta \log n),$$

where $C_1 > 0$ and $C_2 > 0$ are positive constants.

From (1)–(5), we see that our upper bound is less than those of Kim and Kim (2007), Nooghabi and Azarnoosh (2009), Xing (2009), Xing et al. (2009), and Xing and Yang (2010).

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