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Complete Convergence for Weighted Sums and Arrays of Rowwise Extended Negatively Dependent Random Variables

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In this article, we study the complete convergence for weighted sums of extended negatively dependent random variables and row sums of arrays of rowwise extended negatively dependent random variables. We apply two methods to prove the results: the first of is based on exponential bounds and second is based on the generalization of the classical moment inequality for extended negatively dependent random variables.

Keywords Complete convergence; Extended negatively dependent sequence; Negatively dependent.

Mathematics Subject Classification 60E15; 60F15.

1. Introduction

The concept of complete convergence was introduced by Hsu and Robbins (1947) as follows. A sequence of random variables $\{U_n, n \geq 1\}$ is said to *converge completely* to a constant C if $\sum_{n=1}^{\infty} P\{|U_n - C| > \epsilon\} < \infty$ for all $\epsilon > 0$. In view of the Borel-Cantelli lemma, this implies that $U_n \to C$ almost surely (a.s.). The converse is true if the $\{U_n, n \geq 1\}$ are independent. Hsu and Robbins (1947) proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Since then many authors studied the complete convergence for partial sums and weighted sums of random variables. The main purpose of the present investigation is to provide the complete convergence results for weighted sums of END random variables and arrays of rowwise END random variables.

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Our main tools are exponential bounds of sub-Gaussian type and a generalization of the classical moment inequality.

To prove the main results, we need to introduce some notions and present some lemmas.

The following dependence structure was introduced in Liu (2009).

Definition 1.1. We say that random variables $\{X_n, n \geq 1\}$ are *extended negatively dependent* (END) if there exists a constant $M > 0$ such that both inequalities

$$P(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n) \leq M \prod_{i=1}^{n} P(X_i > x_i) \tag{1.1}$$

and

$$P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) \leq M \prod_{i=1}^{n} P(X_i \leq x_i). \tag{1.2}$$

hold for each $n \geq 1$ and all real numbers $x_1, x_2, \dots, x_n$.

In the case $M = 1$ the notion of END random variables reduces to the well-known notion of so-called *negatively dependent* (ND) random variables which was introduced by Lehmann (1966) (cf. also Joag-Dev and Proschan, 1983). Not looking that the notion of END seems to be a straightforward generalization of the notion of negative dependence, the extended negative dependence structure is substantially more comprehensive. As is mentioned in Liu (2009), the END structure can reflect not only a negative dependence structure but also a positive one (inequalities from the definition of ND random variables hold both in reverse direction), to some extend. We refer the interested reader to Example 4.1 in Liu (2009) where END random variables can be taken as negatively or positively dependent. Also, Joag-Dev and Proschan (1983) pointed out that negatively associated (NA) random variables are ND and thus NA random variables are END.

Some interesting applications for END sequence have been found. For example, for END random variables with heavy tails Liu (2009) obtained the precise large deviations and Liu (2010) studied sufficient and necessary conditions for moderate deviations. Since the assumption of END for a sequence of random variables is much weaker than an independence, negative association, or negative dependence, a study on a limiting behavior of END sequences is of interest.

2. Preliminaries

The following two lemmas provide us a few important properties of END random variables. The statement of the first lemma we could found in Liu (2010).

Lemma 2.1. *Let random variables $X_1, X_2, \dots, X_n$ be END.*

(i) *If $f_1, f_2, \dots, f_n$ are all non decreasing (or non increasing) functions, then random variables $f_1(X_1), f_2(X_2), \dots, f_n(X_n)$ are END.*
(ii) *For each $n \geq 1$, there exists a constant $M > 0$ such that*

$$E\left(\prod_{j=1}^{n} X_j^+\right) \leq M \prod_{j=1}^{n} EX_j^+. \tag{2.1}$$

Remark 2.1. Note that (2.1) holds only for positive part of random variables. The main idea of its proof is the application of the following well known formula for positive random variables:

$$E(X_1X_2\dots X_n) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty P(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n)dx_1dx_2\dots dx_n.$$

We would like to note that inequality (2.1) does not hold for arbitrary (taking positive and negative values) END random variables for $n > 2$ even for the case $M = 1$. But for $n = 2$ it holds, that is, if X_1 and X_2 are two END random variables, then

$$E(X_1 \cdot X_2) \leq ME(X_1) \cdot E(X_2).$$

This follows from so-called *Hoeffding identity* and we refer the interested reader to Lehmann (1966).

The next lemma is a simple corollary of the previous one.

Lemma 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables, then for each $n \geq 1$ and $t \in \mathbb{R}$, there exists a constant $M > 0$ such that*

$$E\left(\prod_{i=1}^n e^{tX_i}\right) \leq M \prod_{i=1}^n Ee^{tX_i}. \tag{2.2}$$

As we already mentioned, in this article, we study limiting behavior for END random variables through exponential inequalities of sub-Gaussian type.

Definition 2.2. Let δ and τ be two positive constants. A random variable X is said to be (τ, δ)-*sub-Gaussian*, if $E\exp\{tX\} \leq \exp\{\tau t^2/2\}$ for every $t \in (-\delta, \delta)$.

This is a slight modification of the well-known notion of *sub-Gaussian* random variables, which are simply (τ, δ)-sub-Gaussian with $\delta = \infty$. For classical sub-Gaussian random variables we refer for example to Hoffmann-Jørgensen (1994, Sec. 4.29), where this notion is made explicit and where it is substantiated with several important examples.

Next lemma is a simple statement that a mean zero bounded random variable is subgaussian. The proof may be found in the above-mentioned Hoffmann-Jørgensen (1994, Sec. 4.29), or in Serfling (1980, p. 200).

Lemma 2.3. *Let X be a random variable with $EX = \mu$. If $P(l \leq X \leq u) = 1$, then for every real number t,*

$$Ee^{t(X-\mu)} \leq e^{\frac{t^2(u-l)^2}{8}} \quad \textit{and moreover } Ee^{t|X-\mu|} \leq 2e^{\frac{t^2(u-l)^2}{8}}.$$

Hence, the random variable $X - \mu$ is (τ, δ)-sub-Gaussian with $\tau = (u - l)^2/4$ and $\delta = \infty$.

We say that an array $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ of random variables is rowwise END if for each fixed $n \geq 1$ random variables are END and we assume that the constant M from the definition of END is the same for each row.

To prove the complete convergence for arrays of rowwise END random variables, we need the following generalization of the classical moment inequality.

Lemma 2.4. *Let $\{X_n, n \geq 1\}$ be a sequence of END random variables with $EX_i = 0$ and $EX_i^2 < \infty$ for each $i \geq 1$. Then there exists a positive constant C such that*

$$E\left|\sum_{i=1}^{n} X_i\right|^2 \leq C\sum_{i=1}^{n} EX_i^2. \tag{2.3}$$

Proof. Denote $S_n = \sum_{i=1}^n X_i$ and $M_{2,n} = \sum_{i=1}^n EX_i^2$ for each $n \geq 1$. Let F_i be the distribution function of X_i, $i \geq 1$. For any $y > 0$, denote $Y_i = \min(X_i, y)$, $i = 1, 2, \ldots, n$ and $T_n = \sum_{i=1}^n Y_i$, $n \geq 1$.

It is easy to check that for any $x > 0$,

$$\{S_n \geq x\} \subset \{T_n \neq S_n\} \cup \{T_n \geq x\},$$

which implies that for any positive number h,

$$P(S_n \geq x) \leq P(T_n \neq S_n) + P(T_n \geq x) \leq \sum_{i=1}^{n} P(X_i \geq y) + e^{-hx}Ee^{hT_n}. \tag{2.4}$$

Lemma 2.1(i) implies that $Y_1, Y_2, \ldots, Y_n$ are still END random variables. It follows from (2.4) and Lemma 2.2 that

$$P(S_n \geq x) \leq \sum_{i=1}^{n} P(X_i \geq y) + Me^{-hx}\prod_{i=1}^{n} Ee^{hY_i}, \tag{2.5}$$

where M is a positive constant.

Now we estimate Ee^{hY_i}. It is easy to see that $(e^{hu} - 1 - hu)/u^2$ is non decreasing on the real line. Therefore,

$$\begin{aligned}
Ee^{hY_i} &= \int_{-\infty}^{y} e^{hu}dF_i(u) + \int_{y}^{\infty} e^{hy}dF_i(u) \\
&\leq 1 + hEX_i + \int_{-\infty}^{y} \left(e^{hu} - 1 - hu\right) dF_i(u) + \int_{y}^{\infty} \left(e^{hy} - 1 - hy\right) dF_i(u) \\
&= 1 + \int_{-\infty}^{y} \frac{e^{hu} - 1 - hu}{u^2} u^2 dF_i(u) + \int_{y}^{\infty} \left(e^{hy} - 1 - hy\right) dF_i(u) \\
&\leq 1 + \frac{e^{hy} - 1 - hy}{y^2}\left(\int_{-\infty}^{y} u^2 dF_i(u) + \int_{y}^{\infty} y^2 dF_i(u)\right) \\
&\leq 1 + \frac{e^{hy} - 1 - hy}{y^2}EX_i^2 \ \leq \ \exp\left\{\frac{e^{hy} - 1 - hy}{y^2}EX_i^2\right\},
\end{aligned}$$

which implies that

$$\begin{aligned}
P(S_n \geq x) &\leq \sum_{i=1}^{n} P(X_i \geq y) + Me^{-hx}\prod_{i=1}^{n} Ee^{hY_i} \\
&\leq \sum_{i=1}^{n} P(X_i \geq y) + M\exp\left\{\frac{e^{hy} - 1 - hy}{y^2}M_{2,n} - hx\right\}.
\end{aligned}$$

Replacing X_i by $-X_i$, we have

$$P(-S_n \geq x) \leq \sum_{i=1}^{n} P(-X_i \geq y) + M \exp\left\{ \frac{e^{hy} - 1 - hy}{y^2} M_{2,n} - hx \right\}.$$

Therefore,

$$P(|S_n| \geq x) \leq \sum_{i=1}^{n} P(|X_i| \geq y) + 2M \exp\left\{ \frac{e^{hy} - 1 - hy}{y^2} M_{2,n} - hx \right\}. \tag{2.6}$$

If we take $h = \frac{1}{y} \log\left(1 + \frac{xy}{M_{2,n}}\right)$, then

$$P(|S_n| \geq x) \leq \sum_{i=1}^{n} P(|X_i| \geq y) + 2M \exp\left\{ \frac{x}{y} - \frac{x}{y} \log\left(1 + \frac{xy}{M_{2,n}}\right) \right\}. \tag{2.7}$$

Taking $y = \frac{x}{r}$ in (2.7), where $r > 1$, we have

$$P\left(|S_n| \geq x\right) \leq \sum_{i=1}^{n} P\left(|X_i| \geq \frac{x}{r}\right) + 2Me^r \left(1 + \frac{x^2}{rM_{2,n}}\right)^{-r},$$

which implies that

$$\begin{aligned} \int_0^\infty 2xP\left(|S_n| \geq x\right) dx &\leq 2 \sum_{i=1}^{n} \int_0^\infty xP\left(r|X_i| \geq x\right) dx \\ &\quad + 4Me^r \int_0^\infty x \left(1 + \frac{x^2}{rM_{2,n}}\right)^{-r} dx. \end{aligned}$$

That is to say (or see Lemma 2.4 of Petrov, 1995) for $r > 1$,

$$\begin{aligned} ES_n^2 &\leq r^2 \sum_{i=1}^{n} EX_i^2 + 4Me^r \int_0^\infty x \left(1 + \frac{x^2}{rM_{2,n}}\right)^{-r} dx \\ &= r^2 \sum_{i=1}^{n} EX_i^2 + \frac{2rMe^r}{r-1} \sum_{i=1}^{n} EX_i^2 \\ &= \left(r^2 + \frac{2rMe^r}{r-1}\right) \sum_{i=1}^{n} EX_i^2 \doteq C \sum_{i=1}^{n} EX_i^2. \end{aligned}$$

This completes the proof of the lemma.

In this article, we assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of positive numbers and C and M denote positive constants which may be different from place to place.

3. Complete Convergence for Normed Weighted Sums of a Sequence of END Random Variables

With the preliminaries accounted for, we could now present our first result.

Theorem 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of (τ_n, δ_n)-sub-Gaussian END random variables and $\{b_n, n \geq 1\}$ be a sequence of positive numbers. Denote $B_n = \sum_{i=1}^{n} a_{ni}^2 \tau_i / 2$*

and $\phi_n = \min_{1\le i\le n} \delta_i/a_{ni},\ \ n \ge 1$. *If for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \exp\left\{-\frac{b_n^2\varepsilon^2}{4B_n}\right\} < \infty, \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} \exp\left\{-\frac{\phi_n b_n \varepsilon}{2}\right\} < \infty, \tag{3.2}$$

then

$$\frac{1}{b_n}\sum_{i=1}^{n} a_{ni}X_i \to 0 \quad \textit{completely, as } n \to \infty.$$

Proof. For each $n \ge 1$, $1 \le i \le n$ and $|t| \le \phi_n$, we can write

$$Ee^{ta_{ni}X_i} \le e^{t^2 a_{ni}^2 \tau_i/2}. \tag{3.3}$$

By Lemma 2.1(i), we obtain that $\{a_{ni}X_i, 1 \le i \le n, n \ge 1\}$ and $\{-a_{ni}X_i, 1 \le i \le n, n \ge 1\}$ are sequences of END random variables. Therefore, by Markov's inequality, Lemma 2.2 and the inequality above, we can get that for any $x \ge 0$ and $|t| \le \phi_n$, there exists a constant $M > 0$ such that

$$\begin{aligned}
P\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| \ge x\right) &= P\left(\sum_{i=1}^{n} a_{ni}X_i \ge x\right) + P\left(\sum_{i=1}^{n}(-a_{ni}X_i) \ge x\right) \\
&\le e^{-|t|x} E\exp\left\{|t|\sum_{i=1}^{n} a_{ni}X_i\right\} + e^{-|t|x}E\exp\left\{|t|\sum_{i=1}^{n}(-a_{ni}X_i)\right\} \\
&\le e^{-tx} E\exp\left\{|t|\sum_{i=1}^{n} a_{ni}X_i\right\} + e^{-tx}E\exp\left\{|t|\sum_{i=1}^{n}(-a_{ni}X_i)\right\} \\
&= e^{-tx} E\exp\left\{t\sum_{i=1}^{n} a_{ni}X_i\right\} + e^{-tx}E\exp\left\{t\sum_{i=1}^{n}(-a_{ni}X_i)\right\} \\
&\le e^{-tx} M\left(\prod_{i=1}^{n} Ee^{ta_{ni}X_i} + \prod_{i=1}^{n} Ee^{-ta_{ni}X_i}\right) \\
&\le 2M\exp\left\{-tx + t^2 B_n\right\}.
\end{aligned}$$

Hence,

$$P\left(\left|\sum_{i=1}^{n} a_{ni}X_i\right| \ge x\right) \le 2M \min_{|t|\le\phi_n} \exp\left\{-tx + t^2 B_n\right\}.$$

If $0 \le x \le 2B_n\phi_n$, then

$$\min_{|t|\le\phi_n} \exp\left\{-tx + t^2 B_n\right\} = \exp\left\{-\frac{x}{2B_n}x + \frac{x^2}{4B_n^2}B_n\right\} = \exp\left\{-\frac{x^2}{4B_n}\right\}.$$

If $x \geq 2B_n\phi_n$, then note that $\phi_n^2 B_n \leq \phi_n x/2$ and

$$\min_{|t|\leq\phi_n} \exp\left\{-tx + t^2 B_n\right\} = \exp\left\{-\phi_n x + \phi_n^2 B_n\right\} \leq \exp\left\{-\frac{\phi_n x}{2}\right\}.$$

That is, for any $x \geq 0$,

$$P\left(\left|\sum_{i=1}^n a_{ni}X_i\right| \geq x\right) \leq 2M\left(\exp\left\{-\frac{x^2}{4B_n}\right\} + \exp\left\{-\frac{\phi_n x}{2}\right\}\right).$$

Taking $x = \varepsilon b_n$ we obtain

$$\begin{aligned}
&\sum_{n=1}^\infty P\left(\left|\frac{1}{b_n}\sum_{i=1}^n a_{ni}X_i\right| \geq \varepsilon\right) \\
&\quad \leq 2M\left(\sum_{n=1}^\infty \exp\left\{-\frac{b_n^2\varepsilon^2}{4B_n}\right\} + \sum_{n=1}^\infty \exp\left\{-\frac{\phi_n b_n \varepsilon}{2}\right\}\right) < \infty.
\end{aligned}$$

This completes the proof of the theorem.

Now we consider a few special cases that could help us to establish the convergence of the series $\sum_{n=1}^\infty \exp\left\{-\frac{\phi_n b_n \varepsilon}{2}\right\}$ mentioned in Theorem 3.1. First of all, note that this series obviously convergence in the case of classically sub-Gaussian random variables, that is, $\delta_n = \infty$ for all $n \geq 1$. Thus, we can formulate the following result.

Proposition 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of (τ_n, δ_n)-sub-Gaussian END random variables with $\delta_n = \infty$ and $\{b_n, n \geq 1\}$ be a sequence of positive numbers. Denote $B_n = \sum_{i=1}^n a_{ni}^2\tau_i/2$. If for any $\varepsilon > 0$,*

$$\sum_{n=1}^\infty \exp\left\{-\frac{b_n^2\varepsilon^2}{4B_n}\right\} < \infty, \tag{3.4}$$

then

$$\frac{1}{b_n}\sum_{i=1}^n a_{ni}X_i \to 0 \quad \textit{completely, as } n \to \infty.$$

For the next case we consider the following assumption of Bernstein's type inequality. Let $\{X_i, i \geq 1\}$ be a sequence of random variables with $EX_i = 0$ and $EX_i^2 \doteq \sigma_i^2 < \infty$ and suppose that there exists a positive number H such that for any positive integer $m \geq 2$,

$$|E(X_i)^m| \leq \frac{m!}{2}\sigma_i^2 H^{m-2}. \tag{3.5}$$

Then the random variable X_i is (τ_i, δ_i)-sub-Gaussian with $\tau_i = 2\sigma_i^2$, $\delta_i = \frac{1}{2H}$.

Really, for any $n \geq 1$ and $1 \leq i \leq n$ the Bernstein's type inequality mentioned above implies that

$$\begin{aligned}
Ee^{tX_i} &= 1 + \frac{t^2}{2}\sigma_i^2 + \frac{t^3}{6}EX_i^3 + \cdots \\
&\leq 1 + \frac{t^2}{2}\sigma_i^2(1 + H|t| + H^2t^2 + \cdots).
\end{aligned}$$

When $|t| \le \frac{1}{2H}$, it follows that

$$Ee^{tX_i} \le 1 + \frac{t^2\sigma_i^2}{2} \cdot \frac{1}{1 - H|t|} \le 1 + t^2\sigma_i^2 \le e^{t^2\sigma_i^2} \doteq e^{\tau_i t^2/2}. \tag{3.6}$$

That is to say the random variable X_i is (τ_i, δ_i)-subgaussian with $\tau_i = 2\sigma_i^2$, $\delta_i = \frac{1}{2H}$.

Thus, we can state the following result.

Proposition 3.2. *Let $\{X_n, n \ge 1\}$ be a sequence of END random variables with $EX_i = 0$ and $EX_i^2 \doteq \sigma_i^2 < \infty$ and $\{b_n, n \ge 1\}$ be a sequence of positive numbers. Denote $B_n = \sum_{i=1}^n a_{ni}^2\sigma_i^2$ and $\phi_n = \min_{1\le i\le n} 1/a_{ni}$, $n \ge 1$. Assume that there exists a positive number H such that for any positive integer $m \ge 2$,*

$$|E(X_i)^m| \le \frac{m!}{2}\sigma_i^2 H^{m-2}. \tag{3.7}$$

If for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \exp\left\{-\frac{b_n^2\varepsilon^2}{4B_n}\right\} < \infty, \tag{3.8}$$

and

$$\sum_{n=1}^{\infty} \exp\{-\phi_n b_n \varepsilon\} < \infty, \tag{3.9}$$

then

$$\frac{1}{b_n}\sum_{i=1}^n a_{ni}X_i \to 0 \quad \textit{completely, as } n \to \infty.$$

In the next proposition we consider the case of bounded random variables.

Proposition 3.3. *Let $\{X_n, n \ge 1\}$ be a sequence of END bounded random variables with $EX_i = 0$ and $\{b_n, n \ge 1\}$ be a sequence of positive numbers. Let $\{l_n, n \ge 1\}$ and $\{u_n, n \ge 1\}$ be sequences of real numbers such that $P(l_n \le X_n \le u_n) = 1$, $n \ge 1$. Denote $B_n = \sum_{i=1}^n a_{ni}^2(u_i - l_i)^2/8$. If for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \exp\left\{-\frac{b_n^2\varepsilon^2}{4B_n}\right\} < \infty, \tag{3.10}$$

then

$$\frac{1}{b_n}\sum_{i=1}^n a_{ni}X_i \to 0 \quad \textit{completely, as } n \to \infty.$$

Proof. The statement follows immediately from Proposition 3.1 and Lemma 2.3.

4. Complete Convergence for Row Weighted Sums of an Array of Rowwise END Random Variables

The main tool that we use in this section is the generalization of the classical moment inequality presented in Lemma 2.4.

Theorem 4.1. *Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise END random variables with $EX_{ni} = 0$ and $EX_{ni}^2 \doteq \sigma_{ni}^2 < \infty$ for each $1 \le i \le n$ and $n \ge 1$. Let $\{b_n, n \ge 1\}$ be a sequence of positive numbers. If*

$$\sum_{n=1}^{\infty} \frac{1}{b_n^2} \sum_{i=1}^{n} a_{ni}^2 \sigma_{ni}^2 < \infty, \tag{4.1}$$

then

$$\frac{1}{b_n} \sum_{i=1}^{n} a_{ni} X_{ni} \to 0 \quad \text{completely, as } n \to \infty.$$

Proof. Lemma 2.1(i) implies that $\{a_{ni}X_{ni}, 1 \le i \le n\}$ are still END random variables for fixed $n \ge 1$. By Lemma 2.4 we can see that

$$E\left(\sum_{i=1}^{n} a_{ni} X_{ni}\right)^2 \le C \sum_{i=1}^{n} a_{ni}^2 EX_{ni}^2 = C \sum_{i=1}^{n} a_{ni}^2 \sigma_{ni}^2,$$

where C is a positive constant defined in Lemma 2.4. By the assumption, the inequality above, and Markov's inequality, we have that for any $\varepsilon > 0$,

$$\begin{aligned} \sum_{n=1}^{\infty} P\left(\left|\frac{1}{b_n} \sum_{i=1}^{n} a_{ni} X_{ni}\right| > \varepsilon\right) &\le \sum_{n=1}^{\infty} \frac{1}{b_n^2 \varepsilon^2} E\left(\sum_{i=1}^{n} a_{ni} X_{ni}\right)^2 \\ &\le \frac{C}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{b_n^2} \sum_{i=1}^{n} a_{ni}^2 \sigma_{ni}^2 < \infty. \end{aligned}$$

Hence, $\frac{1}{b_n} \sum_{i=1}^{n} a_{ni} X_{ni} \to 0$ completely, as $n \to \infty$.

Taking $b_n = n^\alpha$, $\alpha > 0$ and $a_{ni} \equiv 1$, $1 \le i \le n$, $n \ge 1$, we can get the following corollary.

Corollary 4.1. *Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise END random variables with $EX_{ni} = 0$ and $EX_{ni}^2 \doteq \sigma_{ni}^2 < \infty$ for each $1 \le i \le n$ and $n \ge 1$. If for some $\alpha > 0$,*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \sum_{i=1}^{n} \sigma_{ni}^2 < \infty,$$

then

$$\frac{1}{n^\alpha} \sum_{i=1}^{n} X_{ni} \to 0 \quad \text{completely, as } n \to \infty.$$

Proposition 4.1. *Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise END random variables with $EX_{ni} = 0$ and $EX_{ni}^2 \doteq \sigma_{ni}^2 < \infty$ for each $1 \le i \le n$ and $n \ge 1$. Suppose that there exists a positive constant C such that $a_{ni}^2 \sigma_{ni}^2 \le C a_{ii}^2 \sigma_{ii}^2$ for each $1 \le i \le n$ and $n \ge 1$. If for some $\alpha > 1/2$,*

$$\sum_{i=1}^{\infty} \frac{a_{ii}^2 \sigma_{ii}^2}{i^{2\alpha-1}} < \infty, \tag{4.2}$$

then

$$\frac{1}{n^\alpha}\sum_{i=1}^{n} a_{ni}X_{ni} \to 0 \quad \textit{completely, as } n \to \infty.$$

Proof. Take $b_n = n^\alpha$, then the assumption from Theorem 4.1 can be estimated as follows:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{b_n^2}\sum_{i=1}^{n} a_{ni}^2\sigma_{ni}^2 &\le C\sum_{n=1}^{\infty}\frac{1}{n^{2\alpha}}\sum_{i=1}^{n} a_{ii}^2\sigma_{ii}^2\\ &= C\sum_{i=1}^{\infty} a_{ii}^2\sigma_{ii}^2\sum_{n=i}^{\infty}\frac{1}{n^{2\alpha}}\\ &\le C\sum_{i=1}^{\infty}\frac{a_{ii}^2\sigma_{ii}^2}{i^{2\alpha-1}} < \infty.\end{aligned}$$

The conclusion follows from Theorem 4.1 immediately.

The last result of this article deals with arrays with uniformly bounded second moments.

Proposition 4.2. *Let $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of rowwise END random variables satisfying*

$$EX_{ni} = 0 \quad \textit{and} \quad EX_{ni}^2 \le A \tag{4.3}$$

for all $1 \le i \le n$ and $n \ge 1$, where A is a positive constant. Suppose that $\sum_{i=1}^{n} a_{ni}^2 = O(n^\delta)$ for some $\delta > 0$. Then for all $\alpha > \frac{1+\delta}{2}$,

$$\frac{1}{n^\alpha}\sum_{i=1}^{n} a_{ni}X_{ni} \to 0 \quad \textit{completely, as} \quad n \to \infty.$$

Proof. Take $b_n = n^\alpha$, then the assumption from Theorem 4.1 can be estimated as follows.

$$\sum_{n=1}^{\infty}\frac{1}{b_n^2}\sum_{i=1}^{n} a_{ni}^2\sigma_{ni}^2 \le A\sum_{n=1}^{\infty}\frac{1}{n^{2\alpha}}n^\delta = A\sum_{n=1}^{\infty}\frac{1}{n^{2\alpha-\delta}} < \infty.$$

Remark 4.1. Hanson and Wright (1971) and Wright (1973) obtained a bound on tail probabilities for quadratic forms in independent random variables using the following condition. There exist $C > 0$ and $\gamma > 0$ such that for all $1 \le i \le n$, $n \ge 1$ and all $x > 0$, we have

$$P(|X_{ni}| \ge x) \le C\int_x^{+\infty} e^{-\gamma t^2}dt. \tag{4.4}$$

Note that if (4.4) is true, then for all $1 \le i \le n$ and $n \ge 1$,

$$\begin{aligned}EX_{ni}^2 = \int_\Omega X_{ni}^2 dP \;&=\; \int_\Omega\Big[\int_0^{|X_{ni}|} 2x dx\Big]dP\\ &= \int_\Omega\Big[\int_0^\infty 2I(|X_{ni}| \ge x)x dx\Big]dP = \int_0^\infty\Big[2x\int_\Omega I(|X_{ni}| \ge x)dP\Big]dx\end{aligned}$$

$$= \int_0^{+\infty} 2xP(|X_{ni}| \geq x)dx \leq \int_0^{+\infty} 2x \left(C \int_x^{+\infty} e^{-\gamma t^2} dt \right) dx$$
$$= C \int_0^{+\infty} e^{-\gamma t^2} \left(\int_0^t 2x dx \right) dt = C \int_0^{+\infty} t^2 e^{-\gamma t^2} dt = \frac{C\sqrt{\pi}}{4\gamma^{3/2}}.$$

Hence, Proposition 4.2 remains true under the condition (4.4) considered in Hanson and Wright (1971) and Wright (1973). For more details about condition (4.4), one can refer to Hanson (1967a,b).

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