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Complete moment convergence for weighted sums of sequences of independent random elements in Banach spaces

Dehua Qiu · Henar Urmeneta · Andrei Volodin

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Abstract In this paper, we obtain complete moment convergence results for weighted sums of sequences of independent random elements in a real separable Banach spaces without any geometric conditions imposed on the Banach space. Our results improve and extend some well known results from the literature. Furthermore, we obtain a complete moment convergence results for weighted sums of arrays of arbitrary random elements.

Keywords Complete convergence · Complete moment convergence · Random element · Weighted sums

Mathematics Subject Classification (2000) 60F15

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D. Qiu (✉)
School of Mathematics and Statistics, Guangdong University of Business Studies,
Guangzhou 510320, Peoples Republic of China
e-mail: qdh20130118@163.com

H. Urmeneta
Department of Statistics and Operations Research, Public University of Navarre,
31016 Pamplona, Spain
e-mail: henar@unavarra.es

Andrei Volodin
Department of Mathematics and Statistics, University of Regina,
Regina, SK S4S 0A2, Canada
e-mail: andrei.volodin@uregina.ca

1 Introduction

The concept of complete convergence was introduced by Hsu and Robbins [20] as follows:

A sequence of random variables $\{X_n, n \geq 1\}$ is said to converge completely to a constant C if $\sum_{n=1}^{\infty} P(|X_n - C| > \epsilon) < \infty$, for all $\epsilon > 0$. From then on, many authors have devoted their study to complete convergence (see [2–12]). In the following, Let $\{\Omega, \mathscr{F}, P\}$ be a probability space and let B be a separable real Banach space with norm $\|\cdot\|$. A random element is defined to be an $\mathscr{F}$-measurable mapping of Ω into B equipped with the Borel σ-algebra (that is, the σ-algebra generated by the open sets determined by $\|\cdot\|$). The expects value of a B-valued random element X is defined to be the Bochner integral and denoted by EX. A sequence of Banach space valued random elements is said to converge completely to the zero element of the Banach space if the corresponding sequence of norms converges completely to 0. There are many papers investigating complete convergence in a Banach space setting (see [2,4–8,10–12]).

When $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables, Chow [13] first investigated the complete moment convergence, which is more exact than complete convergence. He obtained the following result. Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX = 0$. For $1 \leq p < 2$ and $r \geq 1$, if $E\{|X|^{rp} + |X| \log(1 + |X|)\} < \infty$, then

$$\sum_{n=1}^{\infty} n^{r-2-1/p} E\left\{\left|\sum_{k=1}^{n} X_k\right| - \epsilon n^{1/p}\right\}_+ < \infty, \quad \text{for all} \quad \epsilon > 0,$$

where and in the following $x_+ = x$ if $x \geq 0$ and $x_+ = 0$ if $x < 0$, x_+^q mean $(x_+)^q$. Chow's result has been generalized and extended in several directions. For example, Wang and Su [14] extended and generalized Chow's result to a Rademacher type p Banach space, Wang and Zhao [15] for negatively associated (NA) random variables, Li and Zhang [16], Chen et al. [17] for moving-average processes based on NA random variables, Chen [18] for i.i.d. random elements in a Banach space without any geometric conditions imposed on the Banach space. For sums of negatively associated random variables, Liang et al. [19] provided necessary and sufficient conditions for complete moment convergence to hold and they showed that these conditions are equivalent to a form of complete integral convergence.

In this paper, we obtain complete moment convergence results for weighted sums of sequences of independent random elements in a separable real Banach spaces B without any geometric conditions imposed on the Banach space B. Our results improve and extend some well known results from the literature. Furthermore, we obtain a complete moment convergence results for weighted sums of arrays of arbitrary random elements.

We recall that the sequence, $\{X_n, n \geq 1\}$, of random elements taking values in B is said to be stochastically dominated by a random element X taking values in B, if there exists a constant $D > 0$ such that

$$P(\|X_n\| > x) \leq DP(\|X\| > x), \quad \text{for all} \quad x > 0.$$

In this case we write $\{X_n, n \geq 1\} \prec X$.

In the following, we consider the sequence of weighted sums $S_n \equiv \sum_{i=1}^{\infty} a_{ni} X_i, n \geq 1$, where $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of real constants (called weights) and $\{X_n, n \geq 1\}$ is a sequence of random elements taking values in B.

Throughout this paper, we assume without explicit mention that each series S_n converges a.s. if such almost sure convergence is not automatic from the hypotheses. C always stands for a positive constant which may differ from one place to another, the symbol $[x]$ denotes the greatest integer in x, and for a finite set A, the symbol $\sharp A$ denotes the number of elements in the set A.

2 Main results

Now we present the main results of the paper. The proofs will be given in the next section.

Theorem 2.1 *Let θ, β be real constants, $p > 0, \alpha+\beta+1 > 0, 0 < v = \theta + p(\alpha+\beta+1) < 1, 0 < q < 1$. Let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers satisfying*

$$\sup_{i\ge 1} |a_{ni}| = O(n^{-1/p}) \tag{2.1}$$

and

$$\sum_{i=1}^{\infty} |a_{ni}|^{\theta} = O(n^{\alpha}). \tag{2.2}$$

Let $\{X_n, n \ge 1\}$ be a sequence of random elements taking values in B with $\{X_n, n \ge 1\} \prec X$. If

$$\begin{cases} E\|X\|^{v} < \infty, & \text{if } q < v, \\ E\|X\|^{v}\log(1+\|X\|) < \infty, & \text{if } q = v, \\ E\|X\|^{q} < \infty, & \text{if } q > v, \end{cases} \tag{2.3}$$

then

$$\sum_{n=1}^{\infty} n^{\beta} E\{\|S_n\| - \epsilon\}_{+}^{q} < \infty, \quad \text{for all } \epsilon > 0. \tag{2.4}$$

Theorem 2.2 *Let θ be a real constant, $p > 0, \beta \ge -1, \alpha+\beta+1 > 0, v = \theta + p(\alpha+\beta+1) \ge 1, q > 0$. Let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers satisfying (2.1) and (2.2). If $v \ge 2$, moreover, we assume that $\{a_{ni}, i \ge 1, n \ge 1\}$ satisfies*

$$\sum_{i=1}^{\infty} |a_{ni}|^{2} = O(n^{\gamma}) \text{ for some } \gamma < 0. \tag{2.5}$$

Let $\{X_n, n \ge 1\}$ be a sequence of independent random elements taking values in B with $\{X_n, n \ge 1\} \prec X$. If $S_n \xrightarrow{P} 0$ and (2.3) holds, then (2.4) holds.

Theorem 2.3 *Let $p > 0, 0 < \theta < 1, 0 < q < 1$. Let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers satisfying (2.1) and (2.2). Let $\{X_n, n \ge 1\}$ be a sequence of random elements taking values in B with $\{X_n, n \ge 1\} \prec X$. If*

$$\begin{cases} E\|X\|^{\theta}\log(1+\|X\|) < \infty, & \text{if } q < \theta, \\ E\|X\|^{\theta}\log^{2}(1+\|X\|) < \infty, & \text{if } q = \theta, \\ E\|X\|^{q} < \infty, & \text{if } q > \theta, \end{cases} \tag{2.6}$$

then (2.4) holds, where $\beta = -1-\alpha$.

Theorem 2.4 *Let $p > 0, \theta \ge 1, q > 0$. Let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers satisfying (2.1) and (2.2). If $\theta \ge 2$, moreover, we assume that $\{a_{ni}, i \ge 1, n \ge 1\}$ satisfies (2.5). Let $\{X_n, n \ge 1\}$ be a sequence of independent random elements taking values in B with $\{X_n, n \ge 1\} \prec X$. If $S_n \xrightarrow{P} 0$ and (2.6) holds, then (2.4) holds, where $\beta = -1-\alpha$.*

Corollary 2.5 *Let $\{X_n, n \geq 1\}$ be a sequence of independent random elements taking values in B with $\{X_n, n \geq 1\} \prec X$. Let $\{b_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers satisfying*

$$\sum_{i=1}^{\infty} |b_{ni}|^{\theta} = O(1) \text{ for some } 2 \leq \theta < p,$$

and

$$\sum_{i=1}^{\infty} |b_{ni}|^{2} = O(n^{\lambda}) \text{ for some } \lambda < 2/p.$$

If

$$\sum_{i=1}^{\infty} b_{ni} X_i \xrightarrow{P} 0 \text{ and } \begin{cases} E\|X\|^p < \infty, & \text{if } q < p, \\ E\|X\|^p \log(1+\|X\|) < \infty, & \text{if } q = p, \\ E\|X\|^q < \infty, & \text{if } q > p, \end{cases}$$

then

$$\sum_{n=1}^{\infty} E\left\{ n^{-1/p} \left\| \sum_{i=1}^{\infty} b_{ni} X_i \right\| - \epsilon \right\}_{+}^{q} < \infty, \text{ for all } \epsilon > 0.$$

Corollary 2.6 *Let $\{X_n, n \geq 1\}$ be a sequence of independent random elements taking values in B with $\{X_n, n \geq 1\} \prec X$. Let $\{b_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers satisfying*

$$\sum_{i=1}^{\infty} |b_{ni}|^{2} = O(1).$$

If

$$\sum_{i=1}^{\infty} b_{ni} X_i \xrightarrow{P} 0 \quad \text{and} \quad \begin{cases} E\|X\|^2 \log(1+\|X\|) < \infty, & \text{if } q < 2, \\ E\|X\|^2 \log^2(1+\|X\|) < \infty, & \text{if } q = 2, \\ E\|X\|^q < \infty, & \text{if } q > 2, \end{cases}$$

then

$$\sum_{n=1}^{\infty} E\left\{ n^{-1/2} \left\| \sum_{i=1}^{\infty} b_{ni} X_i \right\| - \epsilon \right\}_{+}^{q} < \infty, \text{ for all } \epsilon > 0.$$

Corollary 2.7 *Let $\{Y_n, -\infty < n < \infty\}$ be a sequence of independent random elements taking values in B with $\{Y_n, -\infty < n < \infty\} \prec X$. Let $\{a_n, -\infty < n < \infty\}$ be a sequence of real numbers such that $\sum_{n=-\infty}^{\infty} |a_n| < \infty$. Set $X_i = \sum_{j=-\infty}^{\infty} a_{j+i} Y_j$ for each $i \geq 1$.*

1. *If $0 < p < 2, \beta \geq -1, p(\beta+2) > 1$ and*

$$n^{-1/p} \sum_{i=1}^{n} X_i \xrightarrow{P} 0 \text{ and } \begin{cases} E\|X\|^{p(\beta+2)} < \infty, & \text{if } 0 < q < p(\beta+2), \\ E\|X\|^{p(\beta+2)} \log(1+\|X\|) < \infty, & \text{if } q = p(\beta+2), \\ E\|X\|^q < \infty, & \text{if } q > p(\beta+2), \end{cases}$$

Then

$$\sum_{n=1}^{\infty} n^{\beta} E\left\{ n^{-1/p} \left\| \sum_{i=1}^{n} X_i \right\| - \epsilon \right\}_{+}^{q} < \infty, \text{ for all } \epsilon > 0.$$

2. *If* $p = 1, \beta = -1$ *and*

$$n^{-1}\sum_{i=1}^{n} X_i \xrightarrow{P} 0 \quad \text{and} \quad \begin{cases} E\|X\|\log(1+\|X\|) < \infty, & \text{if } q < 1, \\ E\|X\|\log^2(1+\|X\|) < \infty, & \text{if } q = 1, \\ E\|X\|^q < \infty, & \text{if } q > 1, \end{cases}$$

Then

$$\sum_{n=1}^{\infty} n^{-1} E\left\{n^{-1}\left\|\sum_{i=1}^{n} X_i\right\| - \epsilon\right\}_+^q < \infty, \text{ for all } \epsilon > 0.$$

Remark 1 Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random elements taking values in B and $a_{ni} = n^{-1/p}$, if $1 \leq i \leq n$; $a_{ni} = 0$, if $i > n$. From Theorem 2.1 and Theorem 2.2, we get the corresponding results of Chen [18].

Remark 2 Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random elements taking values in B, Corollaries 2.5 and 2.6 extend and improve the corresponding results of Li et al. [6], Corollary 2.7 extends and improves the corresponding results of Sung [9], since complete moment convergence is more exact than complete convergence.

3 Proofs

In order to prove the results, we present the following Lemma.

Lemma 3.1 *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent symmetric random elements,* $\{a_{ni}, i \geq 1, n \geq 1\}$ *be an array of real numbers. For any fixed* $\eta > 0$ *and* $q > 0$*, if*

$$\lim_{n\to\infty} \sup_{x^{1/q}\geq 1} P\left(\left\|\sum_{i=1}^{\infty} a_{ni}X_i I(\|a_{ni}X_i\| \leq \eta x^{1/q})\right\| > x^{1/q}\epsilon\right) = 0, \text{ for all } \epsilon > 0,$$

then

$$\lim_{n\to\infty} \sup_{x^{1/q}\geq 1} E\left\|x^{-1/q}\sum_{i=1}^{\infty} a_{ni}X_i I(\|a_{ni}X_i\| \leq \eta x^{1/q})\right\| < 6\eta.$$

Proof The proof is similar to Lemma 2.1 of Cheng [18], so it is omitted here. □

We prove Theorems 2.1 and 2.2. The Proof of Theorem 2.3 is similar to that of Theorem 2.1 and the Proof of Theorem 2.4 is similar to that of Theorem 2.2; therefore, the Proofs of Theorems 2.3 and 2.4 are omitted.

Proof of Theorem 2.1 By (2.1) and (2.2), without loss of generality, we can assume that

$$|a_{ni}| \leq n^{-1/p}, \text{ for all } i \geq 1, n \geq 1, \tag{3.1}$$

$$\sum_{i=1}^{\infty} |a_{ni}|^{\theta} \leq n^{\alpha}, \text{ for all } n \geq 1. \tag{3.2}$$

For any fixed $\epsilon > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\beta} E\{\|S_n\|-\epsilon\}_+^q &= \sum_{n=1}^{\infty} n^{\beta} \int_0^{\infty} P\left(\|S_n\|-\epsilon > x^{1/q}\right) dx \\ &= \sum_{n=1}^{\infty} n^{\beta} \int_0^{1} P\left(\|S_n\| > \epsilon + x^{1/q}\right) dx \\ &\quad + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P\left(\|S_n\| > \epsilon + x^{1/q}\right) dx \\ &\le \sum_{n=1}^{\infty} n^{\beta} P(\|S_n\| > \epsilon) + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P(\|S_n\| > x^{1/q}) dx \\ &\stackrel{\text{def}}{=} I_1 + I_2.\end{aligned}$$

By Theorem 1 of Qiu [4], we have that $I_1 < \infty$, for all $\epsilon > 0$, so we only need to prove that $I_2 < \infty$. For $i \ge 1$, $n \ge 1$, we define $U_{ni}^{(x)} = a_{ni} X_i I(\|a_{ni} X_i\| \le x^{1/q})$ and $V_{ni}^{(x)} = a_{ni} X_i - U_{ni}^{(x)}$, for all $x > 0$; thus

$$\begin{aligned}I_2 &\le \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P\left(\left\|\sum_{i=1}^{\infty} U_{ni}^{(x)}\right\| > x^{1/q}/2\right) dx + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P\left(\left\|\sum_{i=1}^{\infty} V_{ni}^{(x)}\right\| > x^{1/q}/2\right) dx \\ &\stackrel{\text{def}}{=} I_{21} + I_{22}.\end{aligned}$$

We first show that $I_{22} < \infty$. Since $\alpha + \beta + 1 > 0$, $p > 0$, we take $\delta > 0$ such that $\theta < \delta < v$, we get by Lemma 2 of Qiu et al. [5], (3.1), (3.2), (2.3), the Mean-Value Theorem and standard computation that

$$\begin{aligned}I_{22} &\le \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P\left(\sum_{i=1}^{\infty} \|V_{ni}^{(x)}\| > 0\right) dx \\ &\le \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} \sum_{i=1}^{\infty} P\left(\|a_{ni} X_i\| > x^{1/q}\right) dx \\ &\le C \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} \sum_{i=1}^{\infty} P\left(\|a_{ni} X\| > x^{1/q}\right) dx \\ &\le C \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} \sum_{i=1}^{\infty} x^{-\delta/q} E\|a_{ni} X\|^{\delta} I(\|a_{ni} X\| > x^{1/q}) dx \\ &\le C \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} \sum_{i=1}^{\infty} |a_{ni}|^{\delta} x^{-\delta/q} E\|X\|^{\delta} I(\|X\| > n^{1/p} x^{1/q}) dx\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{\beta+\alpha-(\delta-\theta)/p}\int_{1}^{\infty} x^{-\delta/q}E\|X\|^{\delta}I(\|X\|>n^{1/p}x^{1/q})dx\\
&= C\sum_{n=1}^{\infty} n^{\beta+\alpha+\theta/p-q/p}\int_{n^{q/p}}^{\infty} y^{-\delta/q}E\|X\|^{\delta}I(\|X\|>y^{1/q})dy\\
&\qquad (\text{ letting } y=n^{q/p}x)\\
&= C\sum_{n=1}^{\infty} n^{(v-q)/p-1}\sum_{m=n}^{\infty}\int_{m^{q/p}}^{(m+1)^{q/p}} x^{-\delta/q}E\|X\|^{\delta}I(\|X\|>x^{1/q})dx\\
&\le C\sum_{n=1}^{\infty} n^{(v-q)/p-1}\sum_{m=n}^{\infty} m^{-1+(q-\delta)/p}E\|X\|^{\delta}I(\|X\|>m^{1/p})\\
&\le \begin{cases} C\sum_{m=1}^{\infty} m^{-1+(v-\delta)/p}E\|X\|^{\delta}I(\|X\|>m^{1/p}), & \text{if } q<v\\ C\sum_{m=1}^{\infty} m^{-1+(q-\delta)/p}\log m\, E\|X\|^{\delta}I(\|X\|>m^{1/p}), & \text{if } q=v\\ C\sum_{m=1}^{\infty} m^{-1+(q-\delta)/p}E\|X\|^{\delta}I(\|X\|>m^{1/p}), & \text{if } q>v \end{cases}\\
&\le \begin{cases} CE\|X\|^{v}, & \text{if } q<v\\ CE\|X\|^{v}\log(1+\|X\|), & \text{if } q=v\\ CE\|X\|^{q}, & \text{if } q>v \end{cases}\\
&< \infty.
\end{aligned}
$$

Let $I_{nk}=\{i:(k+1)^{-1/p}n^{-1/p}<|a_{ni}|\le k^{-1/p}n^{-1/p}\}, k\ge 1, n\ge 1$, then $\cup_{k=1}^{\infty} I_{nk}=\mathbb{N}$, for all $n\ge 1$, where $\mathbb{N}$ is the set of positive integers. Let $t>v>\theta$, we have $k^{(t-\theta)/p}>j^{(t-\theta)/p}$, for all $j\in\mathbb{N}$ with $j<k$. If $\theta>0$, then

$$
\begin{aligned}
n^{\alpha} &\ge \sum_{i=1}^{\infty}|a_{ni}|^{\theta}=\sum_{k=1}^{\infty}\sum_{i\in I_{nk}}|a_{ni}|^{\theta}\ge\sum_{k=1}^{\infty}(\sharp I_{nk})(k+1)^{-\theta/p}n^{-\theta/p}\\
&\ge \sum_{k=j}^{\infty}(\sharp I_{nk})(k+1)^{-t/p}(j+1)^{(t-\theta)/p}n^{-\theta/p}\\
&> 2^{-\theta/p}\sum_{k=j}^{\infty}(\sharp I_{nk})k^{-t/p}j^{(t-\theta)/p}n^{-\theta/p}.
\end{aligned}
$$

If $\theta<0$, then

$$
\begin{aligned}
n^{\alpha} &\ge \sum_{i=1}^{\infty}|a_{ni}|^{\theta}=\sum_{k=1}^{\infty}\sum_{i\in I_{nk}}|a_{ni}|^{\theta}\ge\sum_{k=1}^{\infty}(\sharp I_{nk})k^{-\theta/p}n^{-\theta/p}\\
&\ge \sum_{k=j}^{\infty}(\sharp I_{nk})k^{-t/p}j^{(t-\theta)/p}n^{-\theta/p}.
\end{aligned}
$$

Therefore,

$$\sum_{k=j}^{\infty}(\sharp I_{nk})k^{-t/p} \le Cn^{\alpha+\theta/p}j^{-(t-\theta)/p}, \quad \forall j \ge 1. \tag{3.3}$$

We now prove that $I_{21} < \infty$. Taking t such that $\max\{v, q\} < t < 1$, By Markov's inequality, C_r-inequality, Lemma 2 of Qiu et al. [5] and the proof of $I_{22} < \infty$, we have

$$\begin{aligned}
I_{21} &\le C\sum_{n=1}^{\infty} n^{\beta}\int_{1}^{\infty} x^{-t/q}\sum_{i=1}^{\infty} E\,\|a_{ni}X_i\|^t\, I(\|a_{ni}X_i\| \le x^{1/q})dx \\
&\le C\sum_{n=1}^{\infty} n^{\beta}\int_{1}^{\infty} x^{-t/q}\sum_{i=1}^{\infty}\left\{E\,\|a_{ni}X\|^t\, I(\|a_{ni}X\| \le x^{1/q}) + x^{t/q}P(\|a_{ni}X\| > x^{1/q})\right\}dx \\
&= C\sum_{n=1}^{\infty} n^{\beta}\int_{1}^{\infty} x^{-t/q}\sum_{j=1}^{\infty}\sum_{i\in I_{nj}} E\,\|a_{ni}X\|^t\, I(\|a_{ni}X\| \le x^{1/q})dx + C \\
&\le C\sum_{n=1}^{\infty} n^{\beta}\int_{1}^{\infty} x^{-t/q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-t/p}n^{-t/p}E\|X\|^t I(\|X\|^q \le x(j+1)^{q/p}n^{q/p})dx + C \\
&= C\sum_{n=1}^{\infty} n^{\beta-q/p}\int_{n^{q/p}}^{\infty} y^{-t/q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-t/p}E\|X\|^t I(\|X\|^q \le y(j+1)^{q/p})dy + C \\
&\qquad (\text{ letting } y = n^{q/p}x) \\
&\le C\sum_{n=1}^{\infty} n^{\beta-q/p}\int_{n^{q/p}}^{\infty} x^{-t/q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-t/p}\sum_{k=0}^{[x(j+1)^{q/p}]} E\|X\|^t I(k < \|X\|^q \le k+1)dx + C \\
&= C\sum_{n=1}^{\infty} n^{\beta-q/p}\int_{n^{q/p}}^{\infty} x^{-t/q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-t/p}\sum_{k=0}^{[2x]} E\|X\|^t I(k < \|X\|^q \le k+1)dx \\
&\quad + C\sum_{n=1}^{\infty} n^{\beta-q/p}\int_{n^{q/p}}^{\infty} x^{-t/q}\sum_{j=1}^{\infty}(\sharp I_{nj})j^{-t/p}\sum_{k=[2x]+1}^{[x(j+1)^{q/p}]} E\|X\|^t I(k < \|X\|^q \le k+1)dx + C \\
&\overset{def}{=} I_{211} + I_{212} + C.
\end{aligned}$$

By (3.3), (2.3), the Mean-Value Theorem and a standard computation, we have that

$$\begin{aligned}
I_{211} &\le C\sum_{n=1}^{\infty} n^{\beta-q/p}\int_{n^{q/p}}^{\infty} x^{-t/q}n^{\alpha+\theta/p}\sum_{k=0}^{[2x]} E\|X\|^t I(k < \|X\|^q \le k+1)dx \\
&\le C\int_{1}^{\infty} y^{\beta+\alpha+\theta/p-q/p}dy\int_{y^{q/p}}^{\infty} x^{-t/q}\sum_{k=0}^{[2x]} E\|X\|^t I(k < \|X\|^q \le k+1)dx \\
&= C\int_{1}^{\infty} u^{-2+v/q}du\int_{u}^{\infty} x^{-t/q}\sum_{k=0}^{[2x]} E\|X\|^t I(k < \|X\|^q \le k+1)dx (\text{ letting } u = y^{q/p})
\end{aligned}$$

$$= C \int_1^\infty x^{-t/q} \sum_{k=0}^{[2x]} E\|X\|^t I(k < \|X\|^q \le k+1) dx \int_1^x u^{-2+v/q} du$$

$$\le \begin{cases} C \int_1^\infty x^{-1+v/q-t/q} \sum_{k=0}^{[2x]} E\|X\|^t I(k < \|X\|^q \le k+1) dx, & \text{if } q < v \\ C \int_1^\infty x^{-t/q} \log x \sum_{k=0}^{[2x]} E\|X\|^t I(k < \|X\|^q \le k+1) dx, & \text{if } q = v \\ C \int_1^\infty x^{-t/q} \sum_{k=0}^{[2x]} E\|X\|^t I(k < \|X\|^q \le k+1) dx, & \text{if } q > v \end{cases}$$

$$\le \begin{cases} C \sum_{k=1}^\infty E\|X\|^t I(k < \|X\|^q \le k+1) \int_{k/2}^\infty x^{-1+v/q-t/q} dx + C, & \text{if } q < v \\ C \sum_{k=2}^\infty E\|X\|^t I(k < \|X\|^q \le k+1) \int_{k/2}^\infty x^{-t/q} \log x dx + C, & \text{if } q = v \\ C \sum_{k=1}^\infty E\|X\|^t I(k < \|X\|^q \le k+1) \int_{k/2}^\infty x^{-t/q} dx + C, & \text{if } q > v \end{cases}$$

$$\le \begin{cases} C \sum_{k=1}^\infty k^{v/q-t/q} E\|X\|^t I(k < \|X\|^q \le k+1) + C, & \text{if } q < v \\ C \sum_{k=2}^\infty k^{1-t/q} \log k E\|X\|^t I(k < \|X\|^q \le k+1) + C, & \text{if } q = v \\ C \sum_{k=1}^\infty k^{1-t/q} E\|X\|^t I(k < \|X\|^q \le k+1) + C, & \text{if } q > v \end{cases}$$

$$\le \begin{cases} CE\|X\|^v + C, & \text{if } q < v \\ CE\|X\|^v \log(1+\|X\|) + C, & \text{if } q = v \\ CE\|X\|^q + C, & \text{if } q > v \end{cases}$$

$$< \infty$$

and

$$I_{212} \le C \sum_{n=1}^\infty n^{\beta-q/p} \int_{n^{q/p}}^\infty x^{-t/q} \sum_{k=[2x]+1}^\infty \sum_{j \ge [(k/x)^{p/q}]-1} (\sharp I_{nj}) j^{-t/p} E\|X\|^t I(k < \|X\|^q \le k+1) dx$$

$$\le C \sum_{n=1}^\infty n^{\beta-q/p} \int_{n^{q/p}}^\infty x^{-t/q} \sum_{k=[2x]+1}^\infty n^{\alpha+\theta/p} (k/x)^{-(t-\theta)/q} E\|X\|^t I(k < \|X\|^q \le k+1) dx$$

$$\le C \int_1^\infty y^{\beta+\alpha+\theta/p-q/p} dy \int_{y^{q/p}}^\infty x^{-\theta/q} \sum_{k=[2x]+1}^\infty k^{-(t-\theta)/q} E\|X\|^t I(k < \|X\|^q \le k+1) dx$$

$$= C \int_1^\infty u^{-2+v/q} du \int_u^\infty x^{-\theta/q} \sum_{k=[2x]+1}^\infty k^{-(t-\theta)/q} E\|X\|^t I(k < \|X\|^q \le k+1) dx$$

(letting $u = y^{q/p}$)

$$= C \int_1^\infty x^{-\theta/q} \sum_{k=[2x]+1}^\infty k^{-(t-\theta)/q} E\|X\|^t I(k < \|X\|^q \le k+1) dx \int_1^x u^{-2+v/q} du$$

$$
\leq \begin{cases} C\int_1^\infty x^{-1+v/q-\theta/q}\sum_{k=[2x]+1}^\infty k^{-(t-\theta)/q}E\|X\|^t I(k<\|X\|^q\leq k+1)dx, & \text{if } q<v\\ C\int_1^\infty x^{-\theta/q}\log x\sum_{k=[2x]+1}^\infty k^{-(t-\theta)/q}E\|X\|^t I(k<\|X\|^q\leq k+1)dx, & \text{if } q=v\\ C\int_1^\infty x^{-\theta/q}\sum_{k=[2x]+1}^\infty k^{-(t-\theta)/q}E\|X\|^t I(k<\|X\|^q\leq k+1)dx, & \text{if } q>v \end{cases}
$$

$$
\leq \begin{cases} C\sum_{k=3}^\infty k^{-(t-\theta)/q}E\|X\|^t I(k<\|X\|^q\leq k+1)\int_1^{k/2} x^{-1+v/q-\theta/q}dx, & \text{if } q<v\\ C\sum_{k=3}^\infty k^{-(t-\theta)/q}E\|X\|^t I(k<\|X\|^q\leq k+1)\int_1^{k/2} x^{-\theta/q}\log x dx, & \text{if } q=v\\ C\sum_{k=3}^\infty k^{-(t-\theta)/q}E\|X\|^t I(k<\|X\|^q\leq k+1)\int_1^{k/2} x^{-\theta/q}dx, & \text{if } q>v \end{cases}
$$

$$
\leq \begin{cases} C\sum_{k=3}^\infty k^{v/q-t/q}E\|X\|^t I(k<\|X\|^q\leq k+1), & \text{if } q<v\\ C\sum_{k=3}^\infty k^{1-t/q}\log k E\|X\|^t I(k<\|X\|^q\leq k+1), & \text{if } q=v\\ C\sum_{k=3}^\infty k^{1-t/q}E\|X\|^t I(k<\|X\|^q\leq k+1), & \text{if } q>v \end{cases}
$$

$$
\leq \begin{cases} CE\|X\|^v, & \text{if } q<v\\ CE\|X\|^v\log(1+\|X\|), & \text{if } q=v\\ CE\|X\|^q, & \text{if } q>v \end{cases}
$$

$$
<\infty.
$$

Thus $I_{21}<\infty$. □

Proof of Theorem 2.2 Let $a_{ni}, U_{ni}^{(x)}, V_{ni}^{(x)}$ be as in proof of Theorem 2.1. Let $\{X_n^{'}, n\geq 1\}$ be an independent copy of $\{X_n, n\geq 1\}$ and set $S_n^{'}=\sum_{i=1}^\infty a_{ni}X_i^{'}, n\geq 1$.

For any fixed $\epsilon>0$, $S_n \xrightarrow{P} 0$ implies that there exists a positive integer $n_0=n_0(\epsilon)$ such that for all $n>n_0$,

$$
P(\|S_n\|\leq\epsilon/2)=1-P(\|S_n\|>\epsilon/2)\geq 1/2.
$$

Hence, by (6.1) of Ledoux and Talagrand [21], when $n>n_0$, uniformly for all $x\geq 0$

$$
P(\|S_n\|>\epsilon+x)\leq 2P(\|S_n-S_n^{'}\|>\epsilon/2+x).
$$

Thus, we have that

$$
\begin{aligned}
\sum_{n=1}^\infty n^\beta E\{\|S_n\|-\epsilon\}_+^q &\leq \sum_{n=1}^{n_0} n^\beta\int_0^\infty P\left(\|S_n\|>\epsilon+x^{1/q}\right)dx\\
&\quad +2\sum_{n=n_0+1}^\infty n^\beta\int_0^\infty P\left(\|S_n-S_n^{'}\|>\epsilon/2+x^{1/q}\right)dx\\
&\leq \sum_{n=1}^{n_0} n^\beta E\|S_n\|^q+2\sum_{n=1}^\infty n^\beta E\left\{\|S_n-S_n^{'}\|-\epsilon/2\right\}_+^q.
\end{aligned}
$$

Therefore, without loss of generality, we can assume that $\{X_n, n \geq 1\}$ are symmetric. Note that

$$\begin{aligned}
14^{-q}\sum_{n=1}^{\infty} n^{\beta} E\{\|S_n\| - \epsilon\}_+^q &= \sum_{n=1}^{\infty} n^{\beta} \int_0^{\infty} P(\|S_n\| - \epsilon > 14x^{1/q})dx \\
&= \sum_{n=1}^{\infty} n^{\beta} \left(\int_0^1 P(\|S_n\| > \epsilon + 14x^{1/q})dx \right. \\
&\quad \left. + \int_1^{\infty} P(\|S_n\| > \epsilon + 14x^{1/q})dx \right) \\
&\leq \sum_{n=1}^{\infty} n^{\beta} P(\|S_n\| > \epsilon) + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P(\|S_n\| > 14x^{1/q})dx \\
&\stackrel{\text{def}}{=} I_3 + I_4.
\end{aligned}$$

By Theorem 1 and Theorem 3 of Qiu [4], we have that $I_3 < \infty$, for all $\epsilon > 0$, so we only need to prove that $I_4 < \infty$. Since

$$\begin{aligned}
I_4 &\leq \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P\left(\left\| \sum_{i=1}^{\infty} U_{ni}^{(x)} \right\| > 7x^{1/q} \right) dx + \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} P\left(\left\| \sum_{i=1}^{\infty} V_{ni}^{(x)} \right\| > 7x^{1/q} \right) dx \\
&\stackrel{\text{def}}{=} I_{41} + I_{42}.
\end{aligned}$$

Similar to the proof of $I_{22} < \infty$ in Theorem 2.1, we have $I_{42} < \infty$. Therefore, in order to prove (2.4), we only need to prove that $I_{41} < \infty$. Using the contraction principle in the form of Lemma 6.5 of Ledoux and Talagrand [21],

$$\sup_{x^{1/q} \geq 1} P\left(\left\| \sum_{i=1}^{\infty} U_{ni}^{(x)} \right\| > x^{1/q}\epsilon \right) \leq \sup_{x^{1/q} \geq 1} P(\|S_n\| > x^{1/q}\epsilon) \leq P(\|S_n\| > \epsilon) \to 0, \quad n \to \infty.$$

Hence by Lemma 3.1, when n is large enough, we have uniformly for all $x^{1/q} \geq 1$

$$P\left(\left\| \sum_{i=1}^{\infty} U_{ni}^{(x)} \right\| > 7x^{1/q} \right) \leq P\left(\left| \left\| \sum_{i=1}^{\infty} U_{ni}^{(x)} \right\| - E \left\| \sum_{i=1}^{\infty} U_{ni}^{(x)} \right\| \right| > x^{1/q} \right). \tag{3.4}$$

Taking t such that

$$\begin{cases} t > \max\{q, v, -2(1+\beta)/\gamma\}, & \text{if } \ v \geq 2, \\ t > q, & \text{if } \ 1 \leq v < 2, q \geq 2, \\ t = 2, & \text{if } \ 1 \leq v < 2, 0 < q < 2. \end{cases}$$

By (3.4), Markov's inequality, Lemma 3.1, Theorem 2.1 of de Acosta [1], we have

$$\begin{aligned}
I_{41} &\leq C \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} x^{-t/q} \left(\sum_{i=1}^{\infty} E\|U_{ni}^{(x)}\|^t \right) dx + C \sum_{n=1}^{\infty} n^{\beta} \int_1^{\infty} x^{-t/q} \left(\sum_{i=1}^{\infty} E\|U_{ni}^{(x)}\|^2 \right)^{t/2} dx \\
&\stackrel{def}{=} I_{411} + I_{412}.
\end{aligned}$$

Similar to the proof of $I_{21} < \infty$ in Theorem 2.1, we have $I_{411} < \infty$. We now prove that $I_{412} < \infty$. Consider three separate cases. If $v \geq 2$, note that in this case $E\|X_i\|^2 \leq CE\|X\|^2 < \infty, \forall\, i \geq 1$. We get by (2.5) that

$$I_{412} \leq C\sum_{n=1}^{\infty} n^{\beta} \int_{1}^{\infty} x^{-t/q} \left\{ \sum_{i=1}^{\infty} E \left\| a_{ni} X_i I\left(\|a_{ni}X_i\| \leq x^{1/q} \right\|^2 \right) \right\}^{t/2} dx$$

$$\leq C\sum_{n=1}^{\infty} n^{\beta} \int_{1}^{\infty} x^{-t/q} (n^{\gamma})^{t/2} dx$$

$$= C\sum_{n=1}^{\infty} n^{\beta + t\gamma/2} < \infty.$$

If $1 \leq v < 2, q \geq 2$, note that in this case $E\|X_i\|^2 \leq CE\|X\|^2 < \infty, \forall\, i \geq 1$. We get by (3.1) and (3.2) that

$$I_{412} \leq C\sum_{n=1}^{\infty} n^{\beta} \int_{1}^{\infty} x^{-t/q} n^{\{\alpha-(2-\theta)/p\}t/2} dx$$

$$= C\sum_{n=1}^{\infty} n^{\beta+\{\alpha-(2-\theta)/p\}t/2} < \infty.$$

If $1 \leq v < 2, q < 2$, we have that $I_{412} = I_{411} < \infty$; thus $I_{41} < \infty$. □

Proof of Corollary 2.5 Clearly $\sup_{i\geq 1, n\geq 1} |b_{ni}| < \infty$. Let $a_{ni} = n^{-1/p} b_{ni}, \beta = 0, \alpha = -\theta/p$, and $\gamma = \lambda - 2/p < 0$. Therefore, the result follows from Theorem 2.2. □

Proof of Corollary 2.6 Clearly $\sup_{i\geq 1, n\geq 1} |b_{ni}| < \infty$. Let $p = 2, a_{ni} = n^{-1/2} b_{ni}, \beta = 0, \theta = 2, \alpha = -1$, and $\gamma = -1$. Therefore, the result follows from Theorem 2.4. □

Proof of Corollary 2.7 Note that $\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \sum_{j=-\infty}^{\infty} a_{j+i} Y_j = \sum_{j=-\infty}^{\infty} \sum_{i=1}^{n} a_{j+i} Y_j$, set $a_{ni} = n^{-1/p} \sum_{j=1}^{n} a_{j+i}$ for all $i \geq 1, n \geq 1$. Since $b = \sum_{n=-\infty}^{\infty} |a_n| < \infty$, we have that

$$\sup_{i\geq 1} |a_{ni}| \leq n^{-1/p} \sup_{i\geq 1} \sum_{j=1}^{n} |a_{j+i}| \leq n^{-1/p} \sum_{n=-\infty}^{\infty} |a_n| = n^{-1/p} b,$$

and

$$\sum_{i=-\infty}^{\infty} |a_{ni}|^{\theta} \leq n^{-1/p} \sum_{i=-\infty}^{\infty} \left(\sum_{j=1}^{n} |a_{j+i}| \right) n^{-(\theta-1)/p} b^{\theta-1}$$

$$= n^{-\theta/p} b^{\theta-1} \sum_{j=1}^{n} \sum_{i=-\infty}^{\infty} |a_{j+i}| = n^{1-\vartheta/p} b^{\theta}, \forall\, \theta \geq 1.$$

Taking $\theta = 1, \alpha = 1 - 1/p, \gamma = 1 - 2/p$, all conditions of Theorems 2.2 and 2.4 are satisfied. Thus the result follows from Theorems 2.2 and 2.4. □

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